Local cohomology of invariant rings

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Introduction

- k be a field.
- $R = k[x_1, \ldots, x_d]$ be a polynomial ring.
- G be a finite subgroup of $GL_d(k)$.
- G acts linearly on R. In other words, G acts on R by degree preserving k-algebra automorphism.
- The ring of invariants of G is defined as

$$R^G = \{r \in R \mid g(r) = r, \text{ for all } g \in G\}.$$

Example

- Let $\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $G = \langle \sigma \rangle$ be a group acting on $R = k[x, y]$.
- Then $\sigma(x) = y$ and $\sigma(y) = x$.

$$\begin{aligned} R^G &= \{r \in R \mid g(r) = r, \text{ for all } g \in G\} \\ &= k[x + y, xy]. \end{aligned}$$

- Consider the action of the permutation group $\mathfrak{S}_d \subseteq GL_d(k)$ on a polynomial ring $R = k[x_1, \ldots, x_d]$, where $\mathfrak{S}_d$ acts by permuting the variables.
- Let $\sigma_1, \ldots, \sigma_d$ denote the elementary symmetric polynomials in $x_1, \ldots, x_d$. Then $R^{\mathfrak{S}_d} = k[\sigma_1, \ldots, \sigma_d]$ and hence is a polynomial ring.

Properties of the invariant ring

- **Question**: For which groups G, is the invariant ring

  • a polynomial ring? • a Cohen-Macaulay ring?

- An element g of G is called a pseudo-reflection if it fixes a codimension 1 subspace. In other words $\mathrm{rank}(g - I) \leq 1$.

Non-modular case (when order of G is invertible in k)

- **(G.C. Shephard-J.A. Todd (1954), C. Chevalley (1955), J.P. Serre (1968))** R^G is a polynomial ring $\iff$ G is generated by pseudo-reflections.
- **(J.A. Eagon - M. Hochster, 1971)** R^G is Cohen-Macaulay.

Properties of the invariant ring: the modular case

Modular case (when $\operatorname{char} k$ divides the order of G)

- **(Serre)** If R^G is a polynomial ring, then G is generated by pseudo-reflections.
- **(H. Nakajima, 1979)** G is a group of order p^3 generated by pseudo-reflections.

$$G = \left\{ \begin{bmatrix} 1 & 0 & a+b & b \\ 0 & 1 & b & b+c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in GL_4(\mathbb{F}_p) \mid a, b, c, \in \mathbb{F}_p \right\}.$$

  But R^G is not a polynomial ring.

- Let $G = \langle (12)(34)(56) \rangle$ act on $R = \mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]$. Then R^G is not a Cohen-Macaulay ring.

Local cohomology modules

- The top local cohomology module $H^d_{R_+}(R) = k.\left\{\left[\frac{1}{x_1^{a_1}\cdots x_d^{a_d}}\right] \mid a_1,\ldots,a_d > 0\right\}$.
- The *a*-invariant, $a(R) = \max\{t \mid H^d_{R_+}(R)_t \neq 0\}$.
- When a group G acts on a ring R and R_+ is a G-stable ideal, then the action of G extends to the modules $H^i_{R_+}(R)$.
- When $|G| \in k^\times$, we have $H^d_{R_+}(R)^G \simeq H^d_{R^G_+}(R^G)$.
- **Example**. Let $G = A_3$ act on $R = \mathbb{F}_3[x,y,z]$. For $\Delta = x^2y + y^2z + xz^2$, $R^G = k[\sigma_1,\sigma_2,\sigma_3,\Delta]/(\Delta^2 - \Delta(\sigma_1\sigma_2 - \sigma_1^3) - \sigma_1^6 + \sigma_1^3\sigma_3 + \sigma_2^3)$. Note that $[\Delta/\sigma_1\sigma_2\sigma_3] \in H^3_{\mathfrak{m}}(R^G)$ but the image of this element is zero in $H^3_{\mathfrak{m}}(R)^G$.

Main results (Goel - Jeffries - Singh)

- **Theorem**. Let G be a finite group acting linearly on $R = k[x_1, \ldots, x_d]$ with no pseudo-reflections. Then the following complex of R^G-modules is exact

$$\bigoplus_{g \in G} H^d_{\mathfrak{m}}(R) \xrightarrow{\sigma} H^d_{\mathfrak{m}}(R) \xrightarrow{\mathrm{Tr}} H^d_{\mathfrak{m}}(R^G) \longrightarrow 0,$$

  where $\sigma((\eta_g)_g) = \sum_{g \in G}(\eta_g - g(\eta_g))$ and $\mathrm{Tr}(\zeta) = \sum_{g \in G} g(\zeta)$.

- Consider the action of the group $G = \langle g = (12) \rangle$ on $R = \mathbb{F}_2[x, y]$ by $g(x) = y$ and $g(y) = x$. Then for $[x/\sigma_1\sigma_2] \in H^2_{\mathfrak{m}}(R)$,

$$\mathrm{Tr}([x/\sigma_1\sigma_2]) = [\sigma_1/\sigma_1\sigma_2] = [0].$$

  But $[x/\sigma_1\sigma_2] \notin \mathrm{Im}(1 - g)$. This is because $\mathrm{rank}(H^2_{\mathfrak{m}}(R)_{-2}) = 1$ and $(1 - g)([x/\sigma_1\sigma_2]) = [0]$ implying that $(1 - g)$ is a zero map on $H^2_{\mathfrak{m}}(R)_{-2}$.

Main results (Goel - Jeffries - Singh)

- **Theorem A**. Let G be a finite group acting linearly on R. Then $a(R^G) = a(R)$ if and only if G has no pseudo-reflections and G is a subgroup of $SL_d(k)$.
- **Lemma**. Let G be a finite subgroup of $GL_d(k)$ and H a subgroup of G, acting naturally on R. Then the inequality $a(R^G) \leq a(R^H)$ holds.
- **Theorem B**. Let G be a finite cyclic group with no pseudo-reflections. Then the Hilbert series of $H^d_{\mathfrak{m}}(R^G)$ and $H^d_{\mathfrak{m}}(R)^G$ coincide.
- There is an exact sequence

$$H^d_{\mathfrak{m}}(R) \xrightarrow{1-g} H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R^G) \to 0.$$

  Since the kernel of the first map is $H^d_{\mathfrak{m}}(R)^G$, the statement follows.

Example

- Consider the representation of the Klein-4 group G, over $\mathbb{F}_2$, determined by the matrices:

$$g = \begin{pmatrix} 1,0,0,0,0,0 \\ 0,1,1,0,0,0 \\ 0,0,1,0,0,0 \\ 0,0,0,1,0,0 \\ 0,0,0,0,1,1 \\ 0,0,0,0,0,1 \end{pmatrix} \qquad \text{and} \qquad h = \begin{pmatrix} 1,0,1,0,0,0 \\ 0,1,1,0,0,0 \\ 0,0,1,0,0,0 \\ 0,0,0,1,0,1 \\ 0,0,0,0,1,1 \\ 0,0,0,0,0,1 \end{pmatrix}.$$

Let G act on the polynomial ring $R = \mathbb{F}_2[u,v,w,x,y,z]$.
Then $\operatorname{rank}\,[H^6_{\mathfrak{m}}(R^G)]_{-7} \;=\; 2$ whereas $\operatorname{rank}\,[H^6_{\mathfrak{m}}(R)^G]_{-7} \;=\; 4$.

Thank You