Polynomial invariant rings in modular invariant theory

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Linear representation of finite group

Let $\mathbb{k}$ be a field and let V be a finite dimensional $\mathbb{k}$-vector space. Let $G \subseteq \mathrm{GL}_{\mathbb{k}}(V)$ be a finite group, i.e., V is a finite dimensional linear representation of G over $\mathbb{k}$.

The action of G on V induces an action on V^*. This action extends to a graded $\mathbb{k}$-algebra automorphism on the symmetric algebra of V^*. We denote this algebra by $S = \mathbb{k}[V]$.

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The ring of invariants of G is the subring of S given by

$$S^G = \{f \in S \mid g.f = f \text{ for all } g \in G\}.$$

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Questions

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Can we characterise all representations $G \longrightarrow \mathsf{GL}(V)$ for which the invariant ring S^G is a polynomial ring?

Pseudo-reflections

An element $g \in G$ is called a pseudo-reflection if $V^g = \{v \in V \mid gv = v\}$ is a codimension 1 subspace of V.

The action of g is not diagoznaliable if and only if the only eigenvalue of for the action is 1, which can happen only when $|G| = 0$ in $\mathbb{k}$. In this case, g is called a transvection.

Theorem (Shephard-Todd, Chevally, Serre)

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The Trace map

- The transfer homomorphism $\mathrm{Tr}^G : S \longrightarrow S^G$ given by $f \mapsto \sum_{g \in G} g.f$ is a map of S^G-modules.
- If $|G|$ is invertible in $\mathbb{k}$, then the Reynold's operator $\frac{1}{|G|}\mathrm{Tr}^G$ is a projection of S onto S^G. Hence S^G is a direct summand of S as an S^G-module.

Theorem (Shank-Wehlau 1999)

If $\mathrm{char}(\mathbb{k})$ *divides* $|G|$, *then the image of the trace map* Tr^G *is a non-zero proper ideal of* S^G, *contained in the ideal generted by the homogeneous invariants of positive degree.*

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Suppose $\mathrm{char}(\mathbb{k}) = p > 0$ *and* $G \subseteq \mathrm{GL}(V)$ *a p-group. Then* S^G *is a polynomial ring* $\Leftrightarrow$ *image of the Transfer homomorphism* Tr^G *is a principal ideal of* S^G.

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Broer's reformulation

$R \hookrightarrow S$ a integral extension of integral domains. K, L are fraction fields of R and S, respectively. $K \hookrightarrow L$ is finite seperable. For $y \in L$, multiplication by y, $L \xrightarrow{.y} L$ is a K-linear map. We denote its trace by $\mathrm{Tr}_{L/K}(y)$.

Then R is a direct summand of $S \Leftrightarrow$ there exists a non-zero principal ideal of S which is mapped by trace map $\mathrm{Tr}_{L/K}$ onto a principal ideal of R.

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Polynomial invariant rings

Nakajima group: class of p-groups generated by pseudoreflections.

Theorem (Nakajima 1980)

If G is a Nakajima group then S^G is a polynomial ring. Suppose $\mathbb{k} = \mathbb{F}_p$. Then S^G is a polynomial ring $\Rightarrow$ G is a Nakajima group.

However, Nakajima's proof cannot be naively extended to bigger fields.

Shank-Wehlau-Broer conjecture holds if

1. (Shank-Wehlau 1999) $G \subseteq \mathrm{GL}_{\mathbb{k}}(V)$ is a Nakajima group.
2. (Broer 2010) G is an abelian p-group.
3. (Braun 2022) $\dim_{\mathbb{k}}(V) = 3$.

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Generalised Nakajima group: a class of p-groups which contains the Nakajima groups as a proper subset and representations $G \subseteq \mathrm{GL}_{\Bbbk}(V)$ not generated by pseudoreflections.

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Theorem (Kummini-M)

Shank-Wehlau-Broer conjecture holds in the following cases:

1. *G is a Generalised Nakajima group.*
2. $\mathbb{k} = \mathbb{F}_p$ *and* $\dim_{\mathbb{k}}(V) = 4$.
3. $|G| = p^3$.

The **Hilbert ideal** $\mathfrak{h}_{G,S} = (S^G)_+ S$ is the ideal of S generated by positive degree invariants.

Theorem (Kummini-M, 2022)

Let $G \subseteq \mathrm{GL}_{\mathbb{k}}(V)$ be a Generalised Nakajima group. Suppose $\dim_{\mathbb{k}}(V) = n$. Then the Hilbert ideal $\mathfrak{h}_{G,S} = (f_1, \ldots, f_n)$ where $f_i \in (S^G)_+$ and $\deg(f_i) \leq |G|$ for each $i = 1, \ldots, n$.

This proves a conjecture of Derksen and Kemper for Generalised Nakajima groups.

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Hilbert ideal

Let $W \subseteq V^G$ be a subspace; $W^\perp = \ker(V^* \to W^*)$, i.e., the subspace of linear forms on V that vanish on W. *The Hilbert ideal of G in S relative to W*$:= \mathfrak{h}_{G,S,W} := (W^\perp S \cap S^G)S$.

We show that $\mathfrak{h}_{G,S,W} = (\mathfrak{h}_{G,S,W} \cap \mathrm{Sym}(W^\perp))S$.

Theorem (Kummini-M, –)

Suppose $\dim_{\Bbbk}(V^G) \geq \dim_{\Bbbk}(V) - 2$*. Then the Hilbert ideal* $\mathfrak{h}_{G,S}$ *is a complete intersection.*

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Suppose S^G is a direct summand of S and suppose $f_1, \ldots, f_n \in S^G$ homogeneous invariants of positive degree such that $\mathfrak{h}_{G,S} = (f_1, \ldots, f_n)$. Then $S^G = \Bbbk[f_1, \ldots, f_n]$.

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Representation of p-groups in characteristic $p > 0$

Suppose $\operatorname{char}(\Bbbk) = p > 0$, $G \subseteq \mathrm{GL}(V)$ a p-group generated by transvections.

G has a composition series of $0 = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_k = G$ such that G_l is a transvection group and G_l/G_{l-1} is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ and is generated by the residue class of a transvection.

Compare $\mathfrak{h}_{G',S,W}$ and $\mathfrak{h}_{G,S,W}$ when $G' \leq G$ are transvection p-groups and $G/G' \simeq \mathbb{Z}/p\mathbb{Z}$.

If $\Bbbk = \mathbb{F}_p$ and $\operatorname{rank}(V) = 4$, then for any $G' \leq G$ as above we have $\operatorname{rank}(V^{G'}) \geq \operatorname{rank}(V) - 2 \implies \mathfrak{h}_{G',S,W}$ is a complete intersection $\implies \mathfrak{h}_{G,S,W}$ is a complete intersection $\implies \mathfrak{h}_{G,S}$ is a complete intersection.

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Thank you for your attention.