Non-stationary Energy in General Relativity

Emel Altas and Bayram Tekin

Karamanoglu Mehmetbey University

Phys. Rev. D 101 no.2, 024035 (2020)

13 July, 2023

Overview

1. Motivation
2. Space+Time split of Spacetime
3. Non-stationary energy with time evolution equations

Motivation

- Dain constructed a geometric invariant that measures the non-stationary energy for an asymptotically flat hypersurface in 3+1 dimensions for the case of time-symmetric initial data. So this invariant is a component of the total ADM energy.
- Let us recall the first observation of the merger of two black holes. According to this observation, two initial black holes with masses (approximately) $36M_\odot$ and $29M_\odot$ merged to produce a single stationary black hole of mass $62M_\odot$ plus gravitational radiation of total energy equivalent to $3M_\odot$. Assuming this system to be isolated, the total initial ADM energy of $65M_\odot$ is conserved. But this total ADM energy of the initial data has a non-stationary part equal to $3M_\odot$. The important question is to identify this non-stationary energy in the initial data.
- We briefly summarize Dain's construction using the constraints and present a new approach using the evolution equations.

Space+Time split of Spacetime

$$ds^2 = (N_i N^i - N^2)dt^2 + 2N_i dt dx^i + \gamma_{ij} dx^i dx^j, \tag{1}$$

$$K_{ij} = \frac{1}{2N}\left(\dot{\gamma}_{ij} - D_i N_j - D_j N_i\right), \tag{2}$$

Under the above decomposition of spacetime, the D-dimensional Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{3}$$

yield the Hamiltonian and momentum constraints on the hypersurface Σ as

$$\begin{aligned} \Phi_0(\gamma, K) &:= -{}^{\Sigma}R - K^2 + K_{ij}^2 + 2\Lambda - 2\kappa T_{nn} = 0, \\ \Phi_i(\gamma, K) &:= -2D_k K_i^k + 2D_i K - 2\kappa T_{ni} = 0, \end{aligned} \tag{4}$$

where $K := \gamma^{ij} K_{ij}$ and $K_{ij}^2 := K^{ij} K_{ij}$.

Bartnik's operator

Denoting $\Phi(\gamma, K)$ to be the constraint covector with components (Φ_0, Φ_i) and $D\Phi(\gamma, K)$ to be its linearization about a given solution (γ, K) to the constraints and $D\Phi^*(\gamma, K)$ to be the formal adjoint map, then following Bartnik, one defines another operator $\mathcal{P}$:

$$\mathcal{P} := D\Phi(\gamma, K) \circ \begin{pmatrix} 1 & 0 \\ 0 & -D^m \end{pmatrix}. \tag{5}$$

Dain invariant

$$\mathscr{I}(N, N^i) := \int_\Sigma dV \; \mathcal{P}^* \begin{pmatrix} N \\ N^k \end{pmatrix} \cdot \mathcal{P}^* \begin{pmatrix} N \\ N^k \end{pmatrix}, \tag{6}$$

The integral (6) is to be evaluated for specific vectors $\xi := (N, N^i)$ that satisfy the fourth-order equation

$$\mathcal{P} \circ \mathcal{P}^* (\xi) = 0, \tag{7}$$

which Dain [1] called the *approximate Killing initial data* (KID) equation.

[1] S. Dain, A New Geometric Invariant on Initial Data for the Einstein Equations, Phys. Rev. Lett. **93**, 23, 231101 (2004).

Can we do better ? : YES !

Phase space variables: γ_{ij} and the canonical momenta π^{ij}; be found from the Einstein-Hilbert Lagrangian

$$\mathscr{L}_{EH} = \frac{1}{\kappa}\sqrt{-g}(R - 2\Lambda) = \frac{1}{\kappa}\sqrt{\gamma}N({}^{\Sigma}R + K_{ij}^2 - K^2 + \Lambda) + \text{boundary terms} \tag{8}$$

$$\pi^{ij} := \frac{\delta \mathscr{L}_{EH}}{\delta \dot{\gamma}_{ij}} = \frac{1}{\kappa}\sqrt{\gamma}(K^{ij} - \gamma^{ij}K). \tag{9}$$

Using the canonical momenta, it pays to recast the densitized versions of the constraints (4) for $T_{\mu\nu} = 0$ and setting $\kappa = 1$ as

$$\begin{aligned} \Phi_0(\gamma, \pi) &:= \sqrt{\gamma}\left(-{}^{\Sigma}R + 2\Lambda\right) + G_{ijkl}\pi^{ij}\pi^{kl} = 0, \\ \Phi_i(\gamma, \pi) &:= -2\gamma_{ik}D_j\pi^{kj} = 0, \end{aligned} \tag{10}$$

where the *DeWitt metric* G_{ijkl} in D is

$$G_{ijkl} = \frac{1}{2\sqrt{\gamma}}\left(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \frac{2}{D-2}\gamma_{ij}\gamma_{kl}\right). \tag{11}$$

Non-stationary energy with time evolution equations

Ignoring the possible boundary terms, the ADM Hamiltonian density turns out to be a sum of the constraints as

$$\mathcal{H} = \int_\Sigma d^{D-1}x \, \langle \mathcal{N}, \Phi(\gamma, \pi) \rangle \,, \tag{12}$$

with $\mathcal{N}$ being the lapse-shift vector with components (N, N^i) which play the role of the Lagrange multipliers; and the angle-brackets denote the usual contraction. Given an $\mathcal{N}$, the remaining evolution equations can be written in a compact form (the Fischer-Marsden form) as

$$\frac{d}{dt}\begin{pmatrix}\gamma \\ \pi\end{pmatrix} = J \circ D\Phi^*(\gamma, \pi)(\mathcal{N}), \tag{13}$$

where the J matrix reads

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{14}$$

Non-stationary energy with time evolution equations

The reason why the formal adjoint of the linearized constraint map $D\Phi^*(\gamma, \pi)$ appears in the time evolution can be seen as follows: the Hamiltonian form of the Einstein-Hilbert action

$$\mathcal{S}_{EH}[\gamma, \pi] = \int dt \int d^{D-1}x \left(\pi^{ij}\dot{\gamma}_{ij} - \langle \mathcal{N}, \Phi(\gamma, \pi)\rangle\right), \tag{15}$$

when varied about a background (γ, π) gives

$$D\mathcal{S}_{EH}[\gamma, \pi] = \int dt \int d^{D-1}x \left(\delta\pi^{ij}\dot{\gamma}_{ij} + \pi^{ij}\delta\dot{\gamma}_{ij} - \langle \mathcal{N}, D\Phi(\gamma, \pi) \cdot (\delta\gamma, \delta\pi)\rangle\right). \tag{16}$$

Here the linearized form of the constraint map can be computed to be

$$D\Phi\begin{pmatrix} h_{ij} \\ p^{ij} \end{pmatrix} = \begin{pmatrix} \sqrt{\gamma}\left({}^{\Sigma}R^{ij}h_{ij} - D^iD^jh_{ij} + \triangle h\right) \\ -hG_{ijkl}\pi^{ij}\pi^{kl} + 2G_{ijkl}p^{ij}\pi^{kl} + 2G_{njkl}h_{im}\gamma^{mn}\pi^{ij}\pi^{kl} \\ \\ -2\gamma_{ik}D_jp^{kj} - \pi^{jk}\left(2D_kh_{ij} - D_ih_{jk}\right) \end{pmatrix}, \quad (17)$$

where $\delta\gamma_{ij} := h_{ij}$, $h := \gamma^{ij}h_{ij}$, $\delta\pi^{ij} := p^{ij}$ and $\triangle := D_kD^k$.

In (16) using integration by parts when necessary and dropping the boundary terms one arrives at the desired result

$$D\mathcal{S}_{EH}\left[\gamma,\pi\right]=\int dt\int d^{D-1}x\left(\delta\pi^{ij}\dot{\gamma}_{ij}-\dot{\pi}^{ij}\delta\gamma_{ij}-\left\langle\left(\delta\gamma,\delta\pi\right),D\Phi^{*}(\gamma,\pi)\cdot\mathcal{N}\right\rangle\right) \tag{18}$$

where the adjoint constraint map appears in the process which reads

$$D\Phi^{*}\begin{pmatrix}N\\N^{i}\end{pmatrix}=\begin{pmatrix}\sqrt{\gamma}\left({}^{\Sigma}R^{ij}-D^{i}D^{j}+\gamma^{ij}\triangle\right)N\\-N\gamma^{ij}G_{klmn}\pi^{kl}\pi^{mn}+2NG_{klmn}\gamma^{ik}\pi^{jl}\pi^{mn}\\+2\pi^{k(i}D_{k}N^{j)}-D_{k}(N^{k}\pi^{ij})\\ \\2NG_{ijkl}\pi^{kl}+2D_{(i}N_{j)}\end{pmatrix}. \tag{19}$$

Evolution equations

Setting the variation (18) to zero one obtains the evolution equations (13) or in more explicit form one has

$$\frac{d\gamma_{ij}}{dt} = 2NG_{ijkl}\pi^{kl} + 2D_{(i}N_{j)}, \tag{20}$$

and

$$\begin{aligned} \frac{d\pi^{ij}}{dt} = & \sqrt{\gamma}\left(-^{\Sigma}R^{ij} + D^i D^j - \gamma^{ij}\triangle\right) N + N\gamma^{ij} G_{klmn}\pi^{kl}\pi^{mn} \\ & -2NG_{klmn}\gamma^{ik}\pi^{jl}\pi^{mn} - 2\pi^{k(i}D_k N^{j)} + D_k(N^k\pi^{ij}). \end{aligned} \tag{21}$$

Together with the constraints (10) these two tensor equations constitute a set of constrained dynamical system for a *given* lapse-shift vector an (N, N^i).

The constraints have a dual role: they determine the viable initial data and also generate time evolution of the initial data once the lapse-shift vector is chosen. As noted above, if $D\Phi^*(\gamma,\pi)(\mathcal{N}) = 0$, namely $\mathcal{N} = \xi$ is a Killing vector field then the time evolution is trivial. In particular this would be the case for a stationary Killing vector.

Consider now an $\mathcal{N}$ which is *not* a Killing vector, which means $D\Phi^*(\gamma,\pi)(\mathcal{N}) \neq 0$; and in particular directly from the evolution equations we can find how much $D\Phi^*(\gamma,\pi)(\mathcal{N})$ differs from zero (or how much a given $\mathcal{N}$ fails to be a Killing vector) as

$$D\Phi^*(\gamma,\pi)(\mathcal{N}) = J^{-1} \circ \frac{d}{dt} \begin{pmatrix} \gamma \\ \pi \end{pmatrix}. \qquad (22)$$

To get a number from this matrix, first one should note that the units of γ and π are different by a factor of $1/L$ and so a naive approach of taking the "square" of this matrix does not work. At this stage to remedy this, one needs the (adjoint) operator of Bartnik that we have introduced above: so one has

$$\mathcal{P}^*(\mathcal{N}) := \begin{pmatrix} 1 & 0 \\ 0 & D_m \end{pmatrix} \circ D\Phi^*(\gamma, \pi)(\mathcal{N}) = \begin{pmatrix} 1 & 0 \\ 0 & D_m \end{pmatrix} \circ J^{-1} \circ \frac{d}{dt} \begin{pmatrix} \gamma \\ \pi \end{pmatrix}, \quad (23)$$

which yields $\mathcal{P}^*(\mathcal{N}) = (-\dot{\pi}, D_m \dot{\gamma})$.

Non-stationary energy with time evolution equations

Since π is a tensor density to get a number out of this vector, we further define

$$\widetilde{\mathcal{P}}^*(\mathcal{N}) := \begin{pmatrix} \gamma^{-1/2} & 0 \\ 0 & 1 \end{pmatrix} \circ \mathcal{P}^*(\mathcal{N}). \tag{24}$$

Then the integral of $\widetilde{\mathcal{P}}^*(\mathcal{N}) \cdot \widetilde{\mathcal{P}}^*(\mathcal{N})$ over the hypersurface yields

$$\mathscr{I}(\mathcal{N}) = \int_\Sigma dV\, \widetilde{\mathcal{P}}^*(\mathcal{N}) \cdot \widetilde{\mathcal{P}}^*(\mathcal{N}) = \int_\Sigma dV\, \left(|D_m \dot{\gamma}_{ij}|^2 + \frac{1}{\gamma} |\dot{\pi}^{ij}|^2 \right), \tag{25}$$

where $|D_m \dot{\gamma}_{ij}|^2 := \gamma^{mn} \gamma^{ij} \gamma^{kl} D_m \dot{\gamma}_{ik} D_n \dot{\gamma}_{jl}$ and $|\dot{\pi}^{ij}|^2 := \gamma_{ij} \gamma_{kl} \dot{\pi}^{ik} \dot{\pi}^{jl}$.

$$\mathscr{I}(\mathcal{N}) = \int_{\Sigma} dV \Big\{ |D_m V^{ij}|^2 + {}^{\Sigma}R_{ij}^2 N^2 + (D_i D_j N)^2 - 2^{\Sigma} R^{ij} N D_i D_j N$$
$$+(D-3)\triangle N \triangle N + 2Q\triangle N + Q_{ij}^2 + 2^{\Sigma} R_{ij} N Q^{ij} - 2Q^{ij} D_i D_j N$$
$$+2^{\Sigma} R N \triangle N + 4 D_m D_{(i} N_{j)} D^m D^{(i} N^{j)} + 4 D_m D_i N_j D^m V^{ij} \Big\}, \quad (26)$$

where

$$V^{ij} := \frac{2N}{\sqrt{\gamma}} \left(\pi^{ij} - \frac{1}{D-2} \pi \gamma^{ij} \right), \quad (27)$$

and

$$Q^{ij} : = \frac{2N}{\gamma} \left(\pi^i_k \pi^{kj} - \frac{\pi \pi^{ij}}{D-2} \right) - \frac{N}{\gamma} \gamma^{ij} \left(\pi^2_{kl} - \frac{\pi^2}{D-2} \right)$$
$$- \frac{1}{\sqrt{\gamma}} D_k (N^k \pi^{ij}) + \frac{2}{\sqrt{\gamma}} \pi^{k(i} D_k N^{j)}. \quad (28)$$

Conclusions

This is another representation of Dain's invariant [2] which explicitly involves the time derivatives of the canonical fields. We have also not assumed that the cosmological constant vanishes, hence our result is valid for generic spacetimes. [3]. Note that this expression is valid for any $\mathcal{N}$ which is not necessarily an approximate KID, hence given a solution to the constraint equations and a choice of the lapse-shift vector, one can compute this integral. But the volume integral becomes a surface integral when $\mathcal{N}$ is an approximate KID which is the case considered by Dain. Observe that by construction, $\mathscr{I}(\mathcal{N})$ is a non-negative number.

[2] S. Dain, A New Geometric Invariant on Initial Data for the Einstein Equations, Phys. Rev. Lett. **93**, 23, 231101 (2004)

[3] E. Altas and B. Tekin, Nonstationary energy in general relativity, Phys. Rev. D **101** no.2, 024035 (2020)

The End