Thm [BIM] R Noeth, $p \in J(R)$
Jacobson radical

R regular $\iff$ $\exists$ fflat map $R \longrightarrow A$ w/ A perfectoid

Easiest perfectoids:

- If $\operatorname{char} A = p$, A perfectoid $\iff$ A perfect
- If A p-torsion free, A perfectoid $\iff$ A p-adically complete + separated, $p = u \varpi^p$, u unit, $\varpi \in A$, and $A/\varpi \xrightarrow{F} A/p$ an isomorphism

e.g.

$\mathbb{Z}_p[p^{1/p^\infty}]^{\wedge p}$

$\mathbb{Z}_p[p^{1/p^\infty}, x_2^{1/p^\infty}, \dots, x_d^{1/p^\infty}]^{\wedge p}$

(exercise: compute $A^\flat$ for these)

p-typical Witt Vectors: R any ring

$$W(R) = \{ (a_0, a_1, \dots) \mid a_i \in R \}$$

w/ $+ , \cdot$ determined by univ polynomials so that

$$(X_0, X_1, X_2, \dots) \longmapsto X_0^{p^i} + pX_1^{p^{i-1}} + \dots + p^i X_i$$

always determine ring maps

$$\underline{X} + \underline{Y} = (s_0, s_1, \dots)$$

$$x_0 + y_0 = s_0$$

$$(x_0^p + px_1) + (y_0^p + py_1) = s_0^p + ps_1$$
$$= (x_0 + y_0)^p + ps_1$$

$$s_1 = x_1 + y_1 - \frac{(x_0+y_0)^p - x_0^p - y_0^p}{p}$$

$$\underline{x} \cdot \underline{y} = (t_0, t_1, \dots)$$

$$x_0 y_0 = t_0$$

$$(x_0^p + p x_1)\cdot(y_0^p + p y_1) = t_0^p + p t_1$$

$$= x_0^p y_0^p$$

$$t_1 = x_0^p y_1 + y_0^p x_1 + p x_1 y_1$$

$$\begin{aligned} \pi : W(R) &\twoheadrightarrow R \\ (a_0, \dots) &\longmapsto a_0 \end{aligned}$$ surj ring hom

$(W(\_), \pi)$ functorial in R

(and similarly for truncations $W_i(\_)$)

$\Rightarrow$ get lift of Frobenius if char $R = p$

$$\begin{tikzcd} (a_0, \dots) \arrow[r, mapsto] & (a_0^p, a_1^p, \dots) \end{tikzcd}$$

$$\begin{tikzcd} F : W(R) \arrow[r] \arrow[d] & W(R) \arrow[d] \\ F : R \arrow[r] & R \end{tikzcd}$$

(isom if R perfect)

R perfect $\implies$ $W(R)$ unique

p-adically complete

+ separated

+ p torsion free

w/ $W(R) \xrightarrow{\pi} R$

$\ker \pi = (p)$

<u>shift</u>: (Verschiebung)

$V : W(R) \longrightarrow W(R)$

$(a_0, a_1, \dots) \longmapsto (0, a_0, a_1, \dots)$

additive map w/ $F \circ V = V \circ F = p$ (mult by)

<u>Teichmuller Map</u>:

$[\_] : R \cong R^\flat = W(R)^\flat \xrightarrow{\#} W(R)$

$r \longmapsto [r] = (r, 0, 0, \dots)$

$\pi([r]) = r$ (so $[\_]$ inj)

$$[r][s] = [rs] \qquad ([\_] \text{ multiplicative})$$

$$[r]\cdot(s_0, s_1, \dots) = (rs_0, r^p s_1, r^{p^2} s_2, \dots)$$

$$(r_0, r_1, \dots) = \sum_{n=0}^{\infty} V^n([r_n])$$

$$= \sum_{n=0}^{\infty} V^n F^n([r_n^{1/p^n}])$$

$$= \sum_{n=0}^{\infty} [r_n^{1/p^n}]\, p^n \qquad \left(\begin{array}{c}\text{Teichmueller} \\ \text{coords}\end{array}\right)$$

note: $(r_0, r_1, \dots)$ unit in $W(R)$ $\iff$ r_0 unit in R

<u>Def</u>: $d = (d_0, d_1, \dots) \in W(R)$ distinguished if

R is d_0-adically complete

& d_1 is a unit of R, equiv

$$d = [d_0] + pu \quad , \quad \underset{\text{unit}}{u \in W(R)}$$

A is perfectoid if

$$A \cong \frac{W(R)}{(d)} \quad \text{w/} \quad d \in W(R) \text{ distinguished}$$

in A, $\quad p = u^{-1}[d_0]$

$\quad = (\text{unit}) \cdot \varpi^p$

w/ $\varpi = [d_0^{1/p}]$

note: $\varpi^{1/p^n} = [d_0^{1/p^{n+1}}]$ compat system of p-th power roots

<u>Remarks</u>:

① $$A/p \cong \frac{W(R)}{(p, [d_0])} = R/(d_0) \leftarrow \text{Frob surj}$$

$$A^\flat = (A/p)^\flat \cong (R/(d_0))^\flat \cong R^\flat \cong R$$

analog of # assumption, uses d_0-complete

(2) presentation $A \cong \dfrac{W(A^\flat)}{d}$

determined by Fontaine's θ-map

w/ $\{a_i^{1/p^n}\} \subseteq A$

$[a_0] + [a_1]p + \cdots \longmapsto \sum a_n^{1/p^n} p^n$

get

$$\begin{tikzcd} W(A^\flat) \arrow[r, "\theta"] \arrow[d] & A \arrow[d] \\ A^\flat \arrow[r, "pr_0"'] & A/p \end{tikzcd}$$

Fontaine's map

<u>Equiv def</u>: A ring A is perfectoid if

(1) p-complete + sep

(2) $A/p \xrightarrow{F} A/p$ surjective

(3) kernel of $\theta : W(A^\flat) \longrightarrow A$ is principal

(4) $\exists\ \pi \in A$, $u \in A$ unit, $p = \pi^p u$

Rmk: ① Perfectoid $\Longrightarrow$ reduced

② perfectoid $\Longleftrightarrow$ perfect in char p

($d = p$ in this case)

③ A p-tors free $\Longrightarrow$ A perfectoid

iff A p-complete and scp

$$A/\varphi \xrightarrow{\text{Frob}} A/\varphi \quad \text{isom}$$

④ If A perfectoid, $\exists$ ses

$$0 \longrightarrow A \longrightarrow \underbrace{A/\sqrt{pA}}_{\substack{\text{perfect} \\ \text{char } p>0}} \oplus \underbrace{\overline{A}}_{A/A[p^\infty]} \longrightarrow \overline{A}/\sqrt{pA}\cdot\overline{A} \longrightarrow 0$$

maxptf quotient

Examples:

① $\mathbb{Z}_p[p^{1/p^\infty}]^{\wedge p}$, $\mathbb{Z}_p[p^{1/p^\infty}, x_2^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]^{\wedge p}$

② k perfect,

$W(k)[[x_2, \ldots, x_d]][p^{1/p^\infty}, x_2^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]^{\wedge p}$

③ (R, m, k) complete local domain
char $k = p$
perfect

$R^{+^{\wedge p}}$ = perfectoid

(check kernel of Frob gen by $p^{1/p}$)

④ A perfectoid, $f_1^{1/p^\infty}, \ldots, f_n^{1/p^\infty} \in A$

then $A/(f_1^{1/p^\infty}, \ldots, f_n^{1/p^\infty})^{\wedge p}$ perfectoid

[BIM]

<u>Proof sketch in mixed char: ($\Rightarrow$)</u>

Assume $(R, \underset{p}{\underset{\cup}{m}}, k)$ reg local, $k = \bar{k}$

$$R \longrightarrow \hat{R} = \begin{cases} k[[x_1, \dots, x_d]] & ① \\ W(k)[[x_2, \dots, x_d]] & ② \\ \dfrac{W(k)[[x_1, \dots, x_d]]}{(p-f)} & ③ \\ f \in (x_1, \dots, x_d)^2 & \end{cases}$$

all 3 cases, construct

$$\hat{R} \longrightarrow \underset{\text{perfectoid}}{A} \quad \text{fflat}$$

e.g. for ③

$$A = \frac{W(k)[[x_1, \dots, x_d]][x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]^{\wedge p}}{(p-f)}$$

For ($\Leftarrow$) $R \longrightarrow \underset{\text{perfectoid}}{A}$ fflat

to mimic earlier argument, only

need $\mathrm{fd}_A\, A/\sqrt{mA} < \infty$.

Apply Lemma below w/

$p, f_2, \ldots, f_d$ a sop for R. ✓

<u>Lemma:</u> A perfectoid

$I = \sqrt{(p, f_2, \ldots, f_n)}$

$\Rightarrow \mathrm{fd}_A\, A/I \leq n$

<u>Pf:</u> $\sqrt{(p)} = (\varpi^{1/p^\infty})$ flat

$\bar{A} = A/\sqrt{(p)} \Rightarrow \mathrm{fd}_A\, \bar{A} \leq 1$

$\bar{A}$ perfect $\Rightarrow A/I = \bar{A}/\bar{I} = \bar{A}/(f_2^{1/p^\infty}, \dots, f_n^{1/p^\infty})$

$\Longrightarrow \operatorname{fd}_{\bar{A}} A/I \le n-1$

$\Longrightarrow \operatorname{fd}_A A/I \le n$ by spectral sequence

$E^2_{p,q} =$

$\operatorname{Tor}_p^{\bar{A}}(\operatorname{Tor}_q^A(-, \bar{A}), \bar{A}/\bar{I}) \Rightarrow \operatorname{Tor}_{p+q}^A(-, A/I)$ ✓