F-singularities; characteristic p methods in commutative ring theory and and algebraic geometry

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Abstract

In 1985, M. Hochaster and C. Huneke found the notion of **tight closures** of the ideals in rings of characteristic p. This method enables us to make many things clearer and sometimes to generalize in broader settings. In this Chapter, we explain the concept of tight closures and applications of it.

1 Tight closure of ideals, F-regular rings and Theorem of Briançon-Skoda.

In this section, we fix a prime number p and assume all rings are of characteristic p. If the characteristic of a ring A is p and $a, b \in A$, then we have

$$(a+b)^p = a^p + b^b$$

in such rings. Hence the **Frobenius map**

$$F : A \to A, \quad F(a) = a^p$$

is a ring homomorphism. In the following, the letter q will denote always a power of p ($q = p^e$ for some e).

Let us introduce the definition of tight closure.

Definition 1.1. Let A be a Noetherian ring of characteristic p and I be an ideal of A.

1. We denote A° the set of elements of A which are not in any minimal prime ideals of A. For example, if A is an integral domain, then $A^\circ = A \setminus \{0\}$.

2. We put $I^{[q]} = (a^q \mid a \in I)$.

3. We put $I^* = \{x \in A \mid \exists c \in A^\circ, cx^q \in I^{[q]}(\forall q = p^n \gg 1)\}$. We call I^* the **tight closure of** I.

Let us list some fundamental properties of the tight closures. In the following, we always assume that A is a Noetherian ring of characteristic p.

Proposition 1.2. *Let $I \subset J$ be ideals of A and $x \in A$.*

1. I^* is an ideal of A containing I and we have $(I^*)^* = I^*$.

2. If $I \subset J$, then $I^* \subset J^*$.

3. $x \in I^*$ if and only if for every minimal prime $\mathfrak{p}$ of A we have $x + \mathfrak{p} \in ((I + \mathfrak{p})/\mathfrak{p})^*$ in $A/\mathfrak{p}$.

Proof. (1) Let $a, b \in I^*$. It suffices to show that $a + b \in I^*$. If $c, c' \in A^\circ$ and $ca^q, c'b^q \in I^{[q]}$ then $(cc')(a+b)^q \in I^{[q]}$ for all $q \gg 1$. Note that for finite elements $a_1, \ldots, a_n \in I^*$, we can take $c \in A^\circ$ such that $ca_i^q \in I^{[q]}$ for all a_i. Since A is Noetherian, we can take $c \in A^\circ$ so that $c(I^*)^{[q]} \subset I^{[q]}$ for $q \gg 1$. Now if $b \in (I^*)^*$ and $c'b^q \in (I^*)^{[q]}$, then $(cc')b^q \in I^{[q]}$ and $b \in I^*$. (2) is obvious.

(3) From the proof of (1), for a generator system $a_1, \ldots, a_r$ of I^*, we can take $c \in A^0$ so that $ca_i^q \in I^{[q]}$ $(\forall q \gg 0)$, namely, $c(I^*)^{[q]} \subset I^{[q]}$ $(\forall q \gg 0)$. Now, let $b \in (I^*)^*$. Then there exists $c' \in A^0$ so that $c'b^q \in (I^*)^{[q]}$ $(\forall q \gg 0)$. Then $cc'b^q \in I^{[q]}$ $(\forall q \gg 0)$ and $x \in I^*$.

(4) If $x \in I^*$, from $cx^q \in I^{[q]}$, we have $(c+\mathfrak{p})(x+\mathfrak{p})^q \in ((I+\mathfrak{p})/\mathfrak{p})^{[q]}$ and $x + \mathfrak{p} \in ((I + \mathfrak{p})/\mathfrak{p})^*$.

Conversely, assume that for every minimal prime ideal $\mathfrak{p}$ of A, $x + \mathfrak{p} \in ((I + \mathfrak{p})/\mathfrak{p})^*$. Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ be the set of minimal primes of A. Then for every i, we can take $c_i \in A^0$ so that $c_ix^q \in I^{[q]} + \mathfrak{p}_i$. Then by "prime avoidance", we can take $t_i \in A$ so that $t_i \in \cap_{j \neq i}\mathfrak{p}_j, t_i \notin \mathfrak{p}_i$. Put $c = \sum c_it_i$, then for every $q \gg 1$, we have $cx^q \in I^{[q]} + \cap_{i=1}^r \mathfrak{p}_i = I^{[q]} + \mathfrak{N}(A)$. Take q' so that $\mathfrak{N}(A)^{q'} = (0)$ (not depending on x, I), then since $c_{\mathfrak{p}}x^q \in I^{[q]} + \mathfrak{p}$ putting $c = \prod_{\mathfrak{p}} c_{\mathfrak{p}}$, we have $c^{q'}x^{qq'} \in I^{[qq']}$, hence $x \in I^*$. $\square$

Remark 1.3. In the definition of the tight closure; "$\exists c \in A^0, cx^q \in I^{[q]}$ ($\forall q = p^n \gg 1$)", it is natural to ask "Can we change "$\forall q = p^n \gg 1$)" to "$\forall q = p^n \gg 1$" ? Actually, if A is reduced or $\mathrm{ht}(I) > 0$, we can change the definition to "$\exists c \in A^0, cx^q \in I^{[q]}$ ($\forall q = p^n$)". In fact, if for finite q, $cx^q \notin I^{[q]}$, let q_0 be the maximal of such q and if $\mathrm{ht}(I) > 0$, take $c' \in I \cap A^0$. then we have $c(c')^{q_0}x^q \in I^{[q]}$ for all q.

On the other hand, if A has non 0 nilpotent elements and if $\mathrm{Ass}(A) = \mathrm{Assh}(A)$, then since the elements of A^0 are non 0 divisors, a nilpotent element $x \neq 0$ does not satisfy the condition "$\exists c \in A^0$ such that $cx^q = 0$ for all $q = p^e$". But certainly a nilpotent element x is in $(0)^*$ by definition 1.1.

Let us compute some examples.

Example 1.4. Let k be a field of characteristic p.

1. Let $A = k[X, Y]$ and $I = (X^2, Y^2)$. Then $XY \notin I^*$ and since (X^2, XY, Y^2) is the unique minimal ideal strictly containing I, we have $I^* = I$.

2. Let $A = k[X, Y, Z]/(X^n + Y^n + Z^n)$ with $(n, p) = 1$. If $I = (y, z)$, then $x^2 \in I^*$, where x, y, z are the images of X, Y, Z, respectively.

Proof. (1) If $c(XY)^q = cX^qY^q \in I^{[q]} = (X^{2q}, Y^{2q})$, then we must have $c \in (X^q, Y^q)$. Since this must hold for all $q \gg 1$, we have $c = 0$ and hence $XY \notin I^*$.
(2) Let $q \gg 1$ and put $q = ns + r$, $0 \leq r < n$. Then $(x^2)^q = x^{2r}(y^n + z^n)^{2s}$. In the expansion of $(y^n + z^n)^{2s}$, every term except $(yz)^{ns}$ is divisible by either y^q or z^q. Hence putting $c = y^n$, we have $cx^q \in I^{[q]}\ \forall q \gg 0$. Hence we have $x^2 \in I^*$. □

If $(A, \mathfrak{m})$ is a regular local ring, then for every ideal I of A we have $I^* = I$. To prove that the following Theorem by E. Kunz is essential.

Theorem 1.5. *(E.Kunz) For a Noetherian local ring $(A, \mathfrak{m})$ of dimension d, the following conditions are equivalent.*

1. *A is regular.*

2. $F : A \to A$ *is a flat map.*

3. $\forall q,\ l_A(A/\mathfrak{m}^{[q]}) = q^d$

4. $\exists q,\ l_A(A/\mathfrak{m}^{[q]}) = q^d$

Proof. We will explain outline of the proof. Since the statement is equivalent if we replace A with $\widehat{A}$, we assume that A is complete. Also, we can take a faithfully flat extension $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ with $\mathfrak{n} = \mathfrak{m}B$ and $B/\mathfrak{n}$ is a perfect field by 6.2. Since the equivalence in the Theorem holds for A if and only for B, we can assume that $A/\mathfrak{m}$ is perfect. So, we assume that A is complete and $A/\mathfrak{m}$ is a perfect field. Since $A_\mathfrak{m}$ is perfect, by $F : A \to A$ A is a finite A module.

Now, the implication (1)$\Longrightarrow$ (3) $\Longrightarrow$ (4) is obvious. If A is an integral domain, it is easy to see that $\operatorname{rank}_A A^{1/q} = q^d$. Since $\ell_A(A/\mathfrak{m}^{[q]}$ is the number of minimal generators of $A^{1/q}$ over A, (4) implies that $A^{1/q}$ is a free A module, hence we have (4) $\Longrightarrow$ (2).

Next, assume (2) and let $(x_1, \ldots, x_n)$ be a set of minimal generators of $\mathfrak{m}$. Then for every $q = p^e$, we can show that $(x_1^q, \ldots, x_n^q)$ is "independent" in the sense of the following Exercise 1.6 and by multiplicity theory, we can show $n = d$ and hence A is regular. □

Exercise 1.6. Let $(A, \mathfrak{m})$ be a Noetherian local ring. We say a set $(f_1, \ldots, f_s) \subset A$ is **independent** if it satisfy the condition (#).

(#) If $(a_1, \ldots, a_s) \subset A$ and $\sum_{i=1}^s a_i f_i = 0$, then $a_1, \ldots a_s \in (f_1, \ldots, f_s)$.

(1) Let $(x_1, \ldots, x_n)$ be a minimal generator system of $\mathfrak{m}$. If $A \to A^{1/q}$ is flat, then show that $(x_1, \ldots, x_n)$ is independent.

(2) If $(f_1, \ldots, f_s)$ is independent and if $f_1 \in (g_1)$, show that $(g_1, f_2, \ldots, f_s)$ is independent and show also that $(f_2, \ldots, f_s) : g_1 \subset (f_1, f_2, \ldots, f_s)$.

(3) Show that if $(x_1, \ldots, x_n)$ is a set of minimal generators of $\mathfrak{m}$ and if $(x_1, \ldots, x_n)$ is independent, show that $\ell_A A/(x_1^q, \ldots, x_n^q) = q^d$.

The following fundamental fact of tight closure theory follows from Theorem 1.5.

Theorem 1.7. *If* $(A, \mathfrak{m})$ *is a regular local ring, then* $I^* = I$ *for every ideal* I *of* A*.*

Proof. Let $x \in I^*$ and assume $cx^q \in I^{[q]}$. Then since $F : A \to A$ is flat, we have $[I^{[q]} : x^q] = [I : x]^{[q]}$. If $x \notin I$, then $I : x \subset \mathfrak{m}$ and $[I^{[q]} : x^q] \subset \mathfrak{m}^{[q]}$. Hence if $cx^q \in I^{[q]}$ for $q \gg 1$, then $c \in \cap_q \mathfrak{m}^{[q]} = (0)$. This shows that $x \in I$. □

From this Theorem we define F-regular rings, which behave like regular rings with respect to tight closure.

Definition 1.8. A Noetherian ring A of characteristic $p > 0$ is called a **weakly F-regular** ring if $I^* = I$ holds for every ideal I of A. We say A is **F-regular** if every localization $S^{-1}A$ of A is weakly F-regular.

Compatibility of tight closure with localization is a subtle subject and has been a big issue of the theory.

Remark 1.9. Let A be a Noetherian ring and $S \subset A$ be a multiplicatively closed set.

1. For every ideal I of A it is easy to show that $S^{-1}I^* \subset (S^{-1}A)^*$.
2. The converse implication of (1) was a celebrated open problem of tight closure theory for over 20 years. Then H. Brenner and P. Monsky gave a counterexample in the case $p = 2$ in 2007 ([BM]). The counterexample given there is a hypersurface of dimension 3 given by $k[X, Y, Z, T]/(Z^4 + XYZ^2 + X^3Z + Y^3Z + TX^2Y^2)$, where k is the algebraic closure of $\mathbb{F}_2$. But it seems that no counterexamples appeared for $p \geq 3$.
3. If $\sqrt{I} = \mathfrak{m}$ is a maximal ideal, we can show $I^*A_{\mathfrak{m}} = (IA_{\mathfrak{m}})^*$.
4. We can show that A is weakly F-regular if and only if for every maximal ideal $\mathfrak{m}$ and for every $\mathfrak{m}$ primary ideal I we have $I^* = I$. Hence A is weakly F-regular if and only if so is $A_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of A.
5. If A is regular, then A is F-regular.

Proof. (1) If $x \in I^*$ and if $cx^q \in I^{[q]}$, for some $c \in A^\circ$, then $(c/1)(x/1)^q \in I^{[q]}(S^{-1}A)$ and thus $x/1 \in (S^{-1}A)^*$.

(3) If I is $\mathfrak{m}$ primary, so is $I^{[q]}$. Hence if $(c/s)(x/1)^q \in (S^{-1}A)I^{[q]}$, $\exists s' \notin \mathfrak{m}, s'cx^q \in I^{[q]}$. Hence $cx^q \in I^{[q]}$ and we have $x \in I^*$.

(4) Since every ideal in a Noetherian ring is an (infinite) intersection of $\mathfrak{m}$ primary ideals for maximal ideals $\mathfrak{m}$, (4) follows from (3). (5) follows from 1.7. □

Next, we will see the relationship between tight closure and integral closure.

Proposition 1.10. *We denote by $\overline{I}$ the integral closure of I. If I is an ideal of A with* $\mathrm{ht}(I) > 0$*, then the following statements hold.*

1. *$x \in \overline{I}$ if and only if there existc $\in A^\circ$ and an infinite subset Λ of $\mathbb{N}$ such that $\forall N \in \Lambda, cx^N \in I^N$.*
2. *$I^* \subset \overline{I}$.*

Proof. (1) By the definition of $\overline{I}$, $x \in \overline{I} \iff \exists r, \forall n \geq r, (I + xA)^n = (I+xA)^r I^{n-r}$. Hence, if we take $c \in I^r \cap A^0$, then we have $cx^n \in I^n$ $(\forall n \geq r)$. To show the converse, we may assume A is an integral domain considering $I(A/\mathfrak{p})$ in $A/\mathfrak{p}$ for a minimal prime $\mathfrak{p}$ of A. If $cx^n \in I^n$ $(\forall n \gg 0)$, then for every discrete valuation v of the quotient field of A we have $v(x) \geq v(I)$ and then $x \in \overline{I}$ by valuative criterion of integral closure (Proposition 6.1).

(2) If $x \in I^*$, then for some $c \in A^\circ$ and for $q = p^e \gg 1, cx^q \in I^{[q]} \subset I^q$. This is just the condition in (1), taking $\Lambda = \{q = p^e \mid q \gg 1\}$. □

In many cases, I^* is much smaller than $\overline{I}$ (although, in some cases, I^* is much bigger than I). This fact is the reason of the name "tight" closure.

The next Theorem is called "Briançon-Skoda Theorem" and is very important explaining the behavior of integral closure. The proof of this Theorem, which used to be very difficult before, became surprisingly easier by introducing the notion of tight closure.

Theorem 1.11 (Theorem of Briançon-Skoda)**.** *Let A be a Noetherian ring and I be an ideal of A, generated by n elements. Then for every integer $w \geq 1$ we have*

$$\overline{I^{n+w-1}} \subset (I^w)^*.$$

The proof reduces to the following simple Lemma.

Lemma 1.12. *Let I be an ideal generated by n elements. Then for every q and integer $w \geq 1$, we have*

$$I^{q(w-1)+(q-1)n+1} \subset (I^{[q]})^w = (I^w)^{[q]}.$$

Proof. Let $I = (a_1, \ldots, a_n)$. Then $I^{q(w-1)+(q-1)n+1}$ is generated by the products of $q(w-1)+(q-1)n+1$ elements of $a_1, \ldots, a_n$. But if $i_1+i_2+\ldots+i_n = q(w-1)+(q-1)n+1$, then we have $\sum_{j=1}^{n}[i_j/q] \geq w$, which shows our result. □

Proof of Theorem 1.11. Take an integer r so that

$$(\overline{I^{n+w-1}})^{n+r} = (\overline{I^{n+w-1}})^r (I^{n+w-1})^n \quad (\forall n \geq 0).$$

If $c \in (\overline{I^{n+w-1}})^r I \cap A^\circ$, then for every q and for any $x \in (\overline{I^{n+w-1}})^r$

$$cx^q \in I^{(n+w-1)q+1}.$$

Since $(n+w-1)q+1 > (w-1)q+(q-1)n+1$, we can assert $cx^q \in (I^w)^{[q]}$ by Lemma 1.12. This shows $x \in (I^w)^*$. □

In particular, if $n = w = 1$, since $\overline{I} \subset I^*$, we have the following.

Proposition 1.13. *If $x \in A^\circ$ and if $I = (x)$, we have $(x)^* = \overline{(x)}$.*

As a corollary of this Proposition we get the following important result.

Corollary 1.14. *F-regular rings are normal.*

Proof. Let A be F-regular. If $x \in A$ is a non 0 divisor of A and if for $y \in A$, y/x is integral over A, then since $y \in \overline{(x)} = (x)$, we have $y/x \in A$. Hence A is normal. □

Next Theorem is called “Boutot’s Theorem”, which is very important for theory of rational singularities in algebraic geometry over a field of characteristic 0. Again, in the following form, the proof becomes almost trivial.

Theorem 1.15. *Let A be a pure subring of an F-regular ring B and assume that $A^\circ \subset B^\circ$. Then A is also F-regular.*

Proof. Let I be any ideal of A. By our assumption, we have $I^* \subset (IB)^* = IB$ since B is F-regular. Then $I^* \subset (IB)^* \cap A = IB \cap A = I$, since $A \subset B$ is pure. □

In the next section, we will show that F-regular rings are Cohen-Macaulay. Then we get a simple and beautiful proof of the fact “a pure subring of a regular ring is Cohen-Macaulay”.

2 Colon capturing, F-rational rings

Let $(A, \mathfrak{m})$ be a Noetherian local ring and $(x_1, \ldots, x_d)$ be a system of parameters (which we abbreviate as **sop**) of A. Then it is very important to know how much bigger is $(x_1, \ldots, x_i) : x_{i+1}$ compared to $(x_1, \ldots, x_i)$. For example, A ic Cohen-Macaulay if and only if $(x_1, \ldots, x_i) : x_{i+1} = (x_1, \ldots, x_i)$ for every (some) sop $(x_1, \ldots, x_d)$.

We will show that $(x_1, \ldots, x_i) : x_{i+1} \subset (x_1, \ldots, x_i)^*$. We call this property of sop “colon capturing” property. Then we define an **F-rational** ring to be a ring whose parameter ideals are tightly closed. Then we show that F-rational rings are normal and Cohen-Macaulay. Also, we will show later that F-rational rings are closely related to the notion of **rational singularities** in caracteristic 0.

Let us begin with “colon capturing” of sop.

Theorem 2.1. *Let $R \subset A$ be an extension of rings and assume that R is a regular domain and A is finitely generated as R module. Then for every ideals I, J of R, we have*

$$IA : JA \subset ((I : J)A)^*.$$

In particular, if $(x_1,\dots,x_d)$ *is an sop of* R*, then* $(x_1,\dots,x_i)A : x_{i+1} \subset ((x_1,\dots,x_i)A)^*$ *for every* i*.*

Proof. Let K be the quotient field of R and assume $A \otimes_R K \cong K^r$. Then take a basis $\{a_1,\dots,a_r\}$ of $A \otimes_R K$ from A.

Then let F be the free R submodule of A generated by $\{a_1,\dots,a_r\}$. Then, since A is finitely generated as R module, we can take $c \in R, c \neq 0$ such that $cA \subset F$. Also, since the Frobenius map on R is flat, we have $I^{[q]} : J^{[q]} = [I : J]^{[q]}$.

Now, let $x \in A, x \in IA : JA$. Since $xJ \subset IA$, we have $x^q J^{[q]} \subset I^{[q]}A$. If we multiply c to both sides, we get $cx^q J^{[q]} \subset I^{[q]}F$. Since F is a free R module, we have $cx^q \in (I^{[q]}F : J^{[q]}) = (I : J)^{[q]}F \subset ((I : J)A)^{[q]}$ and we get the claim of the Theorem. □

We will give a proof of "colon capturing" in case A is a homomorphic image of a Cohen-Macaulay local ring.

Theorem 2.2. *Assume that a Noetherian local ring* A *is a homomorphic image of a Cohen-Macaulay local ring* $(B,\mathfrak{n})$ *and if* $\dim A = \dim A/\mathfrak{p}$ *holds for every minimal prime ideal* $\mathfrak{p}$ *of* A*, then for every sop* $(x_1,\dots,x_d)$ *and for every* i*,* $0 \leq i \leq d-1$*, we have* $(x_1,\dots,x_i) : x_{i+1} \subset (x_1,\dots,x_i)^*$*.*

Proof. Let $A = B/\mathfrak{a}$ and assume that $\mathrm{ht}(\mathfrak{a}) = r$. Take $y_1,\dots,y_d \in B$ so that $x_i = y_i \bmod \mathfrak{a}$. Also, let $\mathfrak{a} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_s$ be the primary decomposition of $\mathfrak{a}$ in B. We put $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$ $(i = 1,\dots,s)$. Since $(y_1,\dots,y_d) + \mathfrak{a}$ is an $\mathfrak{n}$ primary ideal, we can take $(z_1,\dots,z_r) \subset \mathfrak{a}$ so that $(y_1,\dots,y_d,z_1,\dots,z_r)$ is an sop of B.

Since B is Cohen-Macaulay, $(y_1,\dots,y_d,z_1,\dots,z_r)$ forms a regular sequence.

Let $(z_1,\dots,z_r) = \mathfrak{b} \cap \mathfrak{q}'_1 \cap \dots \cap \mathfrak{q}'_t$ be the primary decomposition of $(z_1,\dots,z_r)$. Here we assume that $\mathrm{Ass}_B(B/\mathfrak{b}) = \{\mathfrak{p}_1,\dots,\mathfrak{p}_s\}$ and $\mathfrak{q}'_i$ $(1 \leq i \leq t)$ are the primary ideals belonging to the other prime ideals. Then take $d \in \mathfrak{q}'_1 \cap \dots \cap \mathfrak{q}'_t$ with $d \notin \mathfrak{p}_i$ $(1 \leq i \leq s)$. Then the image $c \in A$ of d is in A°. Also note that there exists n such that $c(\mathfrak{a})^n \subset (z_1,\dots,z_r)$.

Now, let $u \in (x_1,\dots,x_i) : x_{i+1}$ and put $ux_{i+1} = \sum_{j=1}^{i} a_j x_j$. Choose $v, b_j \in B$ $(1 \leq j \leq i)$ so that the image of v (resp. b_j) in A is u (resp. a_j).

Then we have

$$vy_{i+1} - \sum_{j=1}^{i} b_j y_j \in \mathfrak{a}.$$

Then for a fixed $q' = p^{e'}$ we have $c\mathfrak{a}^{q'} \in (z_1, \ldots, z_r)$ and then for all $q = p^e, e \geq e'$, we have

$$d(vy_{i+1})^q \in (y_1^q, \ldots, y_d^q, z_1, \ldots, z_r).$$

Since $(y_1^q, \ldots, y_d^q, z_1, \ldots, z_r)$ is a regular sequence in B, we have $dv^q \in (y_1^q, \ldots, y_i^q, z_1, \ldots, z_r)$, which means $cu^q \in (x_1, \ldots, x_i)^{[q]}$ and this shows that $u \in (x_1, \ldots, x_i)^*$. □

Remark 2.3. The condition "$\dim A = \dim A/\mathfrak{p}$ for every minimal prime ideal $\mathfrak{p}$ of A" in Theorem 2.2 is necessary.

For example, let k be a field and let $A = k[[X, Y, Z]]/(XY, XZ)$. We let x, y, z be the images of X, Y, Z in A, respectively. Then $(y, x - z)$ is a sop of A. In this case we have $x \in (0) : y$. But since A is reduced, we have $(0)^* = (0)$ and $(0) : y \subset (0)^*$ does not hold. In this case, the minimal primes of A are (x) and (y, z) with $\dim A/(x) = 2 \neq \dim A/(y, z) = 1$.

In Theorem 2.2, we can assert a slightly stronger result $(x_1, \ldots, x_i)^* : x_{i+1} \subset (x_1, \ldots, x_i)^*$, which will be necessary later.

Theorem 2.4. *Let $(B, \mathfrak{n})$ be a Cohen-Macaulay local ring and $A = B/\mathfrak{a}$ as in 2.2. We also assume that $\dim A = \dim A/\mathfrak{p}$ holds for every minimal prime ideal $\mathfrak{p}$ of A. Then for every sop $(x_1, \ldots, x_d)$ and for every i ($0 \leq i \leq d - 1$), we have $(x_1, \ldots, x_i)^* : x_{i+1} \subset (x_1, \ldots, x_i)^*$.*

Proof. We use the same notation as in the proof of 2.2. If $u \in (x_1, \ldots, x_i)^* : x_{i+1}$, then for some $c' \in A^\circ$ and for every $q = p^e \gg 1$, we have $c'u^q \in [(x_1, \ldots, x_i) : x_{i+1}]^{[q]}$. Hence we have $c'u^q x_{i+1}^q \in (x_1, \ldots, x_i)^{[q]}$. Take $d' \in B$ whose image in A is c', then we can take $b_j \in B$ $(1 \leq j \leq i)$ so that

$$d'v^q y_{i+1}^q - \sum_{j=1}^{i} b_j y_j^q \in \mathfrak{a}.$$

Since $d\mathfrak{a}^{[q']} \subset (z_1, \dots, z_r)$ for a fixed $q' = p^{e'}$, we have

$$dd'^{q'} v^{qq'} y_{i+1}^{qq'} - \sum_{j=1}^{i} b_j^{q'} y_j^{qq'} \in (z_1, \dots z_r)$$

and we have $dd'^{q'} v^{qq'} \in (y_1^{qq'}, \dots, y_i^{qq'}, z_1, \dots, z_r)$. Taking the image of this equation in A, we have

$$cc'^{q'} u^{qq'} \in (x_1^{qq'}, \dots, x_i^{qq'})$$

and hence we have $u \in (x_1, \dots, x_i)^*$. □

Let us give the definition of **F-rational** rings. The name "F-rational" means "characteristic p version of rational singularities in characteristic 0" and actually, this "equivalence" was proved by K. Smith, N. Hara and V. Mehta - V. Srinivas.

Definition 2.5. Let $(A, \mathfrak{m})$ be a Noetherian local ring containing a field of characteristic $p > 0$ and we also assume that A is a homomorphic image of a Cohen-Macaulay local ring.

1. We say A is F-rational if A is equidimensional and for every sop $(x_1, \dots, x_d)$, we have $(x_1, \dots, x_d)^* = (x_1, \dots, x_d)$.
2. If A is a Noetherian ring which is a homomorphic image of a Cohen-Macaulay ring B, we call A F-rational if for every prime ideal $\mathfrak{p}$ of A, $A_{\mathfrak{p}}$ is F-rational in the sense of (1).

F-rational rings have many nice properties.

Theorem 2.6. *Let $(A, \mathfrak{m})$ be a local ring and if for some sop $(x_1, \dots, x_d)$ of A $(x_1, \dots, x_d)^* = (x_1, \dots, x_d)$ holds, then we have the following properties;*

1. *For every $i, 0 \le i \le d-1$, we have $(x_1, \dots, x_i)^* = (x_1, \dots, x_i)$.*
2. *A is Cohen - Macaulay.*
3. *If the equality $(x_1, \dots, x_d)^* = (x_1, \dots, x_d)$ holds for some sop $(x_1, \dots, x_d)$ of A, then $(y_1, \dots, y_d)^* = (y_1, \dots, y_d)$ holds for every sop $(y_1, \dots, y_d)$ of A.*

4. *A is an integrally closed integral domain.*

5. *(Y. Nakamura, Hochster-Huneke) For every prime ideal $\mathfrak{p}$ of A, $A_{\mathfrak{p}}$ is also an F-rational ring.*

6. *If A is Gorenstein and if A is F-rational, then A is F-regular. Namely, for Gorenstein rings, "F-rational" and "F-regular" are equivalent conditions.*

Proof. (1) We show that $(x_1, \ldots, x_i)^* = (x_1, \ldots, x_i)$ by descending induction on i. We have the equality for $i = d$ by our assumption. Assume that $(x_1, \ldots x_i, , x_{i+1})^* = (x_1, \ldots, x_i, x_{i+1})$ and assume $u \in (x_1, \ldots, x_i)^*$. Since $(x_1, \ldots, x_i)^* \subset (x_1, \ldots, x_i, x_{i+1})^* = (x_1, \ldots, x_i, x_{i+1})$, we have $u \in (x_1, \ldots, x_i, x_{i+1})$. Put $u = y + vx_{i+1}, y \in (x_1, \ldots, x_i)$. Then $v \in (x_1, \ldots, x_i)^* : x_{i+1} = (x_1, \ldots, x_i)^*$ by 2.4. Hence we have

$$(x_1, \ldots, x_i)^* \subset (x_1, \ldots, x_i) + (x_1, \ldots, x_i)^* x_{i+1}$$

and we have $(x_1, \ldots, x_i)^*(x_1, \ldots, x_i)$ by NAK.

(2) By (1) and "colon capturing" (2.2), $(x_1, \ldots, x_d)$ is a regular sequence and hence A is Cohen-Macaulay.

(3) Let $(y_1, \ldots, y_d)$ be any sop of A. To show $(y_1, \ldots, y_d)^* = (y_1, \ldots, y_d)$, it suffice to show that

$$(y_1, \ldots, y_d)^* \cap (y_1, \ldots, y_d) : \mathfrak{m} = (y_1, \ldots, y_d).$$

First, we will show that if $(x_1, \ldots, x_d)^* = (x_1, \ldots, x_d)$, then for every positive integer n, we have $(x_1^n, \ldots, x_d^n)^* = (x_1^n, \ldots, x_d^n)$. Since A is Cohen-Macaulay by (2), the natural homomorphism

$$\phi : [(x_1, \ldots, x_d) : \mathfrak{m}]/(x_1, \ldots, x_d) \to [(x_1^n, \ldots, x_d^n) : \mathfrak{m}]/(x_1^n, \ldots, x_d^n)$$

given multiplication of $(x_1 x_2 \cdots x_d)^{n-1}$ is bijective (**??**). If $u(x_1 x_2 \cdots x_d)^{n-1} \in (x_1^n, \ldots, x_d^n)^*$, then since $u \in (x_1, \ldots, x_d)^* = (x_1, \ldots, x_d)$, we show that $(x_1^n, \ldots, x_d^n)^* = (x_1^n, \ldots, x_d^n)$.

Now, let $(y_1, \ldots, y_d)$ be any sop. Taking $(x_1^n, \ldots, x_d^n)$ instead of $(x_1, \ldots, x_d)$, we may assume $(y_1, \ldots, y_d) \supset (x_1, \ldots, x_d)$. Take $z \in A$ so that the multiplication by z

$$z : [(y_1, \ldots, y_d) : \mathfrak{m}]/(y_1, \ldots, y_d) \to [(x_1, \ldots, x_d) : \mathfrak{m}]/(x_1, \ldots, x_d)$$

is bijective. Then note that

$$z^q : [(y_1, \ldots, y_d)^{[q]} : \mathfrak{m}]/(y_1, \ldots, y_d)^{[q]} \to [(x_1, \ldots, x_d)^{[q]} : \mathfrak{m}]/(x_1, \ldots, x_d)^{[q]}$$

is also a bijection.

Now take $u \in (y_1, \ldots, y_d) : \mathfrak{m}] \cap (y_1, \ldots, y_d)^*$ and assume that $cu^q \in (y_1, \ldots, y_d)^{[q]}$ for some $c \in A^\circ$. Then since $c(zu)^q \in (x_1, \ldots, x_d)^{[q]}$, we have $zu \in (x_1, \ldots, x_d)^* = (x_1, \ldots, x_d)$. This shows that $u \in (y_1, \ldots, y_d)$ as desired.

(4) For every $x \in A^\circ$, we can find an sop $(x_1, \ldots, x_d)$ with $x = x_1$. Then by (1) and (3), we show that $(x)^* = (x)$. Then A is a normal domain in the same manner as 1.14 since $\overline{(x)} = (x)^*$ by 1.13.

(5) By descending induction on $\operatorname{ht}\mathfrak{p}$, is suffices to assume that $\operatorname{ht}\mathfrak{p} = d-1$. Then for some element $f \in A \setminus \mathfrak{p}$, $\mathfrak{p}$ is a maximal ideal of A_f. If $(f_1, \ldots, f_{d-1})$ is an sop of $A_\mathfrak{p}$, then $(f_1, \ldots, f_{d-1}, f)$ is an sop of A.

It suffices to show $((f_1, \ldots, f_{d-1})A_f)^* = (f_1, \ldots, f_{d-1})^* A_f$ by 1.9 (2). If $x \in A$ and $x/1 \in ((f_1, \ldots, f_{d-1})A_f)^*$, then there exists $c \neq 0$ such that $cf^n x^q \in (f_1, \ldots, f_{d-1})^{[q]}$ $(\forall q \gg 1)$. Since $(f_1, \ldots, f_{d-1}, f)$ is an sop of A and since A is Cohen-Macaulay, we have $\forall q \gg 1, cx^q \in (f_1, \ldots, f_{d-1})^{[q]}$ and then $x \in (f_1, \ldots, f_{d-1})^* = (f_1, \ldots, f_{d-1})$.

(6) It suffices to show that that for $I^* = I$ for every $\mathfrak{m}$ primary ideal I. Since we have $I = J : (J : I)$ for every sop $J \subset I$ (6.3), we have $I^* = I$ be Lemma 2.7. □

Lemma 2.7. *Let I be an ideal of A. If $I^* = I$, then for every ideal $\mathfrak{a}$ of A, we have $[I : \mathfrak{a}]^* = [I : \mathfrak{a}]$.*

Proof. If $\mathfrak{a} = (a_1, \ldots, a_n)$, then $[I : \mathfrak{a}] = \cap_{i=1}^n [I : (a_i)]$. Hence we can assume that $\mathfrak{a} = (a)$. If $u \in [I : (a)]^*$ and if $cu^q \in (I : (a))^{[q]}$, then $cu^q = bv^q$ for some $v \in [I : (a)]$ and $b \in A$. Then we have $c(ua)^q \in I^{[q]}$ and we have $ua \in I^* = I$. □

Remark 2.8. Let A be F-rational and **not** F-regular. We will construct an ideal for which $I^* \neq I$.

The notion of F-injective rings is a weaker version of F-rational rings.

Definition 2.9. Let $(A, \mathfrak{m})$ be a local ring of characteristic $p > 0$ and let $J = (x_1, \dots, x_d)$ be an sop of A. We say that A is F-injective if the multiplication of $(x_1 \cdots x_d)^{q-1} : A/J \to A/J^{[q]}$ is injective. (This condition is equivalent to say that the Frobenius morphism on $\mathrm{H}^d_{\mathfrak{m}}(A)$ is injective (see §3).)

Problem 2.10 (Fedder)**.** Let $(A, \mathfrak{m})$ be a local ring of characteristic $p > 0$ and let $x \in \mathfrak{m}$ be a non 0 divisor of A. Assume that A/xA is Cohen-Macaulay, normal, F- injective and that A_x is F-rational. Then show that A is F-rational.

In Problems 2.11 and 2.12, assume that k is a field of characteristic $p > 0$.

Problem 2.11 (Huneke)**.** Let x, T be variables over k and put $R = k[T, xT^4, x^{-1}T^4, (x-1)^{-1}T^4]$. Then show the following statements.

1. R is F-rational.
2. R is not F -regular.
3. Put $I = (xT^4, x^{-1}T^4)$. Then show that $T^7 \notin I$ and $T^7 \in (xT^4, x^{-1}T^4)^*$.

Problem 2.12. (cf. [Wa97]) Let $A = k[[X, Y, Z]]/(X^2 + Y^3 + Z^4)$ and $\mathfrak{m}$ be the maximal ideal of A. Let $R = A[\mathfrak{m}T] \subset A[T]$ be the Rees algebra of $\mathfrak{m}$ over A. Then show the following statements.

1. Show that for every prime p, R is F-rational.
2. Show that A is F-rational for $p \geq 5$ and not F-rational for $p = 2, 3$.
3. Show that A is a pure subring of R. Consequently, the statement "If A is a pure subring of B and if B is F-rational, then so is A" ("F-version of Boutot's Theorem for rational singularities) does not hold true.

3 Tight Closure of modules, F pure rings

Let A be a ring of characteristic $p > 0$. We define Frobenius functors F^e on the category of A-modules and in a fixed A-module M, we define tight closure N^*_M of a submodule N of M. In particular, the tight closure of submodules of local cohomology module $\mathrm{H}^i_{\mathfrak{m}}(A)$ and injective envelope $E_A(A/\mathfrak{m})$ are very important.

This concept is applied to theory of complexes over the ring like "phantom cohomology" theorem.

In this section, the rings are Noetherian rings of characteristic $p > 0$ and we put $q = p^e$.

Definition 3.1. 1. We write ${}^eA$ the A module A by $F^e : A \to A, F^e(a) = a^q$. Namely, we define

$$a.x = a^q x$$

for $a \in A$ and $x \in {}^eA$.

2. For an A-module M, we define $F^e(M) = M \otimes_A {}^eA$. For an A-module homomorphism $u : M \to M'$, we define $F^e(u) : F^e(M) \to F^e(M')$ by $F^e(u)(x \otimes 1) = u(x) \otimes 1$.

3. For an A submodule N of M, we define

$$N^{[q]} = \mathrm{Im}(F^e(i) : F^e(N) \to F^e(M)).$$

Here $i : N \to M$ is the inclusion map of N in M. (Since F^e is not an exact functor, note that $F^e(i)$ is not injective in general. In particular, if $N = I$ is an ideal of $A = M, N^{[q]} = I^{[q]}$ coincides our definition in 1.1.

Exercise 3.2. Let M be a finitely generated A-module.

1. If M is represented by the following exact sequence ((a_{ij}) is an $m \times n$ matrix).

$$A^m \xrightarrow{(a_{ij})} A^n \longrightarrow M \longrightarrow 0.$$

Tensoring ${}^eA$ to this exact sequence (put $q = p^e$) we have the exact sequence

$$A^m \otimes_A {}^1A \xrightarrow{(a_{ij}^p)} A^n \otimes_A {}^1A \longrightarrow F^e(M) \longrightarrow 0.$$

2. in particular, for an ideal I of A, $F^e(A/I) \cong A/I^{[q]}$.

Let us note the following Theorem by Buchsbaum and Eisenbud. 6.5.

Theorem 3.3. *Let*

$$G_\bullet : 0 \to G_n \stackrel{\varphi_n}{\to} G_{n-1} \stackrel{\varphi_{n-1}}{\to} G_{n-2} \to \cdots \to G_1 \stackrel{\varphi_1}{\to} G_0$$

be a complex of finitely generated free A-modules. Then G is acyclic if and only if so is $F^e(G_.)$. In particular, if $\operatorname{hd}_A I < \infty$ if and only if $\operatorname{hd}_A I^{[q]} < \infty$ for every $q = p^e$.

Proof. The condition (2) of 6.5 for $G_\bullet$ and $F^e(G_\bullet)$ are clearly equivalent. □

Let us define the tight closure of a submodule.

Definition 3.4. Let M be an A-module and N be a submodule of M.

1. Let $x \in M$. We denote by x^q the image of x by the Frobenius morphism $F^e(x) = x \otimes 1$. Then we have equalities

   $$(x + y)^q = x^q + y^q, \text{ and } (ax)^q = a^q x^q$$

   for $x, y \in M$ and $a \in A$.

2. We define the **tight closure N^*_M of N in M** by

   $$x \in N^*_M \iff \exists c \in A^\circ, \forall q \gg 1, cx^q \in N^{[q]}.$$

N^*_M is an A submodule of M containing N.

Remark 3.5. For $x \in M$, $x \in N^*_M$ if and only if the image of x in M/N is an element of $(0)^*_{M/N}$. Namely, we have $N^*_M/N \cong (0)^*_{M/N}$.

In the applications of tight closure of modules, the tight closures in local cohomology modules and injective envelopes are most typical.

Exercise 3.6. Let $(A, \mathfrak{m})$ be a local ring with $\dim A = d$ and let $\underline{x} = (x_1, \ldots, x_d)$ be an sop of A.

1. By 6.4, recall that $\mathrm{H}^d_{\mathfrak{m}}(A) = \mathrm{H}^d_{\underline{x}}(A) \cong A_{x_1 x_2 \cdots x_d} / \operatorname{Im}[C^{d-1} \to C^d]$. Hence we have $F^e(\mathrm{H}^d_{\mathfrak{m}}(A)) = \mathrm{H}^d_{\underline{x^q}}(A) \cong \mathrm{H}^d_{\mathfrak{m}}(A)$.
   If we write by $[\frac{s}{(x_1 x_2 \cdots x_d)}]$ the image of $\frac{s}{(x_1 x_2 \cdots x_d)} \in C^d$ in $\mathrm{H}^d_{\underline{x}}(A)$, then for $\xi = [\frac{s}{(x_1 x_2 \cdots x_d)}]$, we have $\xi^q = [\frac{s^q}{(x_1 x_2 \cdots x_d)^q}]$.
   $\mathrm{H}^d_{\mathfrak{m}}(A) \cong A_{x_1 x_2 \cdots x_d} / \operatorname{Im}[C^{d-1} \to C^d]$

2. If A is Cohen-Macaulay, A is F rational if and only if $(0)^*_{\mathrm{H}^d_{\mathfrak{m}}(A)} = 0$. In fact, for $\xi = [\frac{s}{(x_1 x_2 \cdots x_d)}]$ and $c \in A^\circ$, $c\xi^q = 0$ in $\mathrm{H}^d_{\mathfrak{m}}(A))$ is equivalent to say that $cs^q \in (\underline{x})^q$. Hence $\xi \in (0)^*_{\mathrm{H}^d_{\mathfrak{m}}(A)} \iff s \in (\underline{x})^*$.

3. We will show in 4.4 that A is F-rational if and only if $\mathrm{H}^d_{\mathfrak{m}}(A)$ has no proper submodule which is stable under Frobenius action (we say that $N \subset M$ is stable under Frobenius action if $F(N) \subset N$).

The action of Frobenius endomorphism characterizes F-regular rings.

Theorem 3.7. *Assume that a Noetherian local ring $(A, \mathfrak{m})$, satisfies the condition*

$(*)$ For every $n > 0$, there exists an **irreducible** *$\mathfrak{m}$-primary ideal $J \subset \mathfrak{m}^n$.*

Then the following conditions on A are equivalent.

1. *A is weakly F-regular.*
2. *For every finitely generated A module M and for any submodule N of M, we have $N^*_M = N$.*
3. *Let $E = E_A(A/\mathfrak{m})$ be the injective envelope of $A/\mathfrak{m}$ as A module. Then we have $(0)^{*fg}_E = 0$, where $(0)^{*fg}_E$ is the sum of $(0)^*_M$ for all finitely generated A submodule of E.*

Proof. We have shown in 1.9 that A is weakly F-regular if for every $\mathfrak{m}$-primary ideal I, we have $I^* = I$. Since any $\mathfrak{m}$ primary ideal is an intersection of irreducible $\mathfrak{m}$-primary ideals, it suffices to assume that I is irreducible. Then there is an injection $A/I \to E_A(A/\mathfrak{m})$ and we have (3)$\Longrightarrow$ (1). Next, we show (1)$\Longrightarrow$ (2). Assume that $x \in N_M^*, x \notin N$. Then assuming that N is a maximal submodule with $x \notin N$, we may assume that $\mathfrak{m}x \subset N$ and $M/N \subset E$. Put $I = \mathrm{Ann}_A(M/N)$ and take an irreducible $\mathfrak{m}$ primary ideal $J \subset I$. since $A/J \to A/I$ is surjective, taking Matlis dual, $M/N \cong (A/I)^\vee \subset (A/J)^\vee \cong A/J$. Then we have $x \in J^*/J$, which implies A is not weakly F-regular. Thus we show (1)$\Longrightarrow$ (2). $\square$

Next we will introduce the notion of f-pure rings. This concept is historically the oldest one among the notions appeared in this Chapter.

Definition 3.8. A is called **F-pure**, if A is a pure subring of ${}^1A$ by $F : A \to {}^1A$.

Problem 3.9. For a local ring $(A, \mathfrak{m})$, let $E = E_A(A/\mathfrak{m})$ be the injective envelope of $A/\mathfrak{m}$. Show that A is F-pure if and only if $F : E \to E \otimes_A {}^1A$ is injective.

Remark 3.10. By 3.9, if A is F-regular, then A is F-pure. there is no implications between "F-pure" and "F-rational".

Problem 3.11. Let R be a graded ring over a field $k = R_0$ of characteristic $p > 0$.

1. Show that a face ring $k[\Delta]$ is F-pure.
2. Show that if R is F-pure (resp. F-rational), then $a(R) \leq 0$ (resp. $a(R) < 0$).
3. Moreover, assume that k is algebraically closed. If $\dim R = 1$ and R is F-pure, then R is isomorphic to a face ring (see also [GW2]).

Problem 3.12 (Fedder's ceiterion)**.** Let $(B, \mathfrak{n})$ be a RLR and $A = S/I$.

1. If we put $E = E_S(S/\mathfrak{n})$, then show that $E_A(A/\mathfrak{m}) = [0 :_E I]$ and $E_A(A/\mathfrak{m}) \otimes {}^1A \cong [0 :_E I^{[p]}]/I[0 :_E I^{[p]}]$.
2. (Fedder's criterion) Show that A is F-pure if and only if $\Longleftrightarrow$ $[I^{[p]} : I] \not\subset \mathfrak{n}^{[p]}$.

3. In particular, if I is generated by a regular sequence $(f_1, \ldots, f_r)$, then A is f-pure if and only if $(f_1 f_2 \cdots f_r)^{p-1} \notin \mathfrak{n}^{[p]}$.

4. Let $A = k[X,Y,Z]/(X^3+Y^3+Z^3)$ $(p \neq 3)$. Then A is F-pure if and only if $p \equiv 1 \pmod 3$. Also, show that $A = k[X,Y,Z]/(X^2+Y^4+Z^4)$ $(p \neq 2)$ is F-pure if and only if $p \equiv 1 \pmod 4$.

Example 3.13. ([Wa87], [Wa91]) Let x, T be variables over a field k of characteristic $p > 0$ and put
$R = k[T, xT^a, x^{-1}T^b, (x-1)^{-1}T^c]$, where $a, b, c \geq 2$ are positive integers. Then the following statements are true and give examples of F-rational rings (or rational singularities) which are not F-pure.

1. R is F-rational for any a, b, c.

2. $R/TR \cong k[X,Y,Z]/(XY, YZ, ZX)$ and hence is F-pure.

3. R is F-pure if and only if $1/a + 1/b + 1/c \leq 1$ and R is F-regular if and only if $1/a + 1/b + 1/c < 1$.

Let us introduce the notion of "Phantom homology".

Definition 3.14. Let $\mathbf{G}$ be a complex of finitely generated A modules. Then we say $\mathbf{G}$ has **Phantom Homology** if forevery $i, \mathrm{Ker}(G_i \to G_{i-1}) \subset [\mathrm{Im}(G_{i+1} \to G_i)]^*$.

The following Theorem is a generalization of a Theorem of Buchsbaum-Eisenbud 6.5 in terms of tight closured.

Theorem 3.15 (Phantom Homology)**.** *Let $G_\bullet : 0 \to G_n \overset{\varphi_n}{\to} G_{n-1} \overset{\varphi_{n-1}}{\to} G_{n-2} \to \cdots \to G_1 \overset{\varphi_1}{\to} G_0$ be a complex of finitely generated free A-modules with $G_i \neq (0)$ $(1 \leq i \leq n)$. Then the following conditions for $G_\bullet$ are equivalent.*

1. $G_\bullet$ has phantom homology.

2. Foe every i, $1 \leq i \leq n$, the following 2 conditions are satisfied.

(a) $\mathrm{ht}(I(\varphi_i)) \geq i$.

(b) $\mathrm{rank}(\varphi_i, M) + \mathrm{rank}(\varphi_{i+1}, M) = \mathrm{rank}_A F_i$.

(Here, we put $\mathrm{rank}(\varphi_{n+1}, M) = 0$.

We have seen the theory of positive characteristic arguments very quickly. If you are interested to know more of the theory, refer to the lecture note of C. Huneke [Hu2].

Also such theory is know to be closely related to the geometric properties of singularities of algebraic varieties defined over a field of characteristic 0. For example, a rational singularity A over a field of characteristic 0 is characterized by the property "for infinitely many p, A_p, the reduction modulo p of A are F-rational (K. Smith, N. Hara and V.Mehta-V.Srinivas)[1]. Also, "log-terminal singularities" are characterized by "Q-Gorenstein and A_p is F-regular for infiniyly many p". Also, the "multiplier ideals in characteristic 0 are also known to be computable by characteristic p method by N. Hara - K. Yoshida and S, Takagi.

4 Reduction modulo p, F-rational rings and rational singularities in characteristic 0

First, let us recall the definition of **rational singularities**.

Definition 4.1. Let k be a field of characteristic 0 and let $(A, \mathfrak{m})$ be a normal local ring which is a localization of a finitely generated algebra over k. We put $\Sigma = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid A_{\mathfrak{p}} \text{ is not regular }\}$ be the singular locus of $\mathrm{Spec}(A)$. It is known that Σ is closed in $\mathrm{Spec}(A)$.

1. A k morphism $f : X \to \mathrm{Spec}(A)$ is called a **resolution of** $\mathrm{Spec}(A)$ if f is a projective birational morphism and f is isomorphism over $\mathrm{Spec}(A) \setminus \Sigma$.
2. A is called a "rational singularity" if $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for every $i > 0$.

Remark 4.2. (1) Using Grothendieck duality and since we have $\mathrm{H}^i(X, \omega_X) = 0$ (Grauert-Riemenschneider vanishing Theorem; here ω_X is the dualizing

[1]Later, it turned out that it suffices to show that A_p is F-rational for a single p ([MaSch])

sheaf of X), A is a rational singularity if and (a) A is Cohen-Macaulay and (b) $\mathrm{H}^0(X, \omega_X) = K_A$.

(2) If the field k has positive characteristic, there are counterexamples of GR Vanishing Theorem. But un general, we can assert that if GR Vanishing holds for a birational morphism $f : X \to Y$, where X is a normal Cohen-Macaulay variety then the conditions (a) and (b) above are equivalent. Here we say that GR Vanishing holds for f if $\mathrm{H}^i(X, \omega_X) = 0$ for every $i > 0$.

(3) An important property is Boutot's Theorem ([Bou]) saying that a pure subring of a rational singularity is a rational singularity. It turns out that this is not true for F-rational rings.

(4) ([Fl], [Wa83]) If $R = \bigoplus_{n \geq 0} R_n$ is a normal graded ring with $R_0 = k$ is a field of characteristic 0, then R is a rational singularity if R satisfies the following 3 properties; (1) $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ has rational singularities, (2) R is Cohen-Macaulay, (3) $a(R) < 0$.

To show that F-rational sings are rational singularity, it is convenient to introduce the notion of pseudo-rational rings, which was introduced in [LT].

Definition 4.3. Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of $\dim A = d$. We say that A is a **pseudo-rational** ring if it satisfies the following condition (†).

(†) For every projective birational morphism $f : W \to X := \mathrm{Spec}(A)$ where W is normal, the natural surjective map $\delta_f : \mathrm{H}^d_{\mathfrak{m}}(A) \to \mathrm{H}^d_{\mathbb{E}}(\mathcal{O}_W)$ is injective, where we put $\mathbb{E} = f^{-1}(\mathfrak{m})$.

It is stated in [LT], p. 102, that A is pseudo-rational if and only if for every f as above, there exists projective birational morphism $g : W' \to W$ (we assume $W.W'$ are normal and Cohen-Macaulay), such that δ_{fg} is injective.

Note that if f is isomorphic over $\mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$, from local cohomology long exact sequence, we have an exact sequence

$$\mathrm{H}^{d-1}_{\mathbb{E}}(\mathcal{O}_X) \to \mathrm{H}^{d-1}(X, \mathcal{O}_X) \to \mathrm{H}^{d-1}(X - \mathbb{E})(\mathcal{O}_X) \cong \mathrm{H}^d_{\mathfrak{m}}(A) \to \mathrm{H}^d_{\mathbb{E}}(\mathcal{O}_X) \to 0$$

Remark 1. (1) If A is essentially of finite type over a field k of characteristic 0, "pseudo rational" and "rational singularity" are equivalent conditions.

(2) We saw Briançon-Skoda Theorm at 1.11 using tight closure. We have also Briançon-Skoda Theorem for pseudo-rational rings. Namely if $(A, \mathfrak{m})$ is a pseudo-rational local ring of dimension d, then for every ideal I in A, we have $\overline{I^d} \subset I$, where $\overline{I^d}$ is the integral closure of I^d.

We can show that F-rational rings are pseudo-rational.

Theorem 4.4. *(K. Smith) Let $(A, \mathfrak{m})$ be a complete normal Cohen-Macaulay local ring of characteristic $p > 0$, $\dim A = d$. Then;*

1. *A is F rational if and only if $\mathrm{H}^d_{\mathfrak{m}}(A)$ has no proper submodule closed under the Frobenius action. (We say that an A submodule N of $\mathrm{H}^d_{\mathfrak{m}}(A)$ is closed under the action of Frobenius if for every $x \in N$ we have $x^q \in N$.)*

2. *If A is F-rational, then A is pseudo-rational.*

Proof. (1) If A is F-rational, then $(0)^*_{\mathrm{H}^d_{\mathfrak{m}}(A)}$ is closed under the action of Frobenius. So the condition imply that A is F-rational. Conversely, if A is F-rational and if N is a proper A submodule of $\mathrm{H}^d_{\mathfrak{m}}(A)$, the Matlis dual $N^\vee$ of N is a proper quotient of K_A. Then we can take $c \in A^\circ$ such that $cN^\vee = 0$, which imply that $\forall x \in N, cx^q = 0$ and then $N \subset (0)^*_{\mathrm{H}^d_{\mathfrak{m}}(A)}$.

(2) Let $f : X \to \mathrm{Spec}(A)$ be a birational morphism with X normal. Put $E = f^{-1}(\mathfrak{m})$. X has Frobenius action $\mathcal{O}_X \to \mathcal{O}_X^{1/p}$. Since the map δ_f : $\mathrm{H}^d_{\mathfrak{m}}(A) \to \mathrm{H}^d_E(\mathcal{O}_X)$ is compatible with Frobenius action of each component. Hence $\mathrm{Ker}(\delta_f)$ is an F-stable submodule of $\mathrm{H}^d_{\mathfrak{m}}(A)$ and hence must be (0). □

Now we introduce the notion of "reduction modulo p" for varieties of finite type over a field k of characteristic 0.

Definition 4.5. Let k be a field of characteristic 0 and consider a local ring which is essentially of finite type over k. There is an affine k algebra $R = k[x_1, \ldots, x_n]/(f_1 \ldots, f_m)$ such that $(A, \mathfrak{m})$ is a localization of R. Since the coefficients of $f_i, \ldots, f_m$ are finite, we can take a finitely generated $\mathbb{Z}$ subalgebra A of k, and finitely generated A algebra R_A such that $R \cong$

$R_A \otimes_A k$. Moreover, by "generic flat theorem", we can take $a \in A$ so that $R_A[a^{-1}]$ is flat over A_a. Then considering A_a instead of A, we can assume that R_A is flat over A. We say R_A a model of R over A. Since A is finitely generated over Z, for any maximal ideal μ of A, A/μ is a field finitely generated over $\mathbb{Z}$ and hence is a finite field and in particular, positive characteristic. If the characteristic of A/μ is p, We say $R_A \otimes A/\mu$ a **reduction of R modulo p**.

Example 4.6. If $R = k[x, y, z]/(x^a + y^b + z^c)$, we can take $A = \mathbb{Z}, R_A = \mathbb{Z}[x, y, z]/(x^a + y^b + z^c)$. and $\mathbb{Z}/(p)[x, y, z]/(x^a + y^b + z^c)$ is a reduction modulo p of R.

Using the notion of reduction modulo p, we can define the notion of **F - P -type** for a property **P** of a ring. For most property **P** and extension $k \subset k'$ of fields, R satisfies **P** if and if $R \otimes_k k'$ satisfies **P** and hence these property depends only on the characteristic p. In this sense, we say a reduction $R_A \otimes_A A/\mu$ simply R_p is $\mathrm{char}(A/\mu) = p$.

Definition 4.7. Let R be a ring essentially of finite type over a field k of characteristic zero. Suppose we are given a model R_A of R over a finitely generated $\mathbb{Z}$-subalgebra A of k. We say that R is of **open** F-**P** type (resp. **dense**) F-**P** type if R_p satisfies **P** for all but finite p (resp. for infinitely many) p.

For example R is open F-rational type if R_p is F-rational for all but finite p dense F-pure type if R_p is F-pure for infinitely many p.

These definitions are independent of the choice of the model R_A.

Example 4.8. (1) Let $R = \mathbb{C}[X, Y, Z]/(X^3 - YZ(Y + Z))$. Then the ring $R_{\mathbb{Z}} = \mathbb{Z}[X, Y, Z]/(X^3 - YZ(Y + Z))$ is a model of R over $\mathbb{Z}$. We take the dense subset S of $\operatorname{Spec}\mathbb{Z}$ to be $\{p \in \operatorname{Spec}\mathbb{Z} \mid p \equiv 1 \bmod 3\}$. (Show that by Fedder's criterion 3.12, $k[X, Y, Z]/(X^3 - YZ(Y + Z))$ if and only if $p \equiv 2 \pmod 3$.) Hence the ring R is of dense F-pure type and not of open F-pure type.

(2) Let $R = \mathbb{C}[X, Y, Z]/(X^2 + Y^3 + Z^5)$. The ring $R_{\mathbb{Z}} = \mathbb{Z}[X, Y, Z]/(X^2 + Y^3 + Z^5)$ is a model of R over $\mathbb{Z}$. It is easy to see that $x \in (y, z)^*$ if and only if $p = 2, 3, 5$. Since $S = \{p \in \operatorname{Spec}\mathbb{Z} \mid p \geq 7\} \cup \{0\}$ is a dense open

subset of $\operatorname{Spec}\mathbb{Z}$, the ring R is of open strongly F-regular type (and of open F-pure type).

We will show that if R has dense F-rational type, then R is a rational singularity.

Theorem 4.9. $(A, \mathfrak{m})$ *is a local ring essentially of finite type over a field k of characteristic o. Then A is a rational singularity if and only if A is of open f-pure type.*

Proof. First, since rational singularities and F-rational rings are both Cohen-Macaulay, we may assume A is a Cohen-Macaulay ring. Assume that A is not a rational singularity and show that A is not of open f-rational type.Take a resolution of singularity $f : X \to \operatorname{Spec}(A)$. Take the support $Y = \bigcup_{i\geq 1} \operatorname{Supp}_A(\mathrm{H}^i(X, \mathcal{O}_X))$. If $\mathfrak{p}$ is a generic point of Y, since localizing by $\mathfrak{p}$ preserves resolution, we can assume that $Y = \{\mathfrak{m}\}$. Then by Grothendieck duality, we have $H^{d-1}(X, \mathcal{O}_X) \cong [\omega_A/f_*(\omega_X)]' \neq 0$. Since A_p is F-rational for all but all p, we can assume that A_p is F-rational. But then $\mathrm{H}^{d-1}(X, \mathcal{O}_X)$ is a F- stable subset of $\mathrm{H}^d_{\mathfrak{m}}(A)$, which contradicts Smith's Theorem (4.4). Hence A shold be a rational singularity.

The converse needs deeper Theorems of algebraic geometry and proved by N. Hara ([Hara]) and V. B. Mehta - V. Srinivas ([MeSr]). □

5 Log terminal, log canonical singularities and F regular and F pure rings

Log terminal (LT) and log canonical (LC) singularities play very important role in Minimal Model Program (MMP) in algebraic geometry over a field of characteristic 0. Those singularities are defined by the **discrepancy** of canonical divisors.

First we define $\mathbb{Q}$ Gorenstein local rings.

Definition 5.1. Let $(A, \mathfrak{m})$ be a normal local ring which has a canonical module K_A. We call A is $\mathbb{Q}$-Gorenstein if $\operatorname{cl}(K_A) \in \operatorname{Cl}(A)$ is (0 or) a torsion element. Also, if the order of $\operatorname{cl}(K_A)$ is r, we say that "A is r-Gorenstein".

For a normal local ring A, A is Gorenstein if A is 1-Gorenstein and Cohen-Macaulay. Likewise, we say "X is r-Gorenstein if r is the least positive integer such that $\mathcal{O}_X(rK_X)$ is an invertible $\mathcal{O}_X$-Module.

Let us also define the concept of SNC (Simple Normal Crosiings).

Definition 5.2. Let X be a regular scheme of dimension n. A closed subscheme Y of X satisfies the following condition.

(SNC) At any closed point x of X, we can take a regular sop $(t_1, \ldots, t_d)$ of $\mathcal{O}_{X,x}$ so that the defining equation of Y at x is $t_1t_2\cdots t_r$.

By Hironaka's resolution of singularities over a field of characteristic 0, we can take a resolution of singularities $f : X \to Y$ so that the exceptional set of f satisfies the condition (SNC).

Definition 5.3. We write $\mathrm{Irr}^1(X)$ the set of irreducible (reduced) closed subschemes of X.

Now we define the discrepancy of a birational morphism with respect to an element $E \in \mathrm{Irr}^1(X)$.

Definition 5.4. Let k be a field and X, Y be normal algebraic varieties of dimension n and assume Y is r-Gorenstein. Let $f : X \to Y$ be a birational morphism and $E \in \mathrm{Irr}^1(X)$ with $\dim f(E) < n - 1$. Then we call E an "exceptional divisor" of f.

We define the discrepancy $a(E, X)$ of E as follows;

Let $z \in E$ be a general point of E and $(x_1, \ldots, x_n)$ be a regular sop of X at z. Also let y be the defining equation of E at z.

On the other hand, at $w = f(z) \in Y$, let $\mathcal{O}_Y(rK_Y)_w = t\mathcal{O}_{Y,w}$. If $f^*(t) = uy^s(dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n)^r$ at z (where u is a unit of $\mathcal{O}_{X,z}$), we write

$$a(E, X) = \frac{1}{r}s.$$

Since $f^*(\mathcal{O}(rK_Y)) \cong \mathcal{O}_X(rK_X - sE)$ near z, we can write

$$K_X = f^*(K_Y) + \sum_E a(X, E)E,$$

where E moves over all exceptional irreducible divisors of f.

Remark 2. A divisor E defines a valuation v_E of X. Then the discrepancy E depends only on v_E. Namely, for birational normal varieties X and X' and exceptional divisors E, E', if the valuations v_E and $v_{E'}$ are the same, then we have $a(X, E) = a(X', E')$.

Definition 5.5. Let $(Y.y)$ be r-Gorenstein if for any normal variety X, any birational morphism $f : X \to Y$ and any exceptional E with $y \in f(E)$,

1. We call $A = \mathcal{O}_{Y,y}$ a **terminal singularity** if $a(E, X) > 0$.
2. We call $A = \mathcal{O}_{Y,y}$ a **canonical singularity** if $a(E, X) \geq 0$.
3. We call $A = \mathcal{O}_{Y,y}$ a **log terminal singularity** if $a(E, X) > -1$.
4. We call $A = \mathcal{O}_{Y,y}$ a **log canonical singularity** if $a(E, X) \geq -1$.

Remark 3. To show that $A = \mathcal{O}_{Y,y}$ is one of the singularities defined in 5.5, it suffices to show the condition for a resolution of singularities $f : X \to \mathrm{Spec}(A)$.

We will give one easy example of computing $a(X, E)$.

Example 5.6. We give two simple examples of discrepancy.

1. Let $\mathcal{L}$ be an ample invertible sheaf on a normal projective variety Y and assume that $\omega_Y = \mathcal{O}_Y(K_Y) \cong \mathcal{L}^{\otimes a}$. If $R = \oplus_{n\geq 0}\mathrm{H}^0(Y, \mathcal{L}^{\otimes n})$ be a cone over Y. Then a resolution of $\mathrm{Spec}(R)$ is given by the blowing-up X of the maximal ideal of R and the exceptional set is $E \cong Y$. In this case we have $a(E, X) = a + 1$.

2. Let $A = k[x^3, x^2y, xy^2, y^3] \cong k[y, xy, x^2y, x^3y] = k[\sigma^\vee \cap \mathbb{Z}^2]$, where $\sigma^\vee$ is the cone in $\mathbb{R}^2$ generated by $(0, 1)$ and $(3, 1)$. Also, K_A is generated by $\mathrm{Int}(\sigma^\vee) \cap \mathbb{Z}^2$ (where $\mathrm{Int}(\sigma^\vee)$ is the relatively interior of $\sigma^\vee$ and generated by (xy, x^2y). Since $(xy, x^2y)^3 = (x^3y^2)(y, xy, x^2y, x^3y)$, $(K_A)^{(3)}$ is a principal module generated by x^3y^2.

   We use terminology of "toric geometry". Let σ be a convex cone generated by $(-1, 3)$ and $(1, 0)$ in $\mathbb{R}^2$ so that the dual cone

$$\sigma = (\sigma^\vee)^\vee = \{x \in \mathbb{R}^2(x', y') \mid 3x' + y' \geq 0 \text{ and } y' \geq 0\},$$

which shows that σ is generated by $(-1, 3)$ and $(1, 0)$.

Now, the resolution of $\mathrm{Spec}(A)$ is defined by the unimodular decomposition

$$\sigma = \sigma_1 \cup \sigma_2,$$

where $\sigma_1 =< (-1, 3), (0, 1) >, \sigma_2 =< (0, 1), (1, 0) >$ and then the resolution X of $\mathrm{Spec}(\sigma^\vee)$ is given by union of 2 affine open sets $X = U_1 \cup U_2$, $U_1 = \mathrm{Spec}(\sigma_1^\vee) = \mathrm{Spec}(k[x, x^3y])$ and $U_2 = \mathrm{Spec}(k[x, y])$.

Now, on U_1, $(x^3y^2) = [x^{-1} \cdot (x^3y)]^3 \cdot x^{-3}y^{-1}$ and on U_2, $(xy)^3 = (x^3y^2) \cdot y^{-1}$, Hence E is the exceptional curve corresponding to $(0, 1)$, this shows that $a(E, X) = -1/3$. Then we conclude that $\mathrm{Spec}(A)$ is a log terminal singularity but not a canonical singularity.

Properties of $\mathbb{Q}$-Gorenstein rings are frequently reduced to those of 1-Gorenstein rings using **index 1 cover**.

Definition 5.7. Let A be a normal ring and assume that $K_A \cong I$, which has pure height 1. If $I^{(r)} = wA$, we put

$$B = \oplus_{i \in \mathbb{Z}/r\mathbb{Z}} I^{(i)} u^i,$$

here, putting $u^r = w^{-1}$ B is an A-algebra, which we call an **index** 1 **cover of** A.

Notice that the construction of B depends on the choice of w and may exists non-isomorphic index 1 covers. Since $K_B \cong \mathrm{Hom}_A(B, K_A) \cong \oplus_{j \in \mathbb{Z}_r} I^{(1-j)} u^{-j} \cong u^{-1}B$. Hence B is Gorenstein if B is Cohen-Macaulay.

Remark 4. We can define terminal ring etc. in positive characteristic. Then it is natural to ask if they are Cohen-Macaulay, since a rational singularity is Cohen-Macaulay only if in dimension 2 or when GR vanishing holds for a resolution, which fails in positive characteristic in dimension ≥ 3. A example of "terminal singularity" in positive characteristic which is **not** Cohen-Macaulay appeared in [Kov], [La-R], [Tot].

Example 5.8. Let k be a field and put $A = k[x^3, x^2y, xy^2, y^3]$ be the Veronese subring of $k[x,y]$. In this case, since $K_A = (xy \cdot k[x,y])^{(3)} = (x^2y, xy^2)$, we can put $I = (xy^2, y^3)$. then $I^{(3)} = (y^6)$ then index 1 cover is

$$B = A \oplus (x,y)A \oplus (x,y)^2A \cong k[x,y].$$

The property of $\mathbb{Q}$-Gorenstein rings can be discussed using **index** 1 **cover**.

Theorem 5.9. *Let A be a $\mathbb{Q}$-Gorenstein normal ring essentially of finite type over a field of characteristic* 0 *and B be an index* 1 *cover of A. Then we have the following;*

1. *If A is log-terminal (resp. terminal, canonical, log-canonical), so is B.*
2. *If A is log-canonical then A is a rational singularity and hence Cohen-Macaulay.*

The discrepancy of B is an integer and hence if A is log-terminal, then B is canonical.

Theorem 5.10. *([El], [Fl]) If A is a canonical singularity, essentially of finite type over a field of characteristic* 0 *then A is a rational singularity and hence is Cohen-Macaulay.*

Definition 5.11. We say that A is **strongly F-regular**, if for every $c \in A^\circ$, there exists $q = p^e$ such that the map $A \to A^{1/q}$ sending 1 to $c^{1/q}$ splits as A-module homomorphism. Namely, there is a A-module homomorphism $A^{1/q} \to A$ such that $g(c^{1/q}) = 1$.

Next, we review F-regular and F-pure rings from the viewpoint of Frobenius splittings.

Definition 5.12. Let A be a Noetherian ring containing a field k of characteristic $p > 0$. We say that A is F-finite if ${}^1A$ is a finitely generated A-module. It is easy to show that if A is essentially of finite type over k or $(A, \mathfrak{m})$ is complete local and $A/\mathfrak{m}$ is perfect, then A is F-finite. It is shown by E. Kunz that F-finite rings are excellent rings. Note that A is F-pure if and only if $A \to A^{1/q}$ splits.

Remark 5. If A is strongly F-regular, then A is weakly F-regular. In fact, if $x \in I^*$ and $cx^q \in I^{[q]}$ for any q, take q so that the exists $g : A^{1/q} \to A$ such that $g(c^{1/q}) = 1$. Then $x = xg(c^{1/q}) = g(cx^q) \in g(I^{[q]}) = I$ and we have $I^* = I$. Actually, for quite many F-finite cases, weakly and strongly F-regular rings are known to be equivalent and as far as I know, there is no example of F-finite weakly F-regular ring which is not strongly F-regular.

Until the end of this section, we assume that all the rings are F-finite.

The next is the most important Lemma in what follows.

Lemma 5.13. *Let A be a normal ring of characteristic $p > 0$ with the canonical module K_A (resp. X be a normal algebraic variety over a field k with* $\mathrm{char}(k) = p$ *with canonical divisor K_X). Then for $q = p^e$, we have*

$$\mathrm{Hom}_A(A^{1/q}, A) \cong [K_A^{(1-q)}]^{1/q},$$

$$(\text{resp.}\quad \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{1/q}, \mathcal{O}_X) \cong [\mathcal{O}_X((1-q)K_X)]^{1/q}).$$

Theorem 5.14. *[HaWa] Let $(A, \mathfrak{m})$ be a normal local ring of dimension d essentially of finite type over a field k of characteristic $p > 0$ and assume that $K_A^{(r)}$ is principal for a positive integer r. Let $f : X \to Y = \mathrm{Spec}(A)$ be a birational morphism and assume that X is normal. If E is an irreducible reduced closed subvariety of X with* $\dim f(E) < d - 1$*. then we have*

1. *If A is F-pure, then we have $a(E, X) \geq -1$.*
2. *If A is strongly F-regular, then $a(E, X) > -1$.*

Proof. Assume that A is F-pure and let $\phi : A^{1/q} \to A$ satisfy $\phi \circ i = 1_A$, where i is the injection $A \to A^{1/q}$. Let L be the quotient field of A. Then we can think $\phi : L^{1/q} \to L$. Hence ϕ deefines $\tilde{\phi} : \mathcal{O}_X^{1/q} \to \mathcal{O}_X$, for which $\tilde{\phi} \circ \tilde{i} = 1_{\mathcal{O}_X}$, where $\tilde{i} : \mathcal{O}_X \to \mathcal{O}_X^{1/q}$ is the injection.

Now, assume $a(E, X) < -1$ and deduce a contradiction. At the generic point ξ of E, since $\phi \circ i$ is the identity. On the other hand, by Lemma 5.13, $\tilde{\phi} \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{1/q}, \mathcal{O}_X) \cong [\mathcal{O}_X((1-q)K_X)]^{1/q}$. On the other hand, since $\tilde{\phi}$ is a pull back of ϕ,

$$\tilde{\phi} \in f^*(\mathcal{O}_Y((1-q)K_Y)) = \mathcal{O}_X((1-q)K_X)(-(1-q)a(E, X)E).$$

Since for large enough q, $(1-q)a(E,X) \geq q$, $\tilde{\phi}(\mathcal{O}_{X,\xi}^{1/q})$ is included in the maximal ideal of $\mathcal{O}_{X,\xi}$ and contraticts the fact that $\tilde{\phi} \cdot \tilde{i}$ is the identity. □

From this we show;

Theorem 5.15. *If A is a local ring essentially of finite type over a field of characteristic 0. Assume also that $K_A^{(r)}$ is principal for some r. If A is of dense F-pure type (resp. of dense F-regular type), then A is log-canonical (resp. log-terminal).*

6 Appendix. Referred Theorems.

Proposition 6.1. *Let A be a Noetherian normal ring with quotient field K and I be an integrally closed ideal of A. Then there are finitely many discrete valuations $v_1, \ldots, v_r$ of K with $v_i(A) \geq 0$ and positive integers $a_1, \ldots, a_r$ such that for an element $u \in A$, $u \in I$ if and only if $v_i(u) \geq a_i$ for $i = 1, \ldots, r$.*

Proposition 6.2. *For a Noetherian local ring $(A, \mathfrak{m})$ and for any field K containing $A/\mathfrak{m}$, we can construct a complete local ring $(B, \mathfrak{n})$ which is faithfully flat over A and satisfying the properties*

1. $\mathfrak{m}B = \mathfrak{n}$,
2. $B/\mathfrak{n} \cong K$.

Proposition 6.3. *For an Artin local ring $(A, \mathfrak{m})$, the following conditions are euivalent.*

1. *A is Gorenstein. Namely, A is an injective R-module.*
2. *(0) is an irreducible idea of A.*
3. $\mathrm{Hom}_A(A/\mathfrak{m}, A) \cong A/\mathfrak{m}$.
4. *For every ideal I of A, we have $\ell_A(A/I) = \ell_A([0 :_A I])$.*
5. *For every I of A, we have $I = [0 :_R [0 :_R I]]$.*

6. *For every finitely generated A module M, we have* $\ell_A(M) = \ell_A(\mathrm{Hom}_A(M, A))$.

Proposition 6.4. *Let R be a Noetherian ring,* $\mathfrak{a}$ *be an ideal of R. If* $\underline{f} = (f_1, f_2, \cdots, f_n) \subset R$ *satisfies* $\sqrt{\mathfrak{a}} = \sqrt{(f_1, f_2, \ldots, f_n)}$*, then for every* $p \in \mathbb{Z}$*, we have an isomorphism*

$$\mathrm{H}^p_{\mathfrak{a}}(*) = \mathrm{H}^p_{\underline{f}}(*) := \mathrm{H}^p(C^\bullet),$$

where the Čech complex $C^\bullet = C^\bullet(\underline{f})$ *is defined by*

$$C^\bullet = [0 \to C^0 \to C^1 \to \ldots \to C^n \to 0],$$

with $C^r = \oplus_{\sharp I = r} \bigwedge R_{f_I} e_I, f_I = \prod_{i \in I} f_i$.

Theorem 6.5. *(Buchsbaum-Eisenbud, [BE]) Let A be a Noetherian ring, M be a finitely generated A-module and*

$$F_\bullet : 0 \to F_n \stackrel{\varphi_n}{\to} F_{n-1} \stackrel{\varphi_{n-1}}{\to} F_{n-2} \to \cdots \to F_1 \stackrel{\varphi_1}{\to} F_0$$

be a complex of finitely generated A free modules and assume $\varphi_i \otimes_A M \neq 0$ $(1 \leq i \leq n)$*. Then the following conditions are equivalent.*

1. *The complex*

$$0 \to F_n \otimes M \stackrel{\varphi_n \otimes M}{\longrightarrow} F_{n-1} \otimes M \stackrel{\varphi_{n-1} \otimes M}{\longrightarrow} F_{n-2} \otimes M \longrightarrow \cdots \longrightarrow F_1 \otimes M \stackrel{\varphi_1 \otimes M}{\longrightarrow} F_0 \otimes M$$

   is exact.

2. *For every* $i, 1 \leq i \leq n$*, the following* 2 *conditions are satisfied.*
   (a) $\mathrm{depth}(\mathrm{I}(\varphi_i, M), M) \geq i$.
   (b) $\mathrm{rank}(\varphi_i, M) + \mathrm{rank}(\varphi_{i+1}, M) = \mathrm{rank}_A F_i$*, where we put* $\mathrm{rank}(\varphi_{n+1}, M) = 0$.

Acknowledgement. This manuscript is partially an English translations from my 2 books S. Goto and K. Watanabe, "Commutative Ring Theory", (後藤四郎・渡辺敬一「可換環論」) Nihon Hyouronsha, 2011, Chapter 10. "Methods of Positive Characteristic" and K. Watanabe and F. Hidaka, "Algebraic Methods in Singularity Theory", (渡辺敬一・日高文夫「特異点論の代数的手法」) to be published from Kyouritsu Shuppan, "Higher dimensional Singularities", written in Japanese. I would like to thank the coauthors and the publishers for the collaborations.

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