ICTP-VIASM-CMI-IMU

Summer School in Differential Geometry

2023

Integral Current Spaces

and

Intrinsic Flat Convergence

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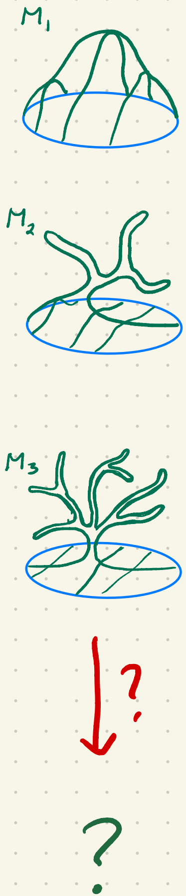
History: How does an Area Minimizing Sequence Converge and what does it converge to?



[FF] Federer - Fleming Annals 1960
defined the Flat and Weak Convergence
of Submanifolds and integral currents in E^N
and proved a Compactness Theorem for them.

[AK] Ambrosio - Kirchheim Acta 2000
defined integral currents in a metric space, Z,
and proved a Compactness Theorem for them.

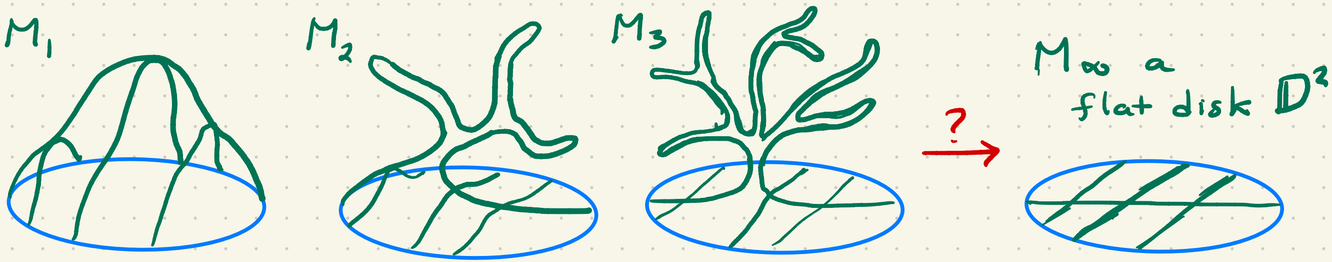
[SW] Sormani - Wenger JDG 2011
defined the intrinsic flat convergence of
Riemannian Manifolds and integral current spaces.

[W] Wenger CVPDE 2011
proved a Compactness Theorem for them.

Federer - Fleming introduced integral currents in $\mathbb{E}^N$ to study the convergence of submanifolds in $\mathbb{E}^N$

Plateau Problem: Is $\inf\{\text{Area}(M) : \partial M = \mathbb{S}^1 \subset \mathbb{E}^3\}$ achieved?

Take $M_j \subset \mathbb{E}^3$ s.t. $\partial M_j = \mathbb{S}^1$ and $\text{Area}(M_j) \to \inf\{\text{Area}(M) : \partial M = \mathbb{S}^1\}$



M_j converge weakly as integral currents to M_∞:

if $\displaystyle\int_{M_j} \omega \to \int_{M_\infty} \omega$ for any differential form

$\omega = a\,dx\wedge dy + b\,dy\wedge dz + c\,dz\wedge dx$

Note that $\partial M_j \to \partial M_\infty$ because

$$\int_{\partial M_j} \eta = \int_{M_j} d\eta \to \int_{M_\infty} d\eta = \int_{\partial M_\infty} \eta \quad \text{for any one form } \eta = A\,dx + B\,dy + C\,dz$$

FF: $M_j \to M_\infty \Rightarrow \liminf_{j\to\infty} \text{Area}(M_j) \geq \text{Area}(M_\infty)$

To handle nonsmooth limits FF defined integral currents, T, which act on forms with boundary defined by $\partial T(\eta) = T(d\eta)$ and mass s.t. $T_j \to T_\infty \Rightarrow \liminf_{j\to\infty} \text{Mass}(T_j) \geq \text{Mass}(T_\infty)$

Federer - Fleming Compactness Theorem:

If Mass $(M_j) \le V$ and Mass $(\partial M_j) \le A$ and $M_j \subset K$ compact

Then a subseq $M_{j_k} \longrightarrow T_\infty$ where T_∞ is an integral current.

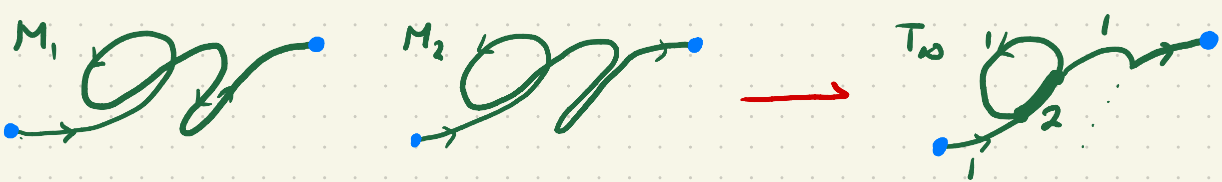
Definition: T_∞ is an integral current if

$\exists$ Borel sets $A_i \subset \mathbb{R}^m$ and Lipschitz $\varphi_i : A_i \to E^N$

and weights $a_i \in \mathbb{Z}$ s.t. $T_\infty(\omega) = \sum_{i=1}^\infty a_i \int_{A_i} \varphi_i^* \omega$

and ∂T has finite mass.

Why do they need weights? Doubling and Cancellation:



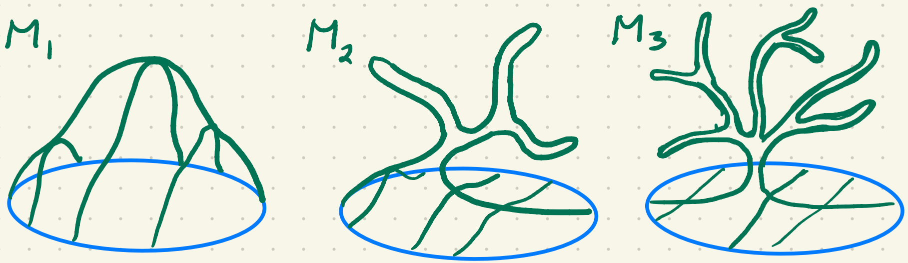
FF Compactness Theorem solves the Plateau Problem:

Plateau Problem: Is $\inf\{\text{Area}(M) : \partial M = S^1 \subset \mathbb{E}^3\}$ achieved?

Take $M_j \subset \mathbb{E}^3$ s.t $\partial M_j = S^1$ and $\text{Area}(M_j) \to \inf\{\text{Area}(M) : \partial M = S^1\}$

Satisfies: $\text{Mass}(\partial M_j) \le A$ and $\text{Mass}(M_j) \le V$ and $M_j \subset K$ compact after chopping

Thus: M_1 M_2 M_3 $\longrightarrow T_\infty$

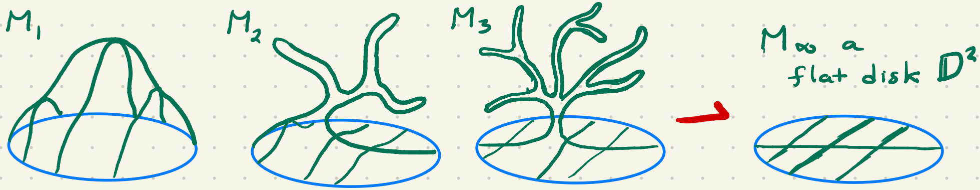


Furthermore: $\text{Mass}(T_\infty) \le \liminf_{j\to\infty} \text{Area}(M_j) = \inf\{\text{Area}(M) : \partial M = S^1\}$

and $\partial T_\infty = \text{weak}\lim \partial M_i = \text{weak}\lim S^1 = S^1$.

So T_∞ achieves the infimum in the Plateau Problem!

How to see convergence to a flat disk?



Must Show:

$$\int_{M_i} \omega \to \int_{\mathbb{D}^2} \omega$$

In the picture above, let B_i = region between M_i and $\mathbb{D}^2$;

$$\int_{M_i} \omega - \int_{\mathbb{D}^2} \omega = \int_{\partial B_i} \omega = \int_{B_i} d\omega \longrightarrow 0 \text{ because } \mathrm{Vol}(B_i) \to 0$$

FF defined the flat ($\flat$) (Whitney) distance between currents

$$|T_j - T_\infty|_\flat = \inf\{\mathrm{Mass}(A) + \mathrm{Mass}(B) : A + \partial B = T_j - T_\infty\}$$

$T_j \xrightarrow{\flat} T_\infty$ implies $T_j \longrightarrow T_\infty$ because
$$T_j(\omega) - T_\infty(\omega) = A_j(\omega) + \partial B_j(\omega)$$
$$= A_j(\omega) + B_j(d\omega) \to 0$$

FF proved the converse when $\mathrm{Mass}(T_j) + \mathrm{Mass}(\partial T_j) \le K$.

Textbook Recommendation for FF theory:

Geometric Measure Theory by Morgan

Next we survey:

[AK] Ambrosio-Kirchheim Acta 2000
defined integral currents in a metric space, Z,
and proved a Compactness Theorem for them.

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Ambrosio-Kirchheim defined [AK]

Integral Currents on Metric Spaces

Consider a complete metric space (Z, d_Z)

we have no differential forms so we take

$\omega = (f, \pi_1 \ldots \pi_m)$ to be a tuple of functions

where $f : Z \longrightarrow \mathbb{R}$ bounded and Lipschitz

and $\pi_i : Z \longrightarrow \mathbb{R}$ Lipschitz

Lipschitz submanifolds $\varphi : M^m \longrightarrow Z$ act on tuples:

$$\varphi_* [\![M]\!] (f, \pi_1 \ldots \pi_m) = \int_{M^m} f \circ \varphi \, d(\pi_1 \circ \varphi) \wedge \ldots \wedge d(\pi_m \circ \varphi)$$

They define

$d\omega = (1, f, \pi_1 \ldots \pi_m)$ so that $\varphi_* [\![M]\!] (d\omega) = \varphi_* [\![\partial M]\!] (\omega)$

Ambrosio-Kirchheim: a rectifiable current, T, is a current acting on tuples such that there exists Borel sets $A_i \subset \mathbb{R}^m$ and bi Lipschitz $\varphi_i : A_i \to Z$ disjoint images and weights $a_i \in \mathbb{R}$ s.t. $T = \sum_{i=1}^{\infty} a_i [\![\varphi_{i\#} A_i]\!]$

such that $\sum_{i=1}^{\infty} |a_i| \, \mathcal{H}^m(\varphi_i(A_i)) < \infty$. (finite mass)

T is an integer rectifiable current if $a_i \in \mathbb{Z}$.

So $$T(f, \pi_1, \dots \pi_m) = \sum_{i=1}^{\infty} a_i \int_{A_i} f \circ \varphi_i \, d(\pi_1 \circ \varphi_i) \wedge \dots \wedge d(\pi_m \circ \varphi_i)$$

Note the collection of charts $\varphi_i : A_i \to Z$ is not unique.

$T_1 = T_2 \iff T_1(f, \pi_1, \dots \pi_m) = T_2(f, \pi_1, \dots \pi_m)$ for all tuples

Ambrosio-Kirchheim: an integral current, T, is an integer rectifiable current whose boundary, ∂T, is also integer rectifiable, where $\partial T(f, \pi_1, \dots \pi_m) = T(1, f, \pi_1, \dots \pi_m)$

Weak Convergence as currents:

$$T_j \to T_\infty \iff T_j(\omega) \to T_\infty(\omega) \text{ for all tuples, } \omega.$$

Then as before $T_j \to T_\infty \implies \partial T_j \to \partial T_\infty$.

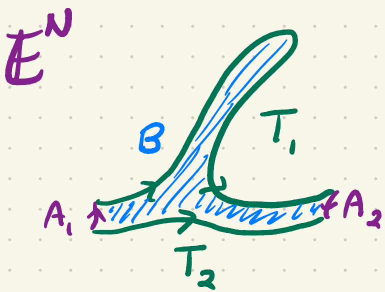
AK define mass so that $\liminf_{j \to \infty} \text{Mass}(T_j) \geq \text{Mass}(T_\infty)$.

AK Compactness Theorem: If Z is a compact metric space and T_j are integral currents s.t $\text{Mass}(T_j) \leq V_0$ and $\text{Mass}(\partial T_j) \leq A_0$ then a subsequence $T_j \to T_\infty$ which is an integral current. possibly 0.

Federer-Fleming had the same compactness theorem for Z compact inside Euclidean Space $\mathbb{E}^N$

Recall FF defined the flat ($\flat$) distance between integral currents
(whitney)

$$|T_1 - T_2|_\flat = \inf\{\text{Mass}(A) + \text{Mass}(B) : A + \partial B = T_1 - T_2\} = d_F^{\mathbb{E}^N}(T_1, T_2)$$



They proved that, under the hypothesis of their compactness theorem,

$$T_j \to T_\infty \iff |T_j - T_\infty|_\flat \to 0$$

Wenger studied the flat (d_F^Z) distance between integral currents in a complete metric space Z and proved the same results for Ambrosio-Kirchheim's notion of integral currents.

Best source for Ambrosio-Kirchheim Theory

is the original article:

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Next we survey:

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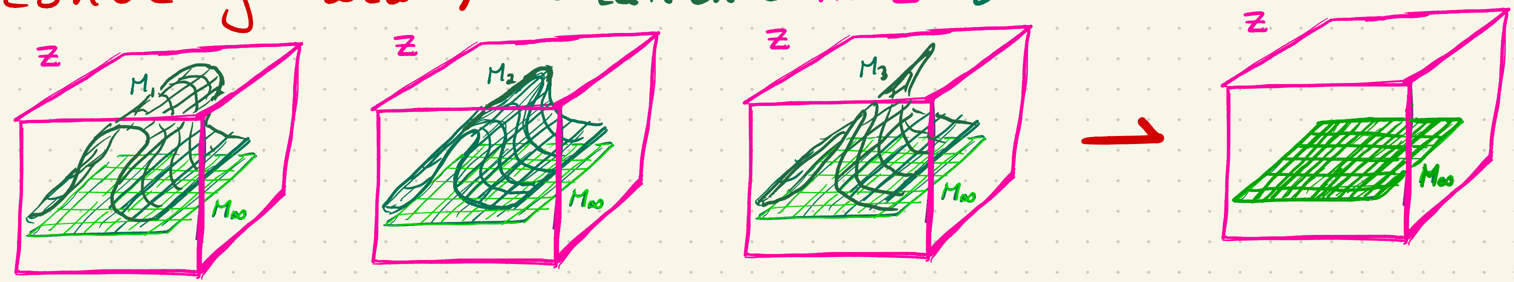
Extrinsic vs Intrinsic

All these notions in [FF] and [AK] are defined relative to an extrinsic space, Z:
Submanifolds, M_j, converge to limits, M_∞, based on how the M_j sit inside Z.

Example: M_j with an increasingly narrow wave in Z converge weakly as currents in Z to the smaller M_∞:



However, viewed intrinsically these $M_j = [0,2]\times[0,1]$ are all the same so they do not converge to $M_\infty = [0,1]\times[0,1]$

Sormani-Wenger defined an intrinsic flat distance and convergence for a sequence of oriented Riemannian Manifolds imitating Gromov's defn of intrinsic Hausdorff distance:

$$M_j \xrightarrow{\mathcal{F}} M_\infty \iff d_{\mathcal{F}}(M_j, M_\infty) \to 0 \text{ where}$$

$$d_{\mathcal{F}}(M_j, M_\infty) = \inf\left\{ d_F^Z(\varphi_{j*}[\![M_j]\!], \varphi_{\infty *}[\![M_\infty]\!]) : \begin{array}{l} \varphi_j : M_j \to Z \\ \text{dist pres} \\ \text{and } Z \text{ complete} \end{array} \right\}$$

where the infimum is over all complete Z and over all distance preserving maps $\varphi_j : M_j \to Z$

$$d_Z(\varphi_j(p), \varphi_j(q)) = d_{M_j}(p, q) \quad \forall p, q \in M_j.$$

SW also defined the intrinsic flat distance for integral current spaces (X, d, T)

which are metric spaces (X, d) with an integral current T defined on $\overline{X}$ (metric completion)

such that $X = \mathrm{set}(T) = \left\{ p \in \overline{X} : \liminf_{r \to 0} \frac{\mathrm{Mass}(B_p(r))}{r^m} > 0 \right\}$

Note that these spaces are rectifiable with oriented weighted Lipshitz charts $\varphi_i : A_i \longrightarrow X$

such that $T = \sum_{i=1}^{m} a_i \, \varphi_{i*}[\![A_i]\!]$ and $\mathcal{H}^m\left(X \setminus \bigcup_{i=1}^{\infty} \varphi_i(A_i)\right) = 0$

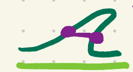
and so are $\partial(X, d, T) = (\mathrm{set}(\partial T), d, \partial T)$.

We also defined the 0^m space $(\emptyset, 0, 0)$ in each dimension.

The intrinsic flat distance between $M_j = (X_j, d_j, T_j)$

$$d_{\mathcal{F}}(M_j, M_\infty) = \inf\left\{ d_F^Z(\varphi_{j*}(T_j), \varphi_{\infty*}(T_\infty)) : \begin{array}{l}\varphi_j : X_j \to Z \\ \text{dist pres} \\ \text{and } Z \text{ complete}\end{array} \right\}$$

where the inf is over all complete metric spaces Z
(so it does not depend on a specific extrinsic Z)

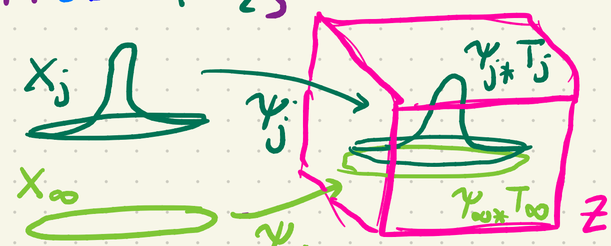
and all distance preserving maps $\varphi_j : X_j \to Z$
(so the spaces are not folded )

The flat distance d_F^Z is defined using currents in Z

$$d_F^Z(T_1, T_2) = \inf\{ \text{Mass}(A) + \text{Mass}(B) \mid A + \partial B = T_1 - T_2 \}$$

is taken between push forwards

$$\varphi_* T(f, \pi_1, \dots \pi_m) = T(f\circ\varphi, \pi_1\circ\varphi, \dots \pi_m\circ\varphi)$$



Note $d_{\mathcal{F}}(0, M)$ is well defined.

Sormani-Wenger $d_{\mathcal{F}}((X_1,d_1,T_1),(X_2,d_2,T_2))=0 \iff$

$\exists$ an isometry $F:(X_1,d_1)\to(X_2,d_2)$ s.t. $F_* T_1 = T_2$.

Sormani-Wenger: If $M_j \xrightarrow{\mathcal{F}} M_\infty$ then $\exists Z$ complete

and there are distance preserving $\varphi_j : M_j \to Z$

such that $d_F^Z(\varphi_{j*}T_j, \varphi_{\infty *}T_\infty)\to 0$ so

$\partial M_j \xrightarrow{\mathcal{F}} \partial M_\infty$ where $\partial(X,d,T)=(\mathrm{set}(\partial T), d, \partial T)$

and $\liminf_{j\to\infty} \mathrm{Mass}(M_j) \geq \mathrm{Mass}(M_\infty)$ where $\mathrm{Mass}(M)=\mathrm{Mass}(T)$

Wenger: If $M_j^m=(X_j,d_j,T_j)$ are integral current spaces

and $\mathrm{Mass}(M_j^m)\leq V_0$, $\mathrm{Mass}(\partial M_j)\leq A_0$, and $\mathrm{Diam}(M_j^m)\leq D_0$

then $\exists M_{j_k}^m \xrightarrow{\mathcal{F}} M_\infty^m$ where M_∞^m is an integral current space (possibly 0^m)

Applications of Intrinsic Flat Convergence
will be discussed in Brian Allen's Lectures.

For links to all these papers and many more
about intrinsic flat convergence see

https://sites.google.com/site/intrinsicflatconvergence/

Thank You!

email me at
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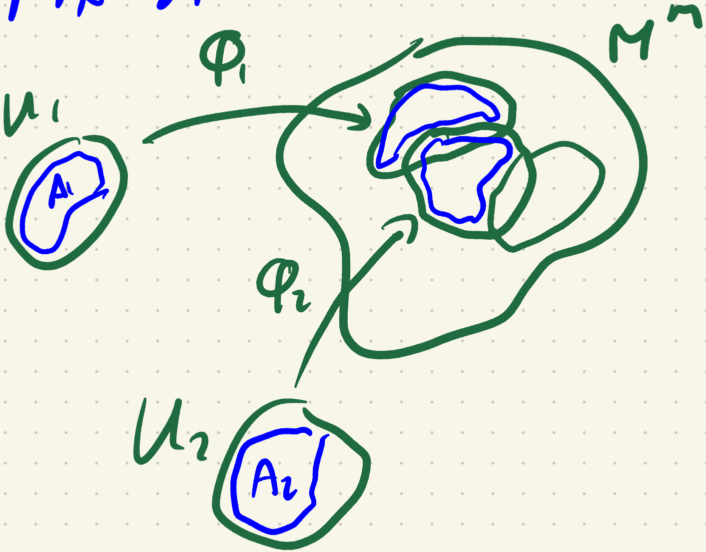
Questions Below:

Question I: Given an oriented Riemannian Manifold, how to define charts $\varphi_i : A_i \to X$ to define an integral current space?

Fix orientation on M^m



Take a collection of smooth oriented charts $\varphi_i : U_i \subseteq \mathbb{R}^m \to M^m$

Then restrict U_i to A_i to ensure $\varphi_i : A_i \subset U_i \subset \mathbb{R}^m \to M^m$ are disjoint

Use weight $1 = a_i$

Get two integral current spaces: one for each orientation

The nice thing about integral current spaces is we have disjoint charts so no transition maps to check.

And φ_i only have to be Lipschitz.

Best Resource is original JDG 2011 article by Sormani Wenger.

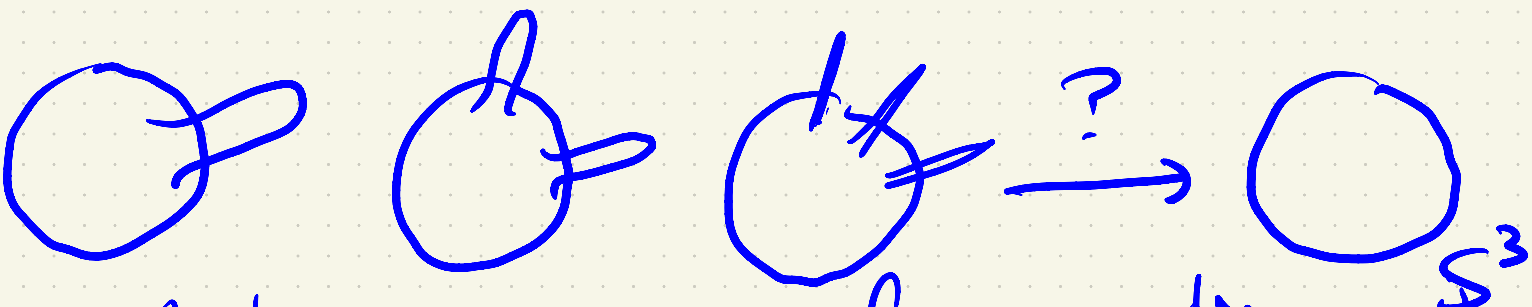
Videos of Courses about Intrinsic Flat Convergence:

https://sites.google.com/site/professorsormani/home/teaching/fourier-s21

https://sites.google.com/site/professorsormani/home/teaching/fields-institute-lectures-2017

Question II: What was the inspiration for intrinsic flat convergence?

Ilmanen Example $M_j^3 = (S^3, g_j)$ with $\mathrm{scal}_j \geq 0$



He asked someone to define a notion of convergence s.t. the limit of his example is S^3. GH convergence fails.

Wenger and I thought this looks like FF flat convergence. So we worked to define an intrinsic version of this notion.

Please email me if you have a questions

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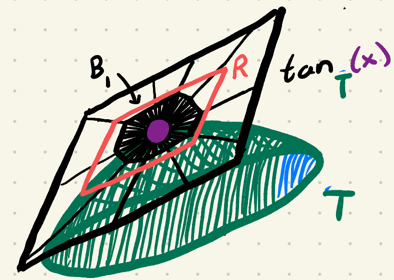
Definition of Mass is below:

[SW] use [AK] Defn: $\overset{\text{fixed } T}{\mathrm{Mass}(T)} = \|T\|(Z)$ where $\|T\|$ is the minimal Borel measure, $\underset{\text{over all measures}}{\mu}$, s.t. $T(f, \pi_1 \dots \pi_m) \le \prod_{i=1}^{m} \mathrm{Lip}(\pi_i) \int_Z |f| \mu$ $\forall$ tuples $(f, \pi_1, \dots, \pi_m)$

Thus: $(X_j, d_j, T_j) \xrightarrow{\mathcal{F}} (X_\infty, d_\infty, T_\infty) \Longrightarrow \liminf_{j \to \infty} \mathrm{Mass}(T_j) \ge \mathrm{Mass}(T_\infty)$

Theorem [AK] $\|T\| = \theta_T \, \lambda_T \, \mathcal{H}^m \llcorner \operatorname{set} T$

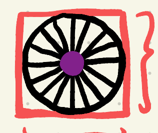


where $\theta_T(x)$ = integer weight of T at x

and $\lambda_T(x)$ = area factor of Banach space $\tan_T(x)$

$$= \frac{2^m}{\omega_m} \sup \left\{ \frac{\mathcal{H}^k(B_1)}{\mathcal{H}^k(R)} : \text{parallelopiped } R \supset B_1 \text{ s.t. } R \subset \tan_T(x) \right\} \subset [c_m, C_m]$$

where B_1 is the unit ball in $\tan_T(x)$

Note that if $\tan_T(x)$ is Hilbert then  $\}^2 \Rightarrow \lambda_T(x) = \frac{2^m}{\omega_m} \frac{\omega_m}{2^m} = 1$