Gröbner deformations and F-singularities

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Let $R = \mathbb{F}_p[X_1, \cdots, X_n]$. Then R is a free module over $F(R)$ with basis $\{X_1^{i_1} \ldots X_n^{i_n} : 0 \leq i_j \leq p-1, j = 1, \cdots, n\}$.

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More generally,

Theorem (Kunz)
A ring R is regular if and only if F is a flat map.

F-singularities

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- Using the action of the Frobenius many kinds of singularities have been defined, all together they are called F-**singularities**.
- Let I be an ideal of R. For any $q = p^e$, define $I^{[q]} := (x^q \mid x \in I)$. **Tight closure of** I is defined by

$$I^* = \{r \in R : \exists c \in R \setminus \cup_{\mathfrak{p} \in Min\ R}\ \mathfrak{p},\ cr^q \in I^{[q]}\ \ \forall\ \ q = p^e \gg 0\}.$$

  An ideal I is called **tightly closed** if $I^* = I$.
- A ring is **weakly** F-**regular** if every ideal is tightly closed. A ring is called F-**regular** if all its localizations are weakly F-regular.
- A ring R is F-**rational** if every parameter ideal is tightly closed.
- A ring R is F-**pure** if the Frobenius homomorphism $F : R \to R$ is a pure map.
- A ring R is F-**injective** if the map $F : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ is injective for any maximal ideal $\mathfrak{m} \subset R$ and $i \in \mathbb{N}$.

F-singularities and examples

$$\begin{array}{ccc} F\text{-regular} & \Longrightarrow & F\text{-rational} \\ \Downarrow & & \Downarrow \\ F\text{-pure} & \Longrightarrow & F\text{-injective} \end{array}$$

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- Summands of a polynomial ring over a field of prime characteristic are F-regular.

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- Normal affine semigroup rings are F-regular.

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Example

- Summands of a polynomial ring over a field of prime characteristic are F-regular.
- Normal affine semigroup rings are F-regular.
- Let $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3)$ with K a field of char $p > 3$. Then R is F-pure if and only if $p \equiv 1 \pmod 3$, but never F-regular.

Initial ideal

- Let $S = K[X_1, \dots, X_n]$ be a polynomial ring over a field K. A **monomial order** $<$ on S is a total order on monomials of S satisfying
  - $1 < u$ for all monomial $u \neq 1$;
  - if u, v monomials with $u < v$, then $uw < vw$ for every monomial w.

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  - $1 < u$ for all monomial $u \neq 1$;
  - if u, v monomials with $u < v$, then $uw < vw$ for every monomial w.
- Let $I \subset S$ be an ideal and $<$ be a monomial order on S. Define the **initial ideal** of I in S as $in_<(I) := (in_<(f) : f \in I)$, where $in_<(f)$ stands for the biggest term of f with respect $<$.
- Given a monomial order $<$, it turns out that it is possible to choose a suitable weight vector $w \in (\mathbb{N}_{>0})^n$ (depending on $<$ and I) such that $in_<(I) = in_w(I)$. Here $in_w(I) = (in_w(f) : f \in I)$, where $in_w(f)$ stands for the sum of the terms of f with maximal w-degree.

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- Let $f = \sum f_j \in S$ be a nonzero polynomial with homogeneous components f_j (with respect to w). Let t be an extra homogenizing variable, define the w-**homogenization** of f, as $f^h = \sum f_j t^{d-j} \in S[t]$, where $d = deg_w f$ and define $hom_w(I) := (f^h : f \in I) \subseteq S[t]$, called the w-**homogenization** of I.
- One note that $hom_w(I)$ is a homogeneous ideal in S[t] with respect to the extended weight $w' = (w_1, \cdots, w_n, 1) \in \mathbb{N}^{n+1}$.

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- One note that $hom_w(I)$ is a homogeneous ideal in S[t] with respect to the extended weight $w' = (w_1, \cdots, w_n, 1) \in \mathbb{N}^{n+1}$.
- Then we say that $R = S[t]/hom_w(I)$ is a **Gröbner deformation**, and we have that:
  - R is a $\mathbb{N}$-graded ring such that $R_0 = K$ and $t \in R$ has degree 1 (the grading is given by $\deg(X_i) = w_i$ and $\deg(t) = 1$).
  - t is a nonzero-divisor on R.
  - $R/tR \cong S/in_w(I)$.
  - $R/(t-1)R \cong S/I$.

Relations between properties of S/I and of $S/in_<(I)$

Theorem

Let I be a homogeneous ideal of S and $<$ be a monomial order on S. Then

(1) $HF(S/I) = HF(S/in_<(I))$, *in particular* $dim(S/I) = dim(S/in_<(I))$.

(2) *depth* $S/I \geq$ *depth* $S/in_<(I)$; *hence if* $S/in_<(I)$ *is CM,* S/I *is so.*

(3) *If* $S/in_<(I)$ *is Gorenstein,* S/I *is so.*

(4) *reg* $S/I \leq$ *reg* $S/in_<(I)$.

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(3) *If* $S/in_<(I)$ *is Gorenstein,* S/I *is so.*
(4) $reg\ S/I \leq reg\ S/in_<(I)$.

A recent work of A. Conca and M. Varbaro states that I and $in_<(I)$ are much more related than usual provided the latter is a squarefree monomial ideal. More precisely,

Theorem (Conca, Varbaro)

Let I be a homogeneous ideal of S such that $in_<(I)$ is a square-free monomial ideal for some monomial order $<$. Then
(1) $depth\ S/I = depth\ S/in_<(I)$; *hence* $S/in_<(I)$ *is CM, if and only if* S/I *is so.*
(2) $reg\ S/I = reg\ S/in_<(I)$.

Questions

Q1. Let I be an ideal of a polynomial ring S over a field K. When is there a monomial order $<$ on S such that $in_<(I)$ is squarefree?

Q2. For which kind of F-singularities do we have that S/I has those F-singularities provided that, for some weight vector $w \in \mathbb{N}^n$, $S/in_w(I)$ has those F-singularities?

Theorem (-, Varbaro)

Let $S = K[X_1, \dots, X_n]$ be the polynomial ring in n variables over a field K (not necessarily of positive characteristic). Let $I \subset S$ be a radical ideal, $<$ a monomial order of S, and call $h = \max\{ht(\mathfrak{p}) : \mathfrak{p} \in Min(I)\}$. If $in_<(I^{(h)})$ contains a squarefree monomial, then $in_<(I)$ is a squarefree monomial ideal.

Idea of the Proof: We first prove for prime characteristic, and then derive over fields of characteristic 0. The proof in prime characteristic uses a suitable version of Fedder criterion.

Negative answers of Q2. for F-regularity

Example

Let $S = K[X_1, \dots, X_5]$ where K has characteristic $p > 2$, and I the ideal generated by the 2-minors of the matrix:

$$\begin{pmatrix} X_4^2 + X_5^3 & X_3 & X_2 \\ X_1 & X_4^2 & X_3^4 - X_2 \end{pmatrix}.$$

- Considering the weight vector $w = (6, 24, 6, 3, 1)$ of $(X_1, X_2, X_3, X_4, X_5)$, one can see that $in_w(I)$ is the ideal generated by the 2-minors of the matrix:

$$\begin{pmatrix} X_4^2 & X_3 & X_2 \\ X_1 & X_4^2 & X_3^4 - X_2 \end{pmatrix}.$$

- By a work of Anurag Singh, $S/in_w(I)$ is F-regular.
- By a work of Anurag Singh it is known that S/I is not F-regular.

Negative answer of Q2. for F-purity

Example

Let $S = K[X_1, \dots, X_5]$ where K has characteristic $p > 3$, and I the ideal generated by the 2-minors of the matrix of previous Example, namely:

$$\begin{pmatrix} X_4^2 + X_5^3 & X_3 & X_2 \\ X_1 & X_4^2 & X_3^4 - X_2 \end{pmatrix}.$$

- If $<$ is the lexicographic monomial order with $X_1 > X_2 > X_3 > X_4 > X_5$, then $in_<(I) = (X_1X_3, X_1X_2, X_2X_3)$.
- $S/in_<(I)$ is F-pure.
- Again by a work of A. Singh, S/I is not F-pure.

Positive answers for F-rationality and strong F-injectivity

$$\text{F-pure} \implies \textbf{strongly } F\textbf{-injective} \implies F\text{-injective.}$$

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Theorem (-, Varbaro)

Let $S = K[X_1, \ldots, X_n]$ be a polynomial ring over a field K of prime characteristic and $w \in (\mathbb{N}_{>0})^n$. If $I \subset S$ is an ideal such that $S/in_w(I)$ is F-rational (resp. strongly F-injective), then S/I is F-rational (resp. strongly F-injective).

Corollary

Let $S = K[X_1, \ldots, X_n]$ be a polynomial ring over a field K of prime characteristic and $<$ be a monomial order on S. Let $I \subset S$ an ideal of S such that $in_<(I)$ is radical, then S/I is strongly F-injective, and so F-injective.

F-singularities of binomial edge algebras

Definition

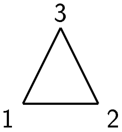
Let $G = (V, E)$ be a simple graph on the vertex set $V = \{1, \cdots, n\}$. Let $S = K[X_1, \cdots, X_n, Y_1 \cdots, Y_n]$, where K is a field. Define the **binomial edge ideal** corresponding to G as

$$I_G := (X_i Y_j - X_j Y_i : \{i, j\} \in E).$$

And the K-algebra S/I_G defined by I_G is called the **binomial edge algebra**.

Example

G =



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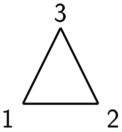
$G =$ 

$$I_G = (X_1 Y_2 - X_2 Y_1, X_2 Y_3 - X_3 Y_2, X_1 Y_3 - X_3 Y_1).$$

Corollary (-, Varbaro)

Let G be a simple graph. Then the corresponding binomial edge algebra S/I_G defined over a field of prime characteristic $p > 0$ is strongly F-injective, and so F-injective.

F-singularities of ASL

Definition

Let $A = \oplus_{i\in\mathbb{N}} A_i$ be a $\mathbb{N}$-graded algebra and let $(H, \prec)$ be a finite poset. Let $H \to \cup_{i>0} A_i$ be an injective function. The elements of H will be identified with their images. Given a chain $h_1 \preceq h_2 \preceq \cdots \preceq h_s$ of elements of H the corresponding product $h_1 \cdots h_s \in A$ is called standard monomial. One says that A is an ***Algebra with straightening laws or ASL*** *on H (with respect to the given embedding H into $\cup_{i>0} A_i$) if three conditions are satisfied:*

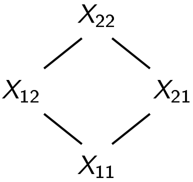
- *The elements of H generate A as a A_0-algebra.*
- *The standard monomials are A_0-linearly independent.*
- *For every pair h_1, h_2 of incomparable elements of H there is a relation*

$$h_1 h_2 = \sum_{j=1}^{u} \lambda_j h_{j1} \cdots h_{jv_j}$$

where $\lambda_j \in A_0 \setminus \{0\}$, the $h_{j1} \cdots h_{jv_j}$ are distinct standard monomials and, assuming that $h_{j1} \preceq \cdots \preceq h_{jv_j}$, one has $h_{j1} \prec h_1$ and $h_{j1} \prec h_2$ for all j.

Example

Let $R = K[X_{11}, X_{12}, X_{21}, X_{22}]/(X_{11}X_{22} - X_{12}X_{21})$. Then R is an ASL with the following poset H:

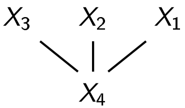


Corollary (-,Varbaro)

Let R be an ASL over a field of prime characteristic $p > 0$. Then R is strongly F-injective, and so F-injective.

An example

- Let $S = K[X_1, X_2, X_3, X_4]$, where K is algebraically closed field of characteristic $p > 0$, and I the ideal generated by the 2-minors of the matrix:
$$\begin{pmatrix} X_4^4 & X_1 & X_3 \\ X_2 & X_4^4 & X_2 - X_3 \end{pmatrix}.$$
- By a work of Hochster and Huneke one can see that S/I is not F-pure.
- The ring S/I can be given an ASL structure on the poset H below:



that is, in the poset H we have $X_4 < X_3, X_2, X_1$ (X_1, X_2 and X_3 are incomparable).

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