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Symmetries of Black Hole Perturbation Theory

based on
arXiv: 2010.00593, 2105.01069, 2203.08832, 2304.03743
and other works in progress in collaboration with
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"Dark Side of the Universe"
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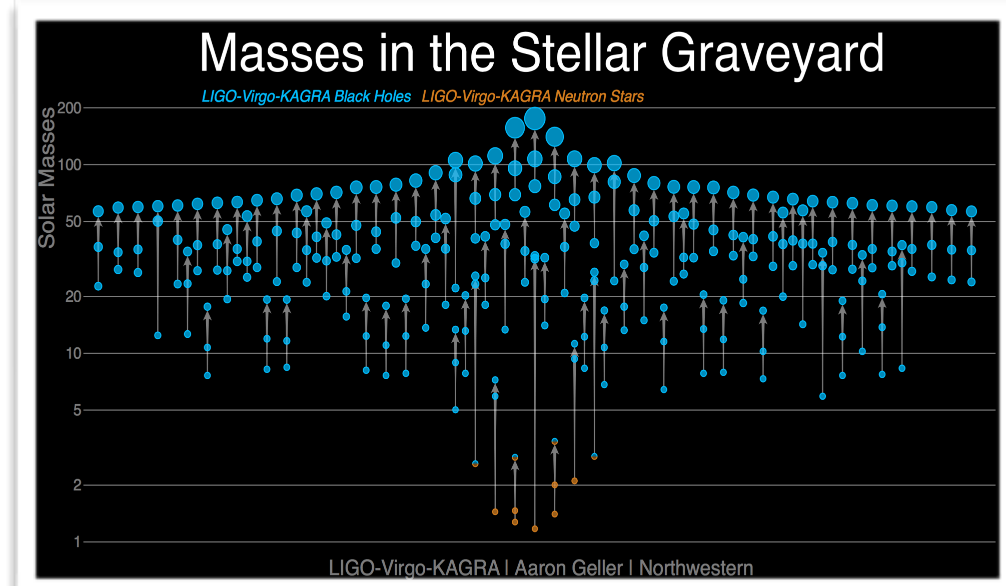
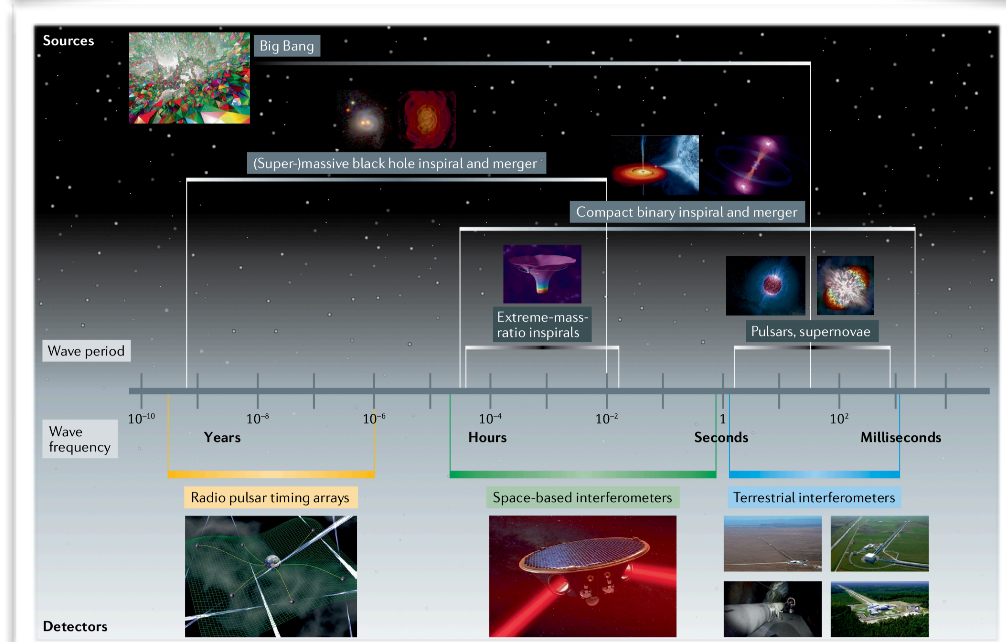
Gravitational-wave astronomy

- The past few years have witnessed a revolution in astronomy: direct detection of gravitational waves.

- Unique opportunity to test general relativity in the strong-field regime, shed light on the fundamental aspects of gravity and black holes, probe the fundamental nature of astrophysical compact objects.

- Extraordinary scientific potential of upgraded detectors and future facilities.





[Nature Reviews Physics, 3, 344–366 (2021)]

- We are witnessing the dawn of the era of precision physics with gravitational waves.
[Berti et al. '15], [Barack et al. '18], [Cardoso and Pani '19], [Baibhav et al. '19], [Barausse et al. '20], [Perkins, Yunes and Berti '20], [Bailes et al. '21], [Berti et al. '22]…

Symmetries of black holes

- Symmetries can help us shed light on the fundamental aspects of black holes and gravity, and constrain broad classes of theories beyond general relativity in a model-independent way.

Symmetries of black holes

- Black hole perturbation theory has a long history starting from the work of Regge and Wheeler, Zerilli, Teukolsky, Chandrasekhar…
- Interestingly, recent investigations suggest the subject has depths yet to be plumbed.

Outline

I will mostly focus on the static response and Love numbers of black holes.

Symmetries of black holes

- "The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time."
  (S. Chandrasekhar, "*The mathematical theory of black holes*")

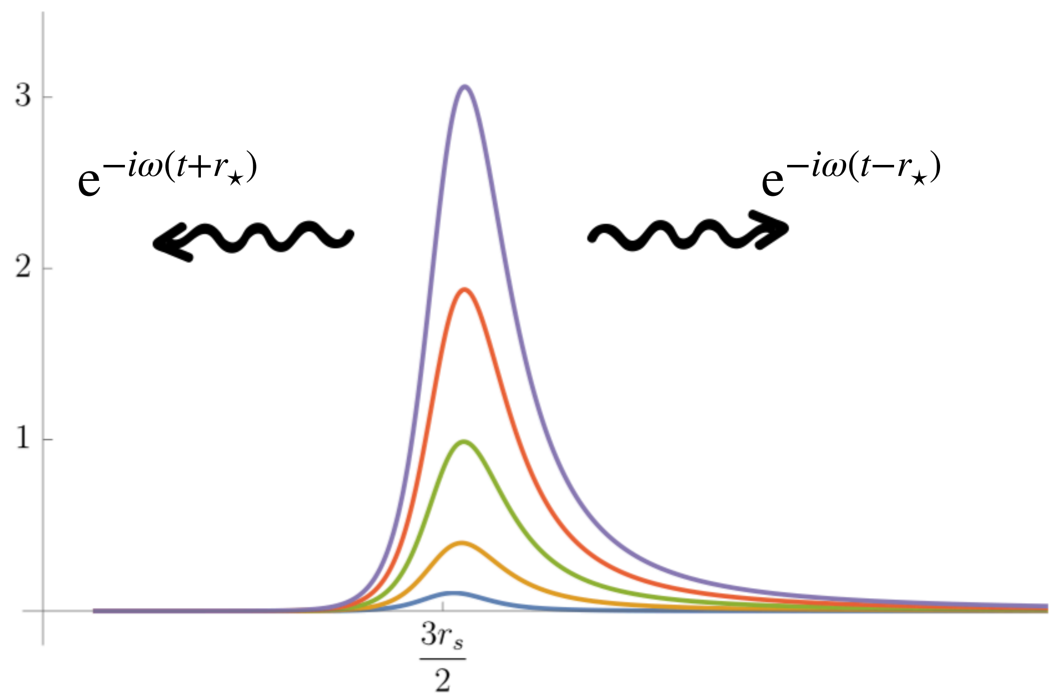
- Black holes are among the simplest and most robust objects in nature: uniquely determined by their mass and spin (and charge).

- This *simplicity* is inherited by the perturbations.

- Some aspects of this *simplicity* are well understood in terms of (hidden) symmetries of general relativity.

Symmetries of black holes

- In general relativity, the two d.o.f. in the gravitational wave emitted by a perturbed black hole have the same characteristic frequencies, i.e. are *isospectral*.
- Isospectrality has been known to follow from a duality of the linearized equations of motion (a.k.a. Chandrasekhar relation) since the 1970s. [Chandrasekhar '75]



- True for massless (spin-0, 1, 2) fields on Schwarzschild/Kerr(-de Sitter) spacetime, as well as for partially-massless spin-2 fields. [Brito, Cardoso and Pani '13], [Rosen and LS '20]
- Symmetry behind the vanishing of the Love numbers unclear until very recently.

Static response and tidal deformability

- The Love numbers are the coefficients encoding the (static) tidal deformability of a compact object (analogous to the electric and magnetic susceptibilities in EM).

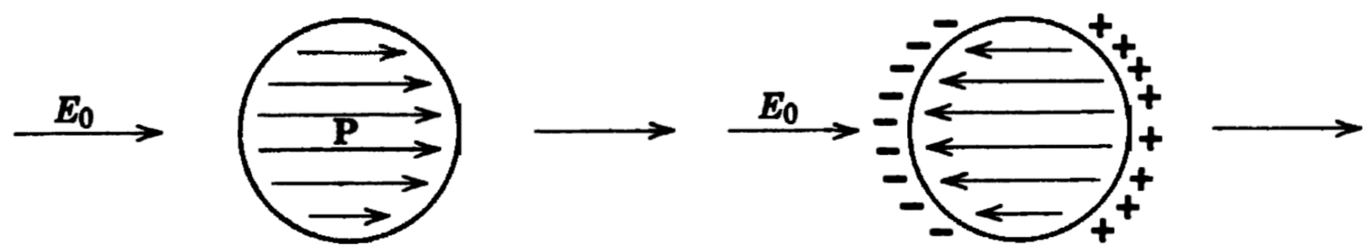


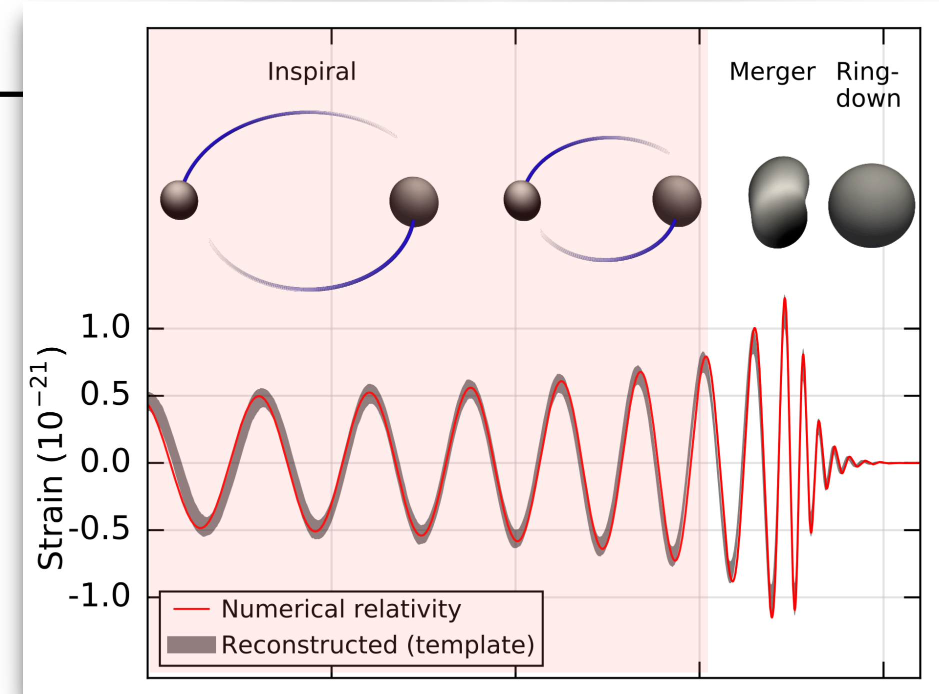
Figure 4.7 Dielectric sphere in a uniform field E_0, showing the polarization on the left and the polarization charge with its associated, opposing, electric field on the right.

- In EM we solve $\vec{\nabla}^2\Phi = 0$:

$$\Phi_{\rm ext} = \sum_\ell A_\ell \left[r^\ell + k_\ell r^{-\ell-1}\right] P_\ell(\cos\theta)\,, \qquad \Phi_{\rm int} = \sum_\ell B_\ell r^\ell P_\ell(\cos\theta)\,.$$

- The boundary condition at $r = +\infty$ fixes A_ℓ, while k_ℓ and B_ℓ are determined by regularity conditions across the surface (continuity of $\vec{E}_\parallel$ and $\vec{D}_\perp$).
- For instance, if $\vec{E}_0 = A_1\hat{z}$, one finds $k_{\ell=1} = -\dfrac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2} r_0^3$ (ϵ_0 and ϵ are the vacuum and dielectric permittivities).
- k_ℓ are the coefficients of the induced response.

Tidal Love numbers



- Tidal deformability affects the dynamics during the inspiral.
- An alteration in the phase of the gravitational-wave signal can be used to constrain the tidal deformability of the objects.
- Love numbers not only provide valuable insights into the properties of known objects, but also potentially indicate the existence of BSM exotic compact objects.
  (See [Chia, Edwards, Wadekar, Zimmerman, Olsen, Roulet, Venumadhav, Zackay and Zaldarriaga '23] for the first matched-filtering search in the O1–O3 LIGO/Virgo data.)
- An explicit calculation in general relativity shows that $k_\ell = 0$ for black holes in *D*=4, as opposed to other types of compact objects and black holes in generic *D*.
  [Fang and Lovelace '05]
  [Binnington and Poisson '09]
  [Damour and Nagar '09]
  [Kol and Smolkin '11]
  [Hui, Joyce, Penco, LS and Solomon '21]
  [Charalambous, Dubovsky and Ivanov '21]
  [Rodriguez, LS, Solomon and Temoche '23]
  [...]

Love numbers for rotating black holes in higher dimensions

- To understand why the vanishing of black hole Love numbers in general relativity is special and *not generic*, consider a higher-dimensional rotating black hole.

- Scalar field on Myers–Perry spacetime (single plane of rotation) in 5 dimensions:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 + (r^2+a^2)\sin^2\theta\,\mathrm{d}\varphi^2 + \frac{\mu}{\Sigma}(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi)^2 + r^2\cos^2\theta\,\mathrm{d}\psi^2\,,$$
$$\Delta = r^2 + a^2 - \mu\,, \qquad \Sigma = r^2 + a^2\cos^2\theta\,.$$

- Given the symmetries of the metric, we shall decompose:

$$\Phi(t,r,\theta,\varphi,\psi) = \mathrm{e}^{-i\omega t}\mathrm{e}^{im_\varphi\varphi}\mathrm{e}^{im_\psi\psi}\phi(r)Y(\theta)\,.$$

- Radial equation in the static limit:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\Delta\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) + \left[\frac{m_\varphi^2a^2}{\Delta} - \frac{m_\psi^2a^2}{r^2} - \ell(\ell+2)\right]\phi = 0\,.$$

Love numbers for rotating black holes in higher dimensions

- The equation can be solved analytically.

- Expanding the solution that is regular at the horizon $r = \sqrt{\mu - a^2}$ at large radii, $\phi \sim r^\ell + \lambda_\ell r^{-\ell-2}$:

$$\lambda_\ell = (-1)^\ell \frac{2\Gamma(\mathfrak{a})\Gamma(\mathfrak{b})}{\ell\,!\,\Gamma(\ell+2)\Gamma(\mathfrak{a}-\ell-1)\Gamma(\mathfrak{b}-\ell-1)} \log\left(\frac{r_0}{r}\right),$$

where

$$\mathfrak{a} = 1+\frac{\ell}{2} + \frac{ia(m_\varphi - m_\psi)}{2\sqrt{\mu - a^2}}, \qquad \mathfrak{b} = 1+\frac{\ell}{2} + \frac{ia(m_\varphi + m_\psi)}{2\sqrt{\mu - a^2}}, \qquad \mathfrak{c} = \ell + 2\,.$$

- The Love numbers are non-vanishing and have log running.

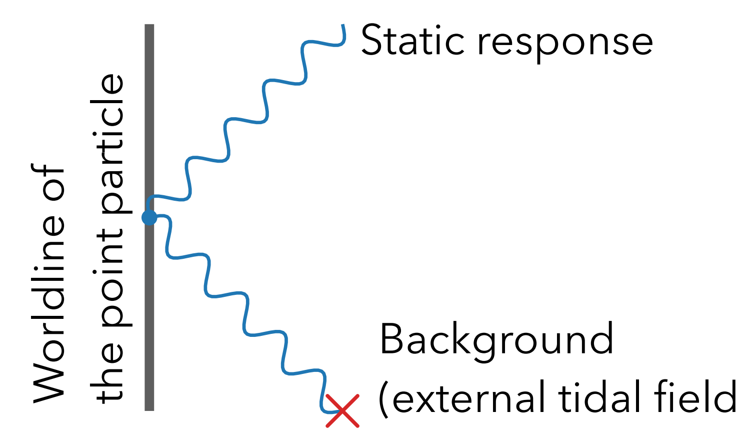
[Rodriguez, LS, Solomon and Temoche '23]
[Charalambous and Ivanov '23]

Vanishing Love Numbers

- A conceptually clean way to define the (conservative) Love numbers is in terms of the worldline effective action [Goldberger and Rothstein '04, '05, …], [Kol and Smolkin '11], [Porto '16].
- At distances large compared to the characteristic size of an object, there is an effective description where the object is modeled as a point particle. Corrections due to the object's finite size and its internal structure are encoded in higher-derivative operators in the effective theory.
- Let's consider e.g. a scalar field around a black hole:

$$S = -\frac{1}{2}\int \mathrm{d}^4x\,(\partial\phi)^2 - M\int \mathrm{d}\tau + \int \mathrm{d}\tau\left[-g\phi + \sum_{\ell=0}^{\infty}\frac{\lambda_\ell}{2\ell\,!}\left(\partial_{(a_1}\cdots\partial_{a_\ell)_T}\phi\right)^2\right].$$

- λ_ℓ are the LN coefficients.
- One generically expects: $\lambda_\ell \sim \mathcal{O}(1) r_s^{2\ell-1}$ and to find (classical) RG running.
- After matching with the UV result: $\lambda_\ell = 0$ in D=4 and no running.
- Generically non-zero in D>4.
  [Kol and Smolkin '11], [Hui, Joyce, Penco, LS and Solomon '21], [Charalambous and Ivanov '23], [Rodriguez, LS, Solomon and Temoche '23]



Vanishing Love Numbers

- Following 't Hooft's naturalness principle, the vanishing of the Love numbers is a naturalness puzzle from an EFT perspective. [Rothstein '14], [Porto '16]

$$S = -\frac{1}{2}\int \mathrm{d}^4x\,(\partial\phi)^2 - M\int \mathrm{d}\tau + \int \mathrm{d}\tau\left[-g\phi + \sum_{\ell=0}^{\infty}\frac{\lambda_\ell}{2\ell!}\left(\partial_{(a_1}\cdots\partial_{a_\ell)_T}\phi\right)^2\right]$$

- Looks like something that can very likely follow from a symmetry in the theory.

Symmetries of vanishing Love Numbers

- In [2105.01069] we showed that the vanishing of the Love numbers is the consequence of linearly realized symmetries governing static perturbations around black holes.

- Let's start from the Teukolsky equation with $\omega = 0$ (static limit):

$$\partial_r \left(\Delta \partial_r \phi_\ell^{(s)} \right) + s(2r - r_s)\partial_r \phi_\ell^{(s)} + \left(\frac{a^2 m^2 + is(2r - r_s)am}{\Delta} - (\ell - s)(\ell + s + 1) \right) \phi_\ell^{(s)} = 0$$

- We can set $s = 0$ by virtue of ladder operators in s (which generalize the Teukolsky-Starobinsky identities).
- In fact we can also set $\ell = 0$ – ladder operators allow to extend the argument to any ℓ.
- I'll set for simplicity $a = 0$ – the generalization to Kerr is straightforward.
  The equation is simply:

$$\partial_r \left(\Delta \partial_r \phi_0 \right) = 0 \,, \qquad \Delta = r(r - r_s) \,,$$

  which is $\Box \phi = 0$ on Schwarzschild with $\omega = 0 = \ell$.
- $P_0 \equiv \Delta \partial_r \phi_0$ is the conserved charge associated with a symmetry of the (static) scalar action.
- It is useful because it allows to connect asymptotics:

$$\phi_0 \sim r^0 \quad \text{as} \quad r \to +\infty \quad \to \quad P_0 = 0 \quad \to \quad \phi_0 \sim \text{const.} \quad \text{as} \quad r \to r_s$$

$$\phi_0 \sim r^{-1} \quad \text{as} \quad r \to +\infty \quad \to \quad P_0 \neq 0 \quad \to \quad \phi_0 \sim \log(r - r_s) \quad \text{as} \quad r \to r_s$$

Symmetries of vanishing Love Numbers

[Hui, Joyce, Penco, LS and Solomon '21]

- Generalization to all ℓs through ladder operators: $\phi_{\ell\pm1} \propto D_\ell^\pm \phi_\ell$

$$D_\ell^+ \equiv -\Delta\partial_r + \frac{\ell+1}{2}(r_s - 2r), \qquad D_\ell^- \equiv \Delta\partial_r + \frac{\ell}{2}(r_s - 2r)$$

$$\phi_\ell \sim r^\ell \quad \text{as} \quad r \to +\infty \quad \to \quad P_\ell = 0 \quad \to \quad \phi_\ell \sim \text{const.} \quad \text{as} \quad r \to r_s$$

$$\phi_\ell \sim r^{-(\ell+1)} \quad \text{as} \quad r \to +\infty \quad \to \quad P_\ell \neq 0 \quad \to \quad \phi_\ell \sim \log(r - r_s) \quad \text{as} \quad r \to r_s$$

- The vanishing of the Love numbers follows from two facts: (1) the purely decaying solution ($\sim 1/r^{\ell+1}$ at large r) is divergent at the horizon, and (2) the solution that is regular at the horizon is a finite polynomial going as $\sim 1 + r + \ldots + r^\ell$.

- The growing branch respects the symmetry, while the decaying branch spontaneously breaks the symmetry.
  (See also [Achour, Livine, Mukohyama, Uzan '22])

- Fact (1) is consistent with the no-hair theorem (a black hole cannot sustain static, scalar profile that decays at infinity [Bekenstein '72]).

From Schwarzschild to AdS, with Love

[Hui, Joyce, Penco, LS and Solomon '21]

- The symmetry has a geometric origin: it arises from the (E)AdS isometries of a dimensionally reduced black hole spacetime.

  Let's consider a static scalar ϕ in a Schwarzschild background,

$$S = \frac{1}{2}\int \mathrm{d}\theta \mathrm{d}\varphi \mathrm{d}r\sqrt{g}\,\phi\,\Box\,\phi\,, \qquad \mathrm{d}s^2 = \mathrm{d}r^2 + \Delta\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right).$$

  After a Weyl rescaling, the metric becomes EAdS$_3$ with

$$\tilde{g}_{ij} = \Omega^2 g_{ij}\,, \qquad \tilde{\phi} = \Omega^{-\frac{1}{2}}\phi\,, \qquad \text{where} \qquad \Omega \equiv L^2/\Delta\,,$$

$$S = \frac{1}{2}\int \mathrm{d}^3x\sqrt{\tilde{g}}\left(\tilde{\phi}\,\tilde{\Box}\,\tilde{\phi} + \frac{r_s^2}{4L^4}\tilde{\phi}^2\right), \qquad \mathrm{d}\tilde{s}^2 = \mathrm{d}r_\star^2 + \frac{4L^4}{r_s^2}\sinh^2\!\left(\frac{r_\star r_s}{2L^2}\right)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)$$

  where $\mathrm{d}r_\star = (L^2/\Delta)\mathrm{d}r$. The space has 6 Killing vectors: 3 rotations and 3 translations (or "boosts"). The translation that mixes $r_\star$ and θ acts on the original ϕ as

$$\delta\phi = -\,2\Delta\cos\theta\partial_r\phi + (r_s - 2r)\partial_\theta(\sin\theta\,\phi)$$

  or, equivalently,

$$\delta\phi_\ell = c_{\ell+1}D^-_{\ell+1}\phi_{\ell+1} - c_\ell D^+_{\ell-1}\phi_{\ell-1}\,.$$

Symmetries of vanishing Love Numbers

[Hui, Joyce, Penco, LS and Solomon '21]

- At large *r*, $\delta\phi$ reduces to a SCT, $\delta\phi = c_i\left(x^i - \vec{x}^2\partial^i + 2x^i\vec{x}\cdot\vec{\partial}\right)\phi.$
- We claim that this is the sought-after infrared symmetry that forbids Love number (and hair) couplings in the point-particle effective action.

Ladder in Kerr: static limit

[Hui, Joyce, Penco, LS and Solomon '21]

- The previous algebraic ladder structure has a direct analog in a Kerr background:

$$\mathrm{d}s^2 = -\frac{\rho^2 - r_s r}{\rho^2}\mathrm{d}t^2 - \frac{2ar_s r\sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\varphi + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\mathrm{d}\varphi^2$$

where $\rho^2 \equiv r^2 + a^2\cos^2\theta$ and $\Delta \equiv r^2 - rr_s + a^2$.

- The static Klein-Gordon equation, has both *ladder* and *horizontal* symmetries,

$$D_\ell^\pm = \mp\,\Delta\partial_r + \frac{1+2\ell\pm 1}{4}(r_s - 2r) \mp iam\ .$$

- The ladder symmetries $D_\ell^\pm$ descend from a CKV of the 3D-static metric:

$$\mathrm{d}s_K^2 = \frac{\rho^2 - rr_s}{\Delta}\left(\mathrm{d}r^2 + \Delta\mathrm{d}\theta^2 + \frac{\Delta^2\sin^2\theta}{\rho^2 - rr_s}\mathrm{d}\varphi^2\right).$$

- The conserved charges P_ℓ associated with the horizontal symmetries, evaluated for the "growing branch", are non-zero (and imaginary), unlike in the Schwarzschild case:

$$P_\ell \propto iq\prod_{k=1}^{\ell}\left(k^2 + 4q^2\right)\ , \qquad q \equiv \frac{am}{r_+ - r_-}\ ,$$

which reproduces the dissipative response [Le Tiec and Casals '20].

Ladder in Spin: From Scalar to Vector and Tensor

[Hui, Joyce, Penco, LS and Solomon '21, '22]

- Ladder operators in the spin, $E_s^{\pm}$, raise and lower s in the Teukolsky equation ($E_s^{\pm}\phi_\ell^{(s)} = \phi_\ell^{(s\pm1)}$),

$$\partial_r\Big(\Delta\partial_r\phi_\ell^{(s)}\Big) + s(2r - r_s)\partial_r\phi_\ell^{(s)} + \Big(\frac{a^2m^2 + is(2r - r_s)am}{\Delta} - (\ell - s)(\ell + s + 1)\Big)\phi_\ell^{(s)} = 0\,.$$

They allow to extend the previous results from scalar to vector and tensor fields.

- $E_s^{\pm}$ are related to what are known as Teukolsky–Starobinsky identities.
In Chandrasekhar's notation,

$$\phi^{(-1)} = \Delta\mathscr{D}_0^\dagger\mathscr{D}_0^\dagger\Delta\phi^{(1)}, \quad \phi^{(1)} = \mathscr{D}_0\mathscr{D}_0\phi^{(-1)}, \quad \phi^{(-2)} = \Delta^2\mathscr{D}_0^\dagger\mathscr{D}_0^\dagger\mathscr{D}_0^\dagger\mathscr{D}_0^\dagger\Delta^2\phi^{(2)}, \quad \phi^{(2)} = \mathscr{D}_0\mathscr{D}_0\mathscr{D}_0\mathscr{D}_0\phi^{(-2)},$$

where $\mathscr{D}_0 \equiv \partial_r + i[am - \omega(r^2 + a^2)]/\Delta$.

The new twist we are adding is that, in the static limit, we can truncate these operations, enabling us to increment s by unity, $E_s^{\pm}\phi_\ell^{(s)} = \phi_\ell^{(s\pm1)}$.

Conclusions

Conclusions and open directions

- The direct detection of GWs is a unique opportunity to test GR in the strong gravity regime.
- Love numbers are an important observable to characterize and constrain the nature and internal structure of compact objects.
- Symmetries are key tools to shed light on the fundamental aspects of gravity and compact objects, and constrain broad classes of theories beyond GR.
- Isospectrality and the vanishing of the Love numbers in GR are examples of properties that follow from hidden symmetries in the theory.

Conclusions and open directions

- Do the symmetries extend beyond linear order, or are they an `accident' of the linearized dynamics?
- What can we learn from (hidden) symmetries of gravity about the regime beyond linear perturbation theory?

Backup slides

Symmetries of vanishing Love Numbers

[Hui, Joyce, Penco, LS and Solomon '22]

- To understand the origin of the ladder symmetries, let's solve a slightly more general problem: scalar field at finite (low) frequency.

- The scalar action is

$$S = \frac{1}{2}\int \mathrm{d}t\mathrm{d}r\mathrm{d}\Omega_{S^2}\left[\frac{r^4}{\Delta}(\partial_t\phi)^2 - \Delta(\partial_r\phi)^2 + \phi\,\nabla^2_{\Omega_{S^2}}\phi\right].$$

- Define the *near-zone* approximation by replacing $(r^4/\Delta)\partial_t^2\phi$ with $(r_s^4/\Delta)\partial_t^2\phi$.

- This has the virtue of preserving the correct singularity as $r \to r_s$, while still accurately capturing the dynamics at larger r, as long as $\omega r \ll 1$.

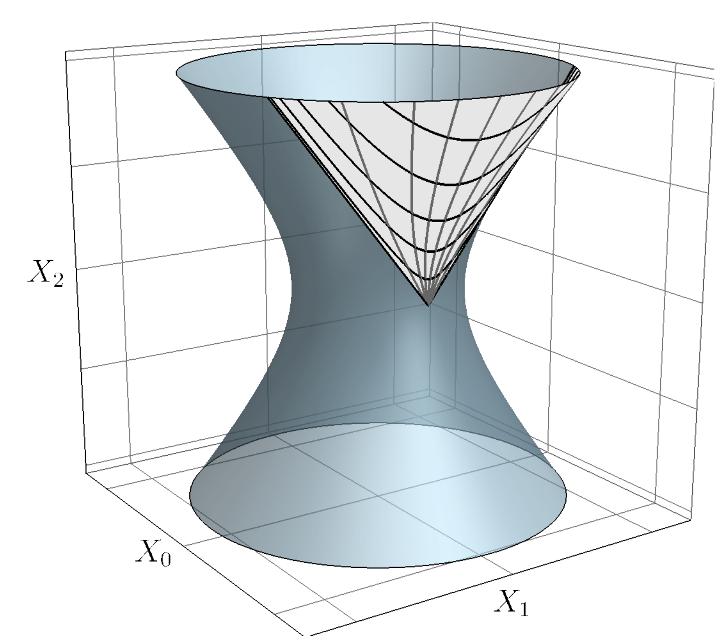
Symmetries of vanishing Love Numbers

[Hui, Joyce, Penco, LS and Solomon '22]

- In this limit, the scalar action is the same as that of a massless scalar minimally coupled to an *effective near-zone metric*:

$$\mathrm{d}s^2_{\text{near-zone}} = -\frac{\Delta}{r_s^2}\mathrm{d}t^2 + \frac{r_s^2}{\Delta}\mathrm{d}r^2 + r_s^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) .$$

- This metric is a conformally-flat $\mathrm{AdS}_2 \times S^2$ spacetime ($\Rightarrow$ 6 KVs + 9 CKVs).



Symmetries of vanishing Love Numbers

[Hui, Joyce, Penco, LS and Solomon '22]

- The 6 KVs and 9 CKVs are:

$$T = 2r_s\,\partial_t$$
$$L_\pm = e^{\pm t/2r_s}(2r_s\,\partial_r\sqrt{\Delta}\partial_t \mp \sqrt{\Delta}\partial_r)$$
$$J_{23} = \partial_\varphi$$
$$J_{12} = \cos\varphi\,\partial_\theta - \cot\theta\sin\varphi\,\partial_\varphi$$
$$J_{13} = \sin\varphi\,\partial_\theta + \cot\theta\cos\varphi\,\partial_\varphi$$

$$J_{01} = -\frac{2\Delta}{r_s}\cos\theta\,\partial_r - \frac{\partial_r\Delta}{r_s}\sin\theta\,\partial_\theta$$
$$J_{02} = -\cos\varphi\left[\frac{2\Delta}{r_s}\sin\theta\,\partial_r + \frac{\partial_r\Delta}{r_s}\left(\frac{\tan\varphi}{\sin\theta}\partial_\varphi - \cos\theta\partial_\theta\right)\right]$$
$$J_{03} = -\sin\varphi\left[\frac{2\Delta}{r_s}\sin\theta\,\partial_r - \frac{\partial_r\Delta}{r_s}\left(\frac{\cot\varphi}{\sin\theta}\partial_\varphi + \cos\theta\partial_\theta\right)\right]$$
$$K_\pm = e^{\pm t/2r_s}\frac{\sqrt{\Delta}}{r_s}\cos\theta\left(\frac{r_s^3}{\Delta}\partial_t \mp \partial_r\Delta\partial_r \mp 2\tan\theta\partial_\theta\right)$$
$$M_\pm = e^{\pm t/2r_s}\cos\varphi\left[\frac{r_s^2}{\sqrt{\Delta}}\sin\theta\partial_t \mp \frac{\sqrt{\Delta}\partial_r\Delta\sin\theta}{r_s}\partial_r \pm \frac{2\sqrt{\Delta}}{r_s}\cos\theta\partial_\theta \mp \frac{2\sqrt{\Delta}}{r_s}\frac{\tan\varphi}{\sin\theta}\partial_\varphi\right]$$
$$N_\pm = e^{\pm t/2r_s}\sin\varphi\left[\frac{r_s^2}{\sqrt{\Delta}}\sin\theta\partial_t \mp \frac{\sqrt{\Delta}\partial_r\Delta\sin\theta}{r_s}\partial_r \pm \frac{2\sqrt{\Delta}}{r_s}\cos\theta\partial_\theta \pm \frac{2\sqrt{\Delta}}{r_s}\frac{\cot\varphi}{\sin\theta}\partial_\varphi\right]$$

- Different perspective on the vanishing of the LNs proposed by [Charalambous, Dubovsky and Ivanov '21].
- This unifies the different sets of symmetries.
- Only T, J_{ij} and J_{0i} remain good symmetries in the static limit ($\omega = 0$).
- J_{01} recovers precisely the ladders: $\delta\phi = \xi^\mu\partial_\mu\phi + \frac{1}{4}\nabla_\mu\xi^\mu\phi$, or equivalently $\delta\phi_\ell = c_{\ell+1}D^-_{\ell+1}\phi_{\ell+1} - c_\ell D^+_{\ell-1}\phi_{\ell-1}$, $D^+_\ell \equiv -\Delta\partial_r + \frac{\ell+1}{2}(r_s - 2r)$ and $D^-_\ell \equiv \Delta\partial_r + \frac{\ell}{2}(r_s - 2r)$.