Epsilon multiplicity in graded dimension two and density functions

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Abstract

Suppose that I is an ideal in Noetherian local ring $(R,\mathfrak{m})$ of Krull dimension d. Ulrich and Validashti have defined the ε-*multiplicity* of I to be

$$\varepsilon(I) := \limsup_{n\to\infty} \frac{l_R\left(H^0_{\mathfrak{m}}(R/I^n)\right)}{n^d/d!}.$$

The ε-multiplicity can be seen as a generalization of the classical Hilbert-Samuel multiplicity. Cutkosky showed that the '$\limsup$' in the definition of ε-multiplicity can be replaced by a limit if the local ring $(R,\mathfrak{m})$ is analytically unramified. An example due to Cutkosky et al. shows that this limit can be an irrational number even in a regular local ring. We restrict ourselves to homogeneous ideals in a standard graded domain over a field of positive characteristic. Inspired by Trivedi's approach to Hilbert-Kunz multiplicity via density functions, we introduce a density function that is related to the epsilon multiplicity by an integral formula. Moreover, we describe a method to compute the epsilon multiplicity of a homogeneous ideal in a two-dimensional normal graded domain over a field. In some situations we have obtained explicit formulas.

Introduction

Suppose that $(R,\mathfrak{m})$ is a Noetherian local ring of dimension d and I is an ideal in R. The ε-*multiplicity of* I is defined to be

$$\begin{aligned}\varepsilon(I) &:= \limsup_{n\to\infty} \frac{d!}{n^d} l_R\left(H^0_{\mathfrak{m}}(R/I^n)\right)\\ &= \limsup_{n\to\infty} \frac{d!}{n^d} l_R\left(\frac{I^n :_R \mathfrak{m}^\infty}{I^n}\right)\end{aligned}$$

This definition is originally due to Kleiman, Ulrich and Validashti. The invariant $\varepsilon(I)$ is always finite. If I is $\mathfrak{m}$-primary then $\varepsilon(I) = e(I)$.

The following result gives a non-vanishing criteria for the ε-multiplicity.

Theorem (Katz and Validashti). *Suppose that $(R,\mathfrak{m})$ is a Noetherian local ring and I is an ideal of R. Then $\varepsilon(I) > 0$ if and only if I has maximal analytic spread.*

The following result is a generalization of Rees' criterion for detecting integral dependence of ideals by means of this new invariant.

Theorem (Katz and Validashti). *Suppose that R is a locally quasi-unmixed Noetherian ring and $J \subseteq I$ be two ideals in R. Then $\overline{J} = \overline{I}$ if and only if $\varepsilon(I_P) = \varepsilon(J_P)$ for all $P \in \operatorname{Spec}(R)$.*

It is important to know when can the ε-multiplicity be realised as a limit.

Theorem (Cutkosky). *Suppose that $(R,\mathfrak{m})$ is an analytically unramified Noetherian local ring and I is an ideal in R. Then $\varepsilon(I)$ exists as a limit.*

In general, the associated length function

$$n \mapsto l_R\left(H^0_{\mathfrak{m}}(R/I^n)\right)$$

may not exhibit any *polynomial-like behaviour*.

Example (Cutkosky, Hà, Srinivasan and Theodorescu). *There exists a height two prime ideal I in $R = \mathbb{C}[[X,Y,Z,W]]$ such that $\varepsilon(I)$ is a positive irrational number.*

If we assume Nagata's conjecture to be true then there exists an ideal I in $R = \mathbb{C}[[X,Y,Z]]$ such that $\varepsilon(I)$ is an irrational number.

Density functions

The concept of denisty functions was introduced by Trivedi to study Hilbert-Kunz multiplicities. Motivated by her construction, we introduce another density function for ε-multiplicities.

The inspiration for the next result comes from an argument due to Cutkosky. The idea is to relate the growth of the graded components of the saturations to that of the growth of global sections of line bundles.

Theorem (, Roy and Trivedi). *Let k be a perfect field of characteristic $p > 0$ and let $R = \bigoplus_{m\geq 0} R_m$ be a standard graded Noetherian k-algebra with graded maximal ideal $\mathfrak{m} = \bigoplus_{m\geq 1} R_m$. Further assume that R is a domain of dimension d and $\operatorname{depth} R_{\mathfrak{m}} \geq 2$. Let $I \subseteq R$ be a graded ideal. Then the limit*

$$g_I(x) := \lim_{s\to\infty} \frac{\dim_k\left(I^{p^s} :_R \mathfrak{m}^\infty\right)_{\lfloor xp^s\rfloor}}{p^{s(d-1)}}$$

exists for every $x \in \mathbb{R}_{\geq 0}$. Moreover, $g_I\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous function and

$$\int_0^c g_I(x)dx = \lim_{s\to\infty} \frac{\sum_{m=0}^{cp^s} \dim_k\left(I^{p^s} :_R \mathfrak{m}^\infty\right)_m}{p^{sd}}$$

for all integers $c > 0$.

The main ingredient in the proof of the following result is the use of a structure theorem for vector partitions due to Strumfels.

Theorem (, Roy and Trivedi). *Let k be a field and let $R = \bigoplus_{m\geq 0} R_m$ be a standard graded Noetherian k-algebra. Further assume that R is a domain of dimension $d \geq 1$. Let $I \subseteq R$ be a graded ideal. Then the limit*

$$f_I(x) := \lim_{n\to\infty} \frac{\dim_k\left(I^n\right)_{\lfloor xn\rfloor}}{n^{d-1}}$$

exists for every $x \in \mathbb{R}_{\geq 0}$. Moreover, the function $f_I\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a piecewise polynomial and

$$\int_0^c f_I(x)dx = \lim_{n\to\infty} \frac{\sum_{m=0}^{cn} \dim_k\left(I^n\right)_m}{n^d}$$

for all integers $c > 0$.

Corollary (, Roy and Trivedi). *Let k be a perfect field of characteristic $p > 0$ and let $R = \bigoplus_{m\geq 0} R_m$ be a standard graded Noetherian k-algebra with graded maximal ideal $\mathfrak{m} = \bigoplus_{m\geq 1} R_m$. Further assume that R is a domain of dimension d and $\operatorname{depth} R_{\mathfrak{m}} \geq 2$. Let $I \subseteq R$ be a graded ideal. Then the limit*

$$\varepsilon_I(x) := \lim_{s\to\infty} \frac{\dim_k\left(H^0_{\mathfrak{m}}\left(R/I^{p^s}\right)\right)_{\lfloor xp^s\rfloor}}{p^{s(d-1)}}$$

exists for every $x \in \mathbb{R}_{\geq 0}$. Moreover, the function $\varepsilon_I\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is compactly supported, continuous everywhere except for finitely many values of x and

$$d!\int_0^\infty \varepsilon_I(x)dx = \lim_{s\to\infty} \frac{\ell_R\left(H^0_{\mathfrak{m}}\left(R/I^{p^s}\right)\right)}{p^{sd}/d!} = \varepsilon(I).$$

Proposition (, Roy and Trivedi). *Let k be a perfect field of characteristic $p > 0$ and let $R = \bigoplus_{m\geq 0} R_m$ be a standard graded Noetherian k-algebra with graded maximal ideal $\mathfrak{m} = \bigoplus_{m\geq 1} R_m$. Further assume that R is a domain of dimension d and $\operatorname{depth} R_{\mathfrak{m}} \geq 2$. Let $J \subseteq I$ be a graded inclusion of graded ideals such that $\overline{J} = \overline{I}$. Then for all $x \in \mathbb{R}_{\geq 0}$,*

$$\varepsilon_I(x) = \varepsilon_J(x).$$

Epsilon multiplicity in graded dimension two

For the next result we use a theorem due to Cutkosky which says that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are two line bundles on a projective curve X (possibly singular) over an algebraically closed field k of characteristic zero then the bigraded Poincàre series

$$\sum_{(n_1,n_2)\in\mathbb{N}^2} h^0\left(X, \mathcal{L}_1^{\otimes n_1} \otimes \mathcal{L}_1^{\otimes n_2}\right) y_1^{n_1} y_2^{n_2}$$

is a rational function.

Theorem (-Dubey-Roy-Verma and -Roy-Trivedi). *Let k be an algebraically closed field of characteristic zero and let $R = \bigoplus_{m\geq 0} R_m$ be a standard graded Noetherian k-algebra with graded maximal ideal $\mathfrak{m} = \bigoplus_{m\geq 1} R_m$. Further assume that R is a two dimensional Cohen-Macaulay domain. Let $I \subseteq R$ be a graded ideal. Then the limit*

$$\varepsilon_I(x) := \lim_{s\to\infty} \frac{\dim_k\left(H^0_{\mathfrak{m}}\left(R/I^n\right)\right)_{\lfloor xn\rfloor}}{n^{d-1}}$$

exists for every $x \in \mathbb{R}_{\geq 0}$. Moreover, the function $\varepsilon_I\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is compactly supported, piecewise polynomial and

$$d!\int_0^\infty \varepsilon_I(x)dx = \lim_{s\to\infty} \frac{\ell_R\left(H^0_{\mathfrak{m}}\left(R/I^n\right)\right)}{n^d/d!} = \varepsilon(I)$$

is a rational number.

The following example shows that it possible for the saturated Rees algebra to be non-Noetherian in the setup of the previous theorem.

Example (Rees). *Suppose that $R = \dfrac{\mathbb{C}[X,Y,Z]}{(X^3+Y^3+Z^3)}$ with graded maximal ideal $\mathfrak{m}$. Then there exist height one graded prime ideal P in R such that the saturated Rees algebra $\bigoplus_{n\geq 0}(P^n :_R \mathfrak{m}^\infty)$ is non-Noetherian.*

We explicitly compute the epsilon multiplicity of the ideal of fat points on Fermat curves.

Example (, Dubey, Roy and Verma). *Suppose that $R = \dfrac{\mathbb{C}[X,Y,Z]}{(X^e+Y^e+Z^e)}$ where $e \geq 3$ is an integer. Consider the graded ideal $I = P_1^{a_1} \cap \cdots \cap P_r^{a_r}$ where $P_1,\ldots,P_r$ are r distict height one graded prime ideals in R and $a_1,\ldots,a_r$ are positive integers. Then*

$$\varepsilon(I) = \left(\sum_{i=1}^r a_i\right)^2 \left(e - 2 + \frac{1}{e}\right).$$