The Hilbert Functions of Artinian Local Complete Intersections

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Hilbert functions

- Let $R = K[\![x_1, \ldots, x_r]\!]$ or $R = K[x_1, \ldots, x_r]$ with degree $x_i = 1$ where K is a field.
- For an ideal (homogeneous) I in R, let $A = R/I$.
- Graded: A is graded if I is a homogeneous ideal in $R = K[x_1, \ldots, x_r]$
- R_i (resp. I_i) be the set of homogeneous polynomial of degree i in (resp. in I)
- Hilbert function: $\mathrm{HF}_A : \mathbb{N} \to \mathbb{N}$

$$i \mapsto \dim_K R_i - \dim_K I_i$$

- $\mathcal{M} = (x_1, \ldots, x_r)$ and $\mathfrak{m} = \mathcal{M}/I$
- If A is local, then $\mathrm{HF}_A : \mathbb{N} \to \mathbb{N}$ is the Hilbert function of the associated graded ring of A, $gr_{\mathfrak{m}}(A) := \oplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$.

Macaulay's theorem

[Macaulay, 1927]: characterized the numerical functions $h : \mathbb{N} \to \mathbb{N}$ that can occur as the Hilbert function of standard graded K-algebras

O-sequence: A sequence of positive integers $h = (h_i : i \geq 0)$ that satisfies Macaulay's condition, that is $h_0 = 1$ and $h_{i+1} \leq h_i^{\langle i \rangle}$ for $i = 1, \ldots, s-1$,

Question: Which O-sequences can occur as the Hilbert function of a graded or local K-algebra with extra properties, for example, domain, reduced, complete intersections etc.

In this talk we are interested in Gorenstein and complete intersection (CI) property

The graded case

Suppose that $A = R/I$ is an Artinian K-algebra and $s := \max\{i : \mathfrak{m}^i \neq 0\}$ is the socle degree of A.

- CI sequence: there exists a CI ring with the Hilbert function h
- Gorenstein sequence: there exists a Gorenstein K-algebra with the Hilbert function h
- [Macaulay] the O-sequence $h = (1, 2, h_2, \ldots, h_{s-1}, h_s = 1)$ is a complete intersection (equivalently, Gorenstein) sequence if and only if

$$|h_i - h_{i-1}| \leq 1 \text{ for all } i = 1, \ldots, s. \tag{1}$$

- [Buchsbaum-Eisenbud, Stanley] Let $h = (1, 3, \ldots, 1)$ be an O-sequence. .Then there exists a graded Gorenstein K-algebra with the Hilbert function h if and only if h is symmetric and $(1, 2, h_2 - h_1, \ldots, h_{\lceil s/2 \rceil)} - h_{\lceil s/2 \rceil) - 1})$ is an O-sequence
- Hilbert functions of graded complete intersection is easy to compute in any codimension

The local case

Which O-sequences can occur as the Hilbert function of Artinian **local** complete intersection K-algebra, more generally Gorenstein K-algebras?

- The problem is open even in codimension three
- The problem is that the Hilbert function of local ring is the Hilbert function of $gr_{\mathfrak{m}}(A)$ which need not have good properties even if A has
- The first open case is when $A = R/I$ is an Artinian ring where $R = K[\![x, y, z]\!]$ and $I \subseteq (x, y, z)^2$ generated by a regular sequence (f, g, p) where f, g, p have nonzero and linearly independent quadratic parts
- Thus $h_1 = h_2 = 3$.
- Therefore h is of the form $(1, 3, 3, \dots, 1)$
- For an O-sequence h, we set $\max h := \max\{h_i : i \geq 0\}$ and

$$\Delta(h) := \max\{|h_i - h_{i-1}| : i = 3, \dots, s\}.$$

Main theorem

Theorem

Let h be a $(1, 3, 3, \ldots, 1)$ be an O-sequence. Then the following statements are equivalent:

(i) *h is a CI sequence;*

(ii) *h is a Gorenstein sequence;*

(iii) *h satisfies one of the following conditions:*
(I) $h_3 \leq 3$;
(II) $h_3 = 4$ and $\Delta(h) = 1$;
(III) $h_3 = 4$, $\Delta(h) = 2$ and h has a unique fall by two at the peak position. In this case, h is of the following form:

$$h_i = \begin{cases} i+1 & \text{for } i \leq d-2 \\ d & \text{for } i = d-1, d, \ldots, d+r-1 \\ d-2 & \text{for } i = d+r \\ h_{i-1} \text{ or } h_{i-1}-1 & \text{for } i \geq d+r+1 \end{cases}$$

for some integers $d = \max h \geq 4$ and $r \geq 0$.

Examples

Example

(1) Consider the O-sequence $h = (1, 3, 3, 4, 5, 4, 4, 2, 1)$. Recently, Iarrobino and Marques proved that h is not a Gorenstein sequence. We show that h is not a Gorenstein sequence using Theorem 1.

(2) The sequence $(1, 3, 3, 4, 3, 3, 3, 1)$ is not Gorenstein

(3) Consider the O-sequence $h = (1, 3, 3, 3, 2, 2, 1, 1)$. Then $I = (yz - x^6, xz - y^5, xy - z^3)$ is a CI ideal with the HF h.

(3) Consider the O-sequence $h = (1, 3, 3, 4, 2, 1)$. Then $I = (xz, yz + x^3, z^2 + y^3)$ is a CI ideal with the Hilbert function $h = (1, 3, 3, 4, 2, 1)$.

Sketch of Proof

- By Macaulay if h is an O-sequence of the form $(1, 3, 3, \ldots, 1)$, then $h_3 \leq 4$.
- If $h_3 \leq 3$, then we explicitly obtain $F \in K_{DP}[X, Y, Z]$ such that $R/\operatorname{Ann}_R(F)$ has the HF h.
- If $h_3 = 4$, then by Macaulay h is of the form $h = (1, 3, 3, 4, 5, \ldots, d, d, d, \ldots, h_t = d, h_{t+1} < d, \ldots, 1)$ where $d = \max h$ and t is the peak position of h
- We reduce to the codimension two case to construct a CI ideal with the HF h.

Thank you!