F-threshold of filtration of ideals

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(Based on a joint work with Dr. Mitra Koley)

Introduction

- Mustata, Takagi and Watanabe (2004): Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$. Let $\mathfrak{a}, I$ be non-zero proper ideals of R such that $\mathfrak{a} \subseteq \sqrt{I}$. For every $e \geq 1$, $\nu_{\mathfrak{a}}^I(p^e) := \max\{r : \mathfrak{a}^r \not\subseteq I^{[p^e]}\}$. Then, the F-threshold of $\mathfrak{a}$ with respect to I is

$$\mathcal{C}^I(\mathfrak{a}) := \lim_{e\to\infty} \frac{\nu_{\mathfrak{a}}^I(p^e)}{p^e}.$$

- Huneke, Mustata, Takagi and Watanabe (2008): Let R be a Noetherian ring of positive characteristic p.

$$\mathcal{C}_+^I(\mathfrak{a}) := \limsup_{e\to\infty} \frac{\nu_{\mathfrak{a}}^I(p^e)}{p^e} \text{ and } \mathcal{C}_-^I(\mathfrak{a}) := \liminf_{e\to\infty} \frac{\nu_{\mathfrak{a}}^I(p^e)}{p^e}.$$

- De Stefani, Betancourt, and Perez (2018): $\mathcal{C}_+^I(\mathfrak{a}) = \mathcal{C}_-^I(\mathfrak{a})$.

Filtration of ideals

Let R be a Noetherian commutative ring of positive characteristic p. A filtration of ideals in R is a collection of ideals $\{\mathfrak{a}_n\}_{n\geq 0}$ which satisfy:

1. $\cdots \subset \mathfrak{a}_{n+1} \subset \mathfrak{a}_n \subset \cdots \subset \mathfrak{a}_1 \subset \mathfrak{a}_0 = R$,
2. $\mathfrak{a}_m\mathfrak{a}_n \subseteq \mathfrak{a}_{m+n}$ for all $m, n \geq 0$.

Standard examples of filtration

Let $\mathfrak{a}$ be a nonzero ideal in R.

- $\mathfrak{a}$-adic filtration: $\mathfrak{a}_n = \mathfrak{a}^n$ for all $n \geq 0$.
- $\mathfrak{a}$-symbolic filtration: take $\mathfrak{a}_n = \mathfrak{a}^{(n)}$ for all $n \geq 0$.
- Normal filtration of $\mathfrak{a}$: take $\mathfrak{a}_n = \overline{\mathfrak{a}^n}$ for all $n \geq 0$.
- Tight closure filtration of $\mathfrak{a}$: take $\mathfrak{a}_n = (\mathfrak{a}^n)^*$ for all $n \geq 0$.

F-threshold of filtration of ideals

- Let I be a non-zero proper ideal of R and $\mathfrak{a}_\bullet = \{\mathfrak{a}_i\}_{i\geq 0}$ be a filtration of ideals in R.
- For every non-negative integer e, we define

$$\nu_{\mathfrak{a}_\bullet}^I(p^e) := \sup\{r \in \mathbb{Z}_{\geq 0} : \mathfrak{a}_r \not\subseteq I^{[p^e]}\}.$$

- We define

$$\mathcal{C}_+^I(\mathfrak{a}_\bullet) := \limsup_{e\to\infty} \frac{\nu_{\mathfrak{a}_\bullet}^I(p^e)}{p^e} \text{ and } \mathcal{C}_-^I(\mathfrak{a}_\bullet) := \liminf_{e\to\infty} \frac{\nu_{\mathfrak{a}_\bullet}^I(p^e)}{p^e}.$$

- If $\mathcal{C}_-^I(\mathfrak{a}_\bullet) = \mathcal{C}_+^I(\mathfrak{a}_\bullet) < \infty$, then we denote it by $\mathcal{C}^I(\mathfrak{a}_\bullet)$ and call it the *F-threshold of $\mathfrak{a}_\bullet$ with respect to I*.
- $\mathcal{C}_+^I(\mathfrak{a}_\bullet) < \infty$ if and only if there exists a positive integer N such that $\mathfrak{a}_{Np^e} \subseteq I^{[p^e]}$ for all $e \gg 0$.

Theorem

Suppose that there exists $N \in \mathbb{N}$ such that $\mathfrak{a}_{Ns} \subseteq I^s$ for all $s \geq 0$. Then $0 \leq \mathcal{C}_\pm^I(\mathfrak{a}_\bullet) \leq N\mu(I)$. Particularly, if R is F-pure ring, then $\mathcal{C}^I(\mathfrak{a}_\bullet)$ exists and

$$0 \leq \mathcal{C}^I(\mathfrak{a}_\bullet) = \sup_{e\geq 0} \frac{\nu_{\mathfrak{a}_\bullet}^I(p^e)}{p^e} \leq N\mu(I).$$

Rees algebra and F-threshold

1. If $\mathfrak{a}_\bullet$ is a Noetherian filtration with $\sqrt{\mathfrak{a}_\bullet} \subseteq \sqrt{I}$, then $0 \leq \mathcal{C}^I(\mathfrak{a}_\bullet) < \infty$, and it is equal to $r\mathcal{C}^I(\mathfrak{a}_r^\bullet)$ for some $r \in \mathbb{N}$.
2. Let I be a nonzero proper ideal of R. Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_i\}_{i\geq 1}$ and $\mathfrak{b}_\bullet = \{\mathfrak{b}_i\}_{i\geq 1}$ be filtrations of ideals in R. If $\mathfrak{a}_\bullet \leq \mathfrak{b}_\bullet$, then $\mathcal{C}_\pm^I(\mathfrak{a}_\bullet) \leq \mathcal{C}_\pm^I(\mathfrak{b}_\bullet)$. Moreover, if $\mathcal{R}(\mathfrak{b}_\bullet)$ is a finitely generated $\mathcal{R}(\mathfrak{a}_\bullet)$-module, then $\mathcal{C}_\pm^I(\mathfrak{a}_\bullet) = \mathcal{C}_\pm^I(\mathfrak{b}_\bullet)$.

Corollary

Let I be a nonzero proper ideal of R and $\mathfrak{a}_\bullet$ be a filtration of ideals in R. Suppose that $\mathcal{R}(\overline{\mathfrak{a}_\bullet})$ is a finitely generated $\mathcal{R}(\mathfrak{a}_\bullet)$-module. Then,

$$\mathcal{C}_\pm^I(\mathfrak{a}_\bullet) = \mathcal{C}_\pm^I((\mathfrak{a}_\bullet)^*) = \mathcal{C}_\pm^I(\overline{\mathfrak{a}_\bullet}).$$

F-threshold of $\mathfrak{a}$-symbolic filtration

We assume that R is a regular ring of positive characteristic p.

Theorem

Let $\mathfrak{a}$ and I be nonzero proper ideals in R such that $\mathfrak{a} \subseteq \sqrt{I}$. Then $\mathcal{C}^I(\mathfrak{a}^{(\bullet)})$ exists, where $\mathfrak{a}^{(\bullet)} = \{\mathfrak{a}^{(i)}\}_{i\geq 0}$ is the symbolic power filtration of $\mathfrak{a}$.

Example

Let $R = \mathbb{K}[x, y, z]$ and $\mathfrak{m} = (x, y, z)$, where $\mathbb{K}$ is a field of characteristic p. Take $\mathfrak{a} = (xy, yz, xz) = (x, y) \cap (y, z) \cap (x, z)$. Then,

$$\mathcal{C}^{\mathfrak{m}}(\mathfrak{a}^{(\bullet)}) = \lim_{e\to\infty} \frac{\nu_{\mathfrak{a}^{(\bullet)}}^{\mathfrak{m}}(p^e)}{p^e} = \lim_{e\to\infty} \frac{2(p^e-1)}{p^e} = 2.$$

$$\mathcal{C}^{\mathfrak{m}}(\mathfrak{a}^{\bullet}) = \lim_{e\to\infty} \frac{\nu_{\mathfrak{a}^{\bullet}}^{\mathfrak{m}}(p^e)}{p^e} = \lim_{e\to\infty} \frac{3(p^e-1)}{2p^e} = \frac{3}{2}.$$

Theorem

Let I and $\mathfrak{a}$ be nonzero proper ideals of R such that $\mathfrak{a} \subseteq \sqrt{I}$. Then,

$$\mathcal{C}^{\mathfrak{m}}(\mathfrak{a}^{(\bullet)}) \leq \mathcal{C}^{\sqrt{I}}(\mathfrak{a}^{(\bullet)}) \leq \mathcal{C}^{\sqrt{\mathfrak{a}}}(\sqrt{\mathfrak{a}}^{(\bullet)}) \leq \text{big-height}(\sqrt{\mathfrak{a}}),$$

for any maximal ideal $\mathfrak{m}$ containing I. In particular, $\mathcal{C}^{\sqrt{I}}(\mathfrak{a}^{\bullet}) \leq \text{big-height}(\sqrt{\mathfrak{a}})$.

Corollary

Let $(R, \mathfrak{m})$ be a regular local ring and let $\mathfrak{a}$ be a nonzero proper ideal of R. Then,

$$\mathcal{C}^{\mathfrak{m}}(\mathfrak{a}^{\bullet}) \leq \mathcal{C}^{\mathfrak{m}}(\mathfrak{a}^{(\bullet)}) \leq ht(\mathfrak{a}).$$

Example

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and $\mathfrak{a} = (x_1^{a_1}, \cdots, x_n^{a_n})$, where $a_1, \ldots, a_n$ are positive integers. Then,

$$\mathcal{C}^{\mathfrak{m}}(\mathfrak{a}^{(\bullet)}) = \frac{1}{a_1} + \cdots + \frac{1}{a_n}.$$

F-threshold of symbolic power filtration

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a standard graded polynomial ring, where $\mathbb{K}$ is a field of prime characteristic p and $\mathfrak{m} = (x_1, \ldots, x_n)$.

Theorem

Let $\mathfrak{a}_\bullet$ be a filtration of nonzero proper homogeneous ideals in R. If $\hat{\alpha}(\mathfrak{a}_\bullet) > 0$, then

$$\mathcal{C}^{\mathfrak{m}}(\mathfrak{a}_\bullet) \leq \frac{n}{\hat{\alpha}(\mathfrak{a}_\bullet)}.$$

Corollary

Let $\mathfrak{a}$ be a nonzero proper homogeneous ideal in R. Then,

1. $\mathcal{C}^{\mathfrak{m}}(\mathfrak{a}^{\bullet}) \leq \dfrac{n}{\alpha(\mathfrak{a})}$.
2. $\mathcal{C}^{\mathfrak{m}}(\mathfrak{a}^{(\bullet)}) \leq \dfrac{n}{\hat{\alpha}(\mathfrak{a}^{(\bullet)})}$.

F-König ideal

Let $\mathfrak{a}$ be a homogeneous radical ideal in R with $\text{ht}(\mathfrak{a}) = h$. We say that $\mathfrak{a}$ is F-König if there exists a homogeneous regular sequence $f_1, \cdots, f_h$ in $\mathfrak{a}$ such that $R/(f_1, \ldots, f_h)$ is F-pure.

Theorem

Let $\mathfrak{a}$ be a F-König ideal in R. Then,

$$\mathcal{C}^{\mathfrak{m}}(\mathfrak{a}^{\bullet}) = \mathcal{C}^{\mathfrak{m}}(\mathfrak{a}^{(\bullet)}) = ht(\mathfrak{a}).$$

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