THE FROBENIUS TEST EXPONENTS IN PRIME CHARACTERISTIC

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Outline

Notations

- $(R, \mathfrak{m})$ is a Noetherian commutative local ring of dimension d and of prime characteristic p (i.e., R contains $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$).
- $R^\circ = R \setminus \bigcup_{\mathfrak{p} \in \operatorname{Min} R} \mathfrak{p}$ is the set of elements of R that are not contained in any minimal prime ideal.
- $I = (x_1, \ldots, x_t)$ is an ideal of R.
- $\mathfrak{q}$ is a parameter ideal of R.
- F: $R \to R$, $x \mapsto x^p$ is the Frobenius endomorphism of R.

The Frobenius action and the relative Frobenius action

1. Let $I = (x_1, \ldots, x_t)$ be an ideal of R. The local cohomology $H^i_I(R)$ may be computed as the cohomology of the Čech complex

$$0 \to R \xrightarrow{d^0} \bigoplus_i R_{x_i} \xrightarrow{d^1} \bigoplus_{i<j} R_{x_i x_j} \xrightarrow{d^2} \cdots \xrightarrow{d^{t-1}} R_{x_1 \ldots x_t} \to 0.$$

The Frobenius endomorphism of R and R_x induce a natural **Frobenius action on local cohomology**

$$F : H^i_I(R) \longrightarrow H^i_{I^{[p]}} \cong H^i_I(R).$$

2. Let $K \subseteq I$ be ideals of R. The Frobenius endomorphism $F : R/K \to R/K$, $F(x + K) = x^p + K$ for all $x \in R$ can be factored as follows:

$$\begin{array}{ccc} R/K & \xrightarrow{\quad F \quad} & R/K \\ & {\scriptstyle F_R} \searrow \quad \nearrow {\scriptstyle \pi} & \\ & R/K^{[p]} & \end{array}$$

where $F_R : R/K \to R/K^{[p]}$, $F_R(x + K) = x^p + K^{[p]}$ for all $x \in R$.
The homomorphism F_R induces **the relative Frobenius actions on local cohomology** $F_R : H^i_I(R/K) \to H^i_I(R/K^{[p]})$ via Čech complexes.

Frobenius closure, tight closure

Let I be an ideal of R, we define

1. The **Frobenius closure** of I is

$$I^F = \{x \mid x^{p^e} \in I^{[p^e]} \text{ for some } e \geqslant 0\},$$

   where $I^{[p^e]} = (x^{p^e} \mid x \in I)$.

2. The **tight closure** of I is

$$I^* = \{x \mid cx^{p^e} \in I^{[p^e]} \text{ for some } c \in R^\circ \text{ and for all } e \gg 0\}.$$

3. The **Frobenius closure of the zero submodule of** $H^i_I(R)$ is

$$0^F_{H^i_I(R)} = \{z \in H^i_I(R) \mid \exists e \geqslant 0, F^e(z) = 0\}$$

   $0^F_{H^i_I(R)}$ is the nilpotent part of $H^i_I(R)$ by the Frobenius action.

Note: $I \subseteq I^F \subseteq I^*$ for all I.

Hartshorne-Speiser-Lyubeznik number

- Since $H^i_{\mathfrak{m}}(R)$ is Artinian for all $i \geqslant 0$, there exists a non-negative integer e such that $0^F_{H^i_{\mathfrak{m}}(R)} = \ker(H^i_{\mathfrak{m}}(R) \xrightarrow{F^e} H^i_{\mathfrak{m}}(R))$. The **Hartshorne-Speiser-Lyubeznik number** of $H^i_{\mathfrak{m}}(R)$ is

$$\mathrm{HSL}(H^i_{\mathfrak{m}}(R)) = \min\{e \mid 0^F_{H^i_{\mathfrak{m}}(R)} = \ker(H^i_{\mathfrak{m}}(R) \xrightarrow{F^e} H^i_{\mathfrak{m}}(R))\}.$$

- The **Hartshorne-Speiser-Lyubeznik number** of a local ring $(R, \mathfrak{m})$ is

$$\mathrm{HSL}(R) = \min\{e \mid 0^F_{H^i_{\mathfrak{m}}(R)} = \ker(H^i_{\mathfrak{m}}(R) \xrightarrow{F^e} H^i_{\mathfrak{m}}(R)) \text{ for all } i = 0, \ldots, d\}.$$

Note: $\mathrm{HSL}(R) < \infty$.

Frobenius test exponent

- By the Noetherianess of R, there is an integer e (depending on I) such that $(I^F)^{[p^e]} = I^{[p^e]}$. The smallest number e satisfying the condition is called the **Frobenius test exponent of** I,

$$\mathrm{Fte}(I) = \min\{e \mid (I^F)^{[p^e]} = I^{[p^e]}\}.$$

- The **Frobenius test exponent for parameter ideals** is

$$\mathrm{Fte}(R) = \min\{e \mid (\mathfrak{q}^F)^{[p^e]} = \mathfrak{q}^{[p^e]} \text{ for all parameter ideals } \mathfrak{q}\},$$

and $\mathrm{Fte}(R) = \infty$ if we have no such integer.

Uniform bound of the Frobenius test exponents for parameter ideals

There is no uniform bound of the Frobenius test exponents for all ideals by Brenner in 2006.

Question 1: Study the existence of an uniform bound of the Frobenius test exponents for some classes of ideals (parameter ideals, ideals generated filter regular sequences).

Question 1.1 (Katzman-Sharp)

Let $(R, \mathfrak{m})$ be an (equidimensional) local ring of prime characteristic p. Then does there exist an uniform bound of the Frobenius test exponents for parameter ideals (i.e., $\mathrm{Fte}(R) < \infty$)?

Theorem

Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p and of dimension d. Then $\mathrm{Fte}(R) < \infty$ in the following cases

1. (Katzman-Sharp, 2006) R is a Cohen-Macaulay ring. Moreover, $\mathrm{Fte}(R) = \mathrm{HSL}(R)$.
2. (Huneke-Katzman-Sharp-Yao, 2006) R is a generalized Cohen-Macaulay ring.
3. (Quy, 2019) R is a weakly F-nilpotent ring, i.e., $H^i_{\mathfrak{m}}(R) = 0^F_{H^i_{\mathfrak{m}}(R)}$ for all $i < d$.
4. (Maddox, 2019) R is a generalized weakly F-nilpotent ring, i.e., $H^i_{\mathfrak{m}}(R)/0^F_{H^i_{\mathfrak{m}}(R)}$ has finite length for all $i < d$.

Uniform bound of the Frobenius test exponents for ideals generated by filter regular sequences

Corollary(Katzman-Sharp, 2006)

Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p and of dimension d and $\underline{x} = x_1, \ldots, x_t$ a fixed regular sequence with $t \leqslant d$. Then there exists an integer $C_{\underline{x}}$ such that $\mathrm{Fte}((x_1^{n_1}, \ldots, x_t^{n_t})) \leqslant C_{\underline{x}}$ for all $n_1, \ldots, n_t \geqslant 1$.

Question 1.2

Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic p and of dimension d, and $t \leqslant \mathrm{depth}(R)$ an integer. Does there exist a positive integer C such that for any regular sequence $x_1, \ldots, x_t$ we have $\mathrm{Fte}((x_1, \ldots, x_t)) \leqslant C$?

Theorem (Huong-Quy)

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and of prime characteristic p, and $t \leqslant d$ a non-negative integer such that $H^j_{\mathfrak{m}}(R)/0^F_{H^j_{\mathfrak{m}}(R)}$ has finite length for all $j < t$. Then there exists a positive integer C such that for any filter regular sequence $x_1, \ldots, x_r$ with $0 < r \leqslant t$ we have $\mathrm{Fte}((x_1, \ldots, x_r)) \leqslant C$.

Note: From this theorem we have aforementioned positive answers of Question 1.1.

An application of the Frobenius test exponent for parameter ideals

Question 2: Let $(R,\mathfrak{m})$ be a Noetherian local ring of prime characteristic p and of dimension d with the embedding dimension v. Suppose $\mathrm{Fte}(R)$ is finite, and let $Q = p^{\mathrm{Fte}(R)}$. Find an upper bound of the Hilbert-Samuel multiplicity $e(R)$ in terms of d, v and $\mathrm{Fte}(R)$.

- In 2015, Huneke and Watanabe proved that:
  If R is F-pure (i.e., $F : R \to R, x \mapsto x^p$ is a pure homomorphism) then $e(R) \leqslant \binom{v}{d}$,
  If R is F-rational (i.e., it is a homomorphic image of a Cohen-Macaulay local ring and $\mathfrak{q}^* = \mathfrak{q}$ for all $\mathfrak{q}$), then $e(R) \leqslant \binom{v-1}{d-1}$.
- In 2019, Katzman and Zhang proved that:
  If R is Cohen-Macaulay then $e(R) \leqslant Q^{v-d}\binom{v}{d}$.

Upper bound of the multiplicity of the ring

Theorem (Huong-Quy)

Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic p and of dimension d with the embedding dimension v. Set $Q = p^{\mathrm{Fte}(R)}$. Then

1. If R is F-nilpotent (i.e., $0^F_{H^i_{\mathfrak{m}}(R)} = H^i_{\mathfrak{m}}(R)$ for all $i \leqslant d-1$ and $0^F_{H^d_{\mathfrak{m}}(R)} = 0^*_{H^d_{\mathfrak{m}}(R)}$) then
$$e(R) \leqslant Q^{v-d}\binom{v-1}{d-1}.$$
2. Suppose $\mathrm{Fte}(R) < \infty$. Then $e(R) \leqslant Q^{v-d}\binom{v}{d}$.


THANK YOU