Epsilon multiplicity in graded dimension two and density functions

Suprajo Das

Indian Institute of Technology, Bombay, India

School on Commutative Algebra and Algebraic Geometry in Prime Characteristics, ICTP, Italy

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Epsilon multiplicity

Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring of dimension d and I is an ideal in R. The *ε-multiplicity of I* is defined to be

$$\begin{aligned}\varepsilon(I) &:= \limsup_{n\to\infty} \frac{d!}{n^d} l_R\left(H^0_{\mathfrak{m}}(R/I^n)\right) \\ &= \limsup_{n\to\infty} \frac{d!}{n^d} l_R\left(\frac{I^n :_R \mathfrak{m}^\infty}{I^n}\right)\end{aligned}$$

This definition is due to Kleiman, Ulrich and Validashti. The invariant $\varepsilon(I)$ is always finite. If I is $\mathfrak{m}$-primary then $\varepsilon(I) = e(I)$.

Properties of epsilon multiplicity

Theorem (Katz and Validashti)

Suppose that $(R, \mathfrak{m})$ *is a Noetherian local ring and* I *is an ideal of* R. *Then* $\varepsilon(I) > 0$ *if and only if* I *has maximal analytic spread.*

Theorem (Katz and Validashti)

Suppose that R *is a locally quasi-unmixed Noetherian ring and* $J \subseteq I$ *be two ideals in* R. *Then* $\overline{J} = \overline{I}$ *if and only if* $\varepsilon(I_P) = \varepsilon(J_P)$ *for all* $P \in \operatorname{Spec}(R)$.

When does epsilon multiplicity exist as a limit?

Theorem (Cutkosky)

Suppose that $(R, \mathfrak{m})$ *is an analytically unramified Noetherian local ring and* I *is an ideal in* R*. Then* $\varepsilon(I)$ *exists as a limit.*

In general, the associated function

$$n \mapsto l_R\left(H^0_{\mathfrak{m}}\left(R/I^n\right)\right)$$

may not exhibit any polynomial-like behaviour.

Example (Cutkosky, Hà, Srinivasan and Theodorescu)

Suppose that $R = \mathbb{C}[[X, Y, Z, W]]$*. Then there exists a height two prime ideal* I *in* R *such that* $\varepsilon(I)$ *is a positive irrational number.*

A density function for saturated powers of an ideal

Theorem (_ , Roy and Trivedi)

Let k be a perfect field of characteristic $p > 0$ and let $R = \bigoplus_{m \geq 0} R_m$ be a standard graded Noetherian k-algebra with graded maximal ideal $\mathfrak{m} = \bigoplus_{m \geq 1} R_m$. Further assume that R is a domain of dimension d and depth $R_{\mathfrak{m}} \geq 2$. *Let $I \subseteq R$ be a graded ideal. Then the limit*

$$g_I(x) := \lim_{s \to \infty} \frac{\dim_k \left(I^{p^s} :_R \mathfrak{m}^{\infty} \right)_{\lfloor xp^s \rfloor}}{p^{s(d-1)}}$$

exists for every $x \in \mathbb{R}_{\geq 0}$. Moreover, $g_I : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous function and

$$\int_0^c g_I(x)dx = \lim_{s \to \infty} \frac{\sum_{m=0}^{cp^s} \dim_k \left(I^{p^s} :_R \mathfrak{m}^{\infty} \right)_m}{p^{sd}}$$

for all integers $c > 0$.

A density function for usual powers of an ideal

Theorem (_ , Roy and Trivedi)

Let k be a field and let $R = \bigoplus_{m\geq 0} R_m$ be a standard graded Noetherian k-algebra. Further assume that R is a domain of dimension $d \geq 1$. Let $I \subseteq R$ be a graded ideal. Then the limit

$$f_I(x) := \lim_{n\to\infty} \frac{\dim_k (I^n)_{\lfloor xn \rfloor}}{n^{d-1}}$$

exists for every $x \in \mathbb{R}_{\geq 0}$. Moreover, the function $f_I \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a piecewise polynomial and

$$\int_0^c f_I(x)dx = \lim_{n\to\infty} \frac{\sum_{m=0}^{cn} \dim_k (I^n)_m}{n^d}$$

for all integers $c > 0$.

A density function for epsilon multiplicity

Corollary (_ , Roy and Trivedi)

Let k be a perfect field of characteristic $p > 0$ and let $R = \bigoplus_{m\geq 0} R_m$ be a standard graded Noetherian k-algebra with graded maximal ideal $\mathfrak{m} = \bigoplus_{m\geq 1} R_m$. Further assume that R is a domain of dimension d and depth $R_{\mathfrak{m}} \geq 2$. *Let $I \subseteq R$ be a graded ideal. Then the limit*

$$\varepsilon_I(x) := \lim_{s\to\infty} \frac{\dim_k \left(H^0_{\mathfrak{m}}\left(R/I^{p^s}\right)\right)_{\lfloor xp^s \rfloor}}{p^{s(d-1)}}$$

exists for every $x \in \mathbb{R}_{\geq 0}$. Moreover, the function $\varepsilon_I : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is compactly supported, continuous everywhere except for finitely many values of x and

$$d! \int_0^\infty \varepsilon_I(x)dx = \lim_{s\to\infty} \frac{\ell_R\left(H^0_{\mathfrak{m}}\left(R/I^{p^s}\right)\right)}{p^{sd}/d!} = \varepsilon(I).$$

A proof sketch of the corollary

By Swanson's theorem there exists an integer $c > 0$ such that

$$(I^n :_R \mathfrak{m}^\infty)_m = (I^n)_m$$

for all integers $n > 0$ and $m > cn$. Therefore,

$$\ell_R\left(H^0_{\mathfrak{m}}\left(R/I^{p^s}\right)\right) = \sum_{m=0}^{cp^s} \dim_k\left(I^{p^s} :_R \mathfrak{m}^\infty\right)_m - \sum_{m=0}^{cp^s} \dim_k\left(I^{p^s}\right)_m.$$

After dividing by p^{sd} and taking limits, we obtain

$$\varepsilon(I) = d! \int_0^c g_I(x)dx - d! \int_0^c f_I(x)dx.$$

Then $\varepsilon_I(x) := g_I(x) - f_I(x)$ satisfies the conclusions of our corollary.

Integral closures and epsilon density functions

Proposition (_ , Roy and Trivedi)

Let k be a perfect field of characteristic $p > 0$ and let $R = \bigoplus_{m \geq 0} R_m$ be a standard graded Noetherian k-algebra with graded maximal ideal $\mathfrak{m} = \bigoplus_{m \geq 1} R_m$. Further assume that R is a domain of dimension d and depth $R_{\mathfrak{m}} \geq 2$. *Let $J \subseteq I$ be a graded inclusion of graded ideals such that $\overline{J} = \overline{I}$. Then*

$$\varepsilon_I(x) = \varepsilon_J(x)$$

for all $x \in \mathbb{R}_{\geq 0}$.

Epsilon multiplicity in graded dimension two

Theorem (_-Dubey-Roy-Verma and _-Roy-Trivedi)

Let k be an algebraically closed field of characteristic zero and let $R = \bigoplus_{m \geq 0} R_m$ be a standard graded Noetherian k-algebra with graded maximal ideal $\mathfrak{m} = \bigoplus_{m \geq 1} R_m$. Further assume that R is a two dimensional Cohen-Macaulay domain. Let $I \subseteq R$ be a graded ideal. Then the limit

$$\varepsilon_I(x) := \lim_{s \to \infty} \frac{\dim_k \left(H^0_{\mathfrak{m}} \left(R/I^n \right) \right)_{\lfloor xn \rfloor}}{n^{d-1}}$$

exists for every $x \in \mathbb{R}_{\geq 0}$. Moreover, the function $\varepsilon_I : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is compactly supported, piecewise polynomial and

$$d! \int_0^\infty \varepsilon_I(x) dx = \lim_{s \to \infty} \frac{\ell_R \left(H^0_{\mathfrak{m}} \left(R/I^n \right) \right)}{n^d / d!} = \varepsilon(I)$$

is a rational number.

Fat points on projectively normal curves

Example (Rees)

Suppose that $R = \dfrac{\mathbb{C}[X,Y,Z]}{(X^3+Y^3+Z^3)}$ *with graded maximal ideal* $\mathfrak{m}$. *Then there exist height one graded prime ideal* P *in* R *such that the saturated Rees algebra* $\bigoplus_{n\geq 0}(P^n :_R \mathfrak{m}^\infty)$ *is non-Noetherian.*

Example (_, Dubey, Roy and Verma)

Suppose that $R = \dfrac{\mathbb{C}[X,Y,Z]}{(X^e+Y^e+Z^e)}$ *where* $e \geq 3$ *is an integer. Consider the graded ideal* $I = P_1^{a_1} \cap \cdots \cap P_r^{a_r}$ *where* $P_1, \ldots, P_r$ *are* r *distict height one graded prime ideals in* R *and* $a_1, \ldots, a_r$ *are positive integers. Then*

$$\varepsilon(I) = \left(\sum_{i=1}^{r} a_i\right)^2 \left(e - 2 + \frac{1}{e}\right).$$

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