all rings comm w/ 1, $p>0$
a fixed prime throughout

often $(R, \mathfrak{m}, k)$ local w/ $p \in \mathfrak{m}$,
i.e. $\operatorname{char} k = p > 0$

two cases

(equal char)
$\operatorname{char} R = p$

e.g. $\mathbb{F}_p[[x_1, \dots, x_d]]$

(mixed char)
$\operatorname{char} R \neq p$

e.g. $\mathbb{Z}_p[[x_2, \dots, x_d]]$
(note: $\Longrightarrow$ $\operatorname{char} R = 0$ if R domain)

<u>Def:</u> R perfect if $\operatorname{char} R = p$
and Frobenius $F: R \longrightarrow R$
$r \longmapsto r^p$

an isomorphism.

Facts: If R perfect, then

① R reduced

② R Noetherian $\iff$ R a finite product of fields

Pf: $(\Rightarrow)$

$Q \subseteq R$ min prime, $S = R/Q$ perfect Noeth domain

$$s \in S \Rightarrow \sqrt{(s)} = (s^{1/p^\infty}) := (s, s^{1/p}, \dots)$$

$$= (s^{1/p^n}) \quad n >> 0$$

$$s^{1/p^{n+1}} = s^{1/p^n} \cdot t$$

$$s^{1/p^{n+1}}(1 - s^{1/p^n - 1/p^{n+1}} t) = 0$$

$\Rightarrow$ $s = 0$ or s is a unit. ✓

Two ways to make rings perfect: char $R = p > 0$

① $R_{perf} := \varinjlim \left(R \xrightarrow{F} R \to \cdots \right)$

$$= \bigcup_{e \geq 0} \left(R_{red}\right)^{1/p^e} = \left(R_{red}\right)^{1/p^\infty}$$

② $R^\flat := \varprojlim \left(\cdots \longrightarrow R \xrightarrow{F} R \right)$

Ψ

$(r_0, r_1, r_2, \ldots)$ w/ $r_{n+1}^p = r_n$

e.g. if R reduced,

$R^\flat \hookrightarrow R$ is the subring of elements w/ arbitrary p-power roots

More generally, for any ring A, put

$$A^\flat := \left(A/p\right)^\flat = \varprojlim \left(\cdots \to A/p \to A/p \to 0 \right)$$

Say A p-complete + sep,

$\exists\ \#: A^\flat \longrightarrow A$ well-defined + multiplicative (check!)

$\{a_n\} \longmapsto \lim_{n\to\infty} a_n^{p^n}$

$a_{n+1}^p \equiv a_n \bmod p$

$\Longrightarrow$ isom of mult monoids (check!)

$$\varprojlim_{x\mapsto x^p} A \longrightarrow A^\flat$$

$\{a_n\}$ $\qquad\qquad$ $\{a_n \bmod p\}$

$a_{n+1}^p = a_n$ $\qquad\qquad$ $a_{n+1}^p \equiv a_n \bmod p$

w/ inverse $\quad A^\flat \longrightarrow \varprojlim_{x\mapsto x^p} A$

$\quad\quad \alpha \longmapsto (\#\alpha, \#\alpha^{1/p}, \#\alpha^{1/p^2}, \dots)$

<u>Rmk:</u> Similarly, R perfect $d_0 \in R$ and R d_0-complete + sep

$$\Rightarrow \quad (R/d_0)^\flat \cong R^\flat \cong R.$$

Thm (Bhatt - Iyengar - Ma)

R Noeth w/ char $R = p > 0$

R regular $\iff \exists\ R \longrightarrow A$

fflat w/ A perfect

Pf: ($\Rightarrow$)

R regular $\overset{\text{Kunz}}{\iff}$ $F: R \to R$ fflat

$\Longrightarrow$ $A = R_{perf}$ fflat

($\Leftarrow$) may assume (R, m, k) local Noeth

$x_1, \ldots, x_d \in R$ sop

$$\sqrt{x_i A} = (x_i^{1/p^n})$$

$$= \varinjlim \left(A \xrightarrow{x_i^{1-1/p}} A \xrightarrow{x_i^{1/p - 1/p^2}} A \longrightarrow \cdots \right)$$

flat A-mod [Aberbach - Hochster]
(check!)

[Hochster] $\prod_j (x_{ij}^{1/p^\infty}) = \bigcap_j (x_{ij}^{1/p^\infty})$

$\Longrightarrow$ can check

$$C_\bullet = \bigotimes_{i=1}^{d} [(x_i^{1/p^\infty}) \longrightarrow A]$$

flat res'n of

$$A/(x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}) = A/\sqrt{mA}$$

of length d $/A$, i.e

$\mathrm{fd}_A A/\sqrt{mA} \leq d.$

$i > d$

$$0 = \mathrm{Tor}_i^A(A \otimes_R k, A/\sqrt{mA}) \overset{A \text{ flat}/R}{=} \mathrm{Tor}_i^R(k, A/\sqrt{mA})$$

On the other hand, $A/\sqrt{mA} = k^{\oplus I}$ as R-mod

so $\mathrm{Tor}_i^R(k, A/\sqrt{mA})$

$$= \mathrm{Tor}_i^R(k,k)^{\oplus I}$$

$\therefore$ $\mathrm{Tor}_i^R(k,k) = 0$ for $i > d$, so R is regular. ✓

Thm [BIM] R Noeth, $p \in J(R)$
Jacobson radical

R regular $\iff$ $\exists$ fflat map $R \longrightarrow A$ w/ A perfectoid

Easiest perfectoids:

- If $\operatorname{char} A = p$, A perfectoid $\iff$ A perfect
- If A p-torsion free, A perfectoid $\iff$ A p-adically complete + separated, $p = u\varpi^p$, u unit, $\varpi \in A$, and $A/\varpi \xrightarrow{F} A/p$ an isomorphism

e.g.

$\mathbb{Z}_p[p^{1/p^\infty}]^{\wedge p}$

$\mathbb{Z}_p[p^{1/p^\infty}, x_2^{1/p^\infty}, \dots, x_d^{1/p^\infty}]^{\wedge p}$

(exercise: compute $A^\flat$ for these)