Bounds for the reduction number of primary ideals in dimension three

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Set up

Suppose $(R, \mathfrak{m})$ is a Noetherian local ring of dimension $d \geqslant 1$ and I is an $\mathfrak{m}$-primary ideal.

- A sequence of ideals $\mathcal{I} = \{I_n\}_{n \in \mathbb{Z}}$ is called an *I-admissible filtration* if
  1. $I_{n+1} \subseteq I_n$,
  2. $I_m I_n \subseteq I_{m+n}$ and
  3. $I^n \subseteq I_n \subseteq I^{n-k}$ for some $k \in \mathbb{N}$.
- A *reduction* of $\mathcal{I}$ is an ideal $J \subseteq I_1$ such that $JI_n = I_{n+1}$ for $n \gg 0$.
  It is called *minimal reduction* if it is minimal with respect to containment among all reductions.
- Minimal reduction of $\mathcal{I}$ exist and is generated by d elements if $R/\mathfrak{m}$ is infinite.
- For a minimal reduction J of $\mathcal{I}$, we define *reduction number of $\mathcal{I}$ with respect to J*

$$r_J(\mathcal{I}) = \sup\{n \in \mathbb{Z} \mid I_n \neq JI_{n-1}\}$$

Set up

- Let $\mathcal{I} = \{I_n\}$ be an I-admissible filtration. Then the Hilbert-Samuel function of $\mathcal{I}$ is defined as

$$H_{\mathcal{I}}(n) = \lambda(R/I_n).$$

- There is a polynomial $P_{\mathcal{I}}(x) \in \mathbb{Q}[x]$ of degree d so that $H_{\mathcal{I}}(n) = P_{\mathcal{I}}(n)$ for all large n and
- there exist integers $e_0(\mathcal{I}), e_1(\mathcal{I}), \ldots, e_d(\mathcal{I})$ such that

$$P_{\mathcal{I}}(x) = e_0(\mathcal{I})\binom{x+d-1}{d} - e_1(\mathcal{I})\binom{x+d-2}{d-1} + \cdots + (-1)^d e_d(\mathcal{I}).$$

- The coefficients $e_i(\mathcal{I})$ are called the Hilbert coefficients of $\mathcal{I}$.

Bounding reduction number

- Consider $\mathcal{I} = \{I^n\}$.
- If R is a one dimensional Cohen-Macaulay local ring then $r(I) \leqslant e_0(I) - 1$.

$$r(I) = \min\{r_J(I) : J \text{ is a minimal reduction of } I.\}$$

- Theorem (Vasconcelos) In a Cohen-Macaulay local ring of dimension $d \geqslant 1$,

$$r(I) \leqslant \frac{d.e_0(I)}{o(I)} - 2d + 1 \qquad (1)$$

where $o(I)$ is the largest positive integer n such that $I \subseteq \mathfrak{m}^n$.
- A non-Cohen-Macaulay version of the above result exists.

Bounding reduction number

- Theorem (Rossi, 1999) Let R be a Cohen-Macaulay local ring of dimension at most two. Then for a minimal reduction $J \subseteq I$

$$r_J(I) \leqslant e_1(I) - e_0(I) + \lambda(R/I) + 1. \tag{2}$$

- When $d \geqslant 3$, it is a conjecture.
- Difficulties
  1. Reduction number of I does not behave well with respect to superficial elements. We have $r_{JR'}(IR') \leqslant r_J(I)$ where $R' = R/(x)$.

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- When $d \geqslant 3$, it is a conjecture.
- Difficulties
  1. Reduction number of I does not behave well with respect to superficial elements. We have $r_{JR'}(IR') \leqslant r_J(I)$ where $R' = R/(x)$.
  2. Ratliff-Rush filtration of I does not behave well with respect to superficial elements i.e. $\widetilde{I^n}R' \neq \widetilde{I^nR'}$ for $n \geqslant 1$.

Known Cases

- Remark Let $(R, \mathfrak{m})$ be a three dimensional Cohen-Macaulay local ring, I an $\mathfrak{m}$ primary ideal and $J \subseteq I$ a minimal reduction of I. Then

$$r_J(I) \leqslant e_1(I) - e_0(I) + \lambda(R/I) + 1$$

if one of the following conditions hold:

1. depth $G(I) \geqslant 1$. (Rossi)
2. depth $G(\mathcal{F}) \geqslant 2$, where $\mathcal{F} = \{\widetilde{I^n}\}$.
3. $e_2(I) = e_3(I) = 0$.
4. $e_2(I) = 0$ and I is asymptotically normal .
5. $e_2(I) = 0$ and $G(I)$ is generalized Cohen-Macaulay.

How to extend to dimension 3?

If $\operatorname{depth} G(I) > 0$ then $r_{J/(x)}(I/(x)) = r_J(I)$. The following examples show that $r_{J/(x)}(I/(x)) = r_J(I)$ may hold even if $\operatorname{depth} G(I) = 0$.

Example 1

(Rossi-Valla) Let $R = Q[[x, y]]$ and $I = (x^4, x^3y, xy^3, y^4)$. Then $x^2y^2 \in I^2 : I \subseteq \widetilde{I}$ but $x^2y^2 \notin I$ which implies $\operatorname{depth} G(I) = 0$.

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How to extend to dimension 3?

- Lemma (Mandal, _)) Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geqslant 1$ and $\operatorname{depth} R > 0$. Let I be an $\mathfrak{m}$ primary ideal and $J \subseteq I$ a minimal reduction of I. If $r_{J/(x)}(I/(x)) < r_J(I)$ for a superficial element $x \in I$, then $\widetilde{I^n} \neq I^n$ for all $r_{J/(x)}(I/(x)) \leqslant n < r_J(I)$.

- Consider
$$\rho(I) = \min\{i \geqslant 1 | \widetilde{I^n} = I^n \text{ for all } n \geqslant i\}.$$

- Theorem (Mandal, _) Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d = 3$ and I an $\mathfrak{m}$ primary ideal. For a minimal reduction J of I, if $\rho(I) \leqslant r_J(I) - 1$, then $r_J(I) \leqslant e_1(I) - e_0(I) + \lambda(R/I) + 1$.

Bounds in dimension three

- Lemma (Mandal, _) Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geqslant 2$ and I an $\mathfrak{m}$-primary ideal with depth $G(I^t) > 0$ for some $t \geqslant 1$. Let $x \in I$ be a superficial element for I and $J \subseteq I$ be a minimal reduction of I. Then

$$r_J(I) \leqslant r_{J/(x)}(I/(x)) + t - 1.$$

- Theorem (Mandal,_) Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d = 3$ and I an $\mathfrak{m}$-primary ideal with depth $G(I^t) > 0$ for some $t \geqslant 1$. Let $J \subseteq I$ be a minimal reduction of I. Then

$$r_J(I) \leqslant e_1(I) - e_0(I) + \lambda(R/I) + t.$$

Furthermore, if $r_J(I) \equiv k \mod t$, $1 \leqslant k \leqslant t-1$, then

$$r_J(I) \leqslant e_1(I) - e_0(I) + \lambda(R/I) + k.$$

- Corollary Suppose depth $G(I^2) > 0$ and $r_J(I)$ is odd. Then

$$r_J(I) \leqslant e_1(I) - e_0(I) + \lambda(R/I) + 1.$$

In this case, $r_J(I) = r_{J/(x)}(I/(x))$.

Example

Example 2

Let $R = k[[x, y, z, u, v, w, t]]/(t^2, tu, tv, tw, uv, uw, vw, u^3 - xt, v^3 - yt, w^3 - zt)$. Then R is a Cohen-Macaulay local ring of dimension 3 and $\operatorname{depth} G(\mathfrak{m}) = 0$. We have $e_0(\mathfrak{m}) = 8$, $e_1(\mathfrak{m}) = 11$, $e_2(\mathfrak{m}) = 4$ and $e_3(\mathfrak{m}) = 0$. We have $\mathfrak{m}^2 \neq \widetilde{\mathfrak{m}^2}$ and $\mathfrak{m}^j = \widetilde{\mathfrak{m}^j}$ for $j \geqslant 3$. Therefore $\operatorname{depth} G(\mathfrak{m}^3) \geqslant 1$. Now $J = (x, y, z)$ is a minimal reduction of $\mathfrak{m}$ and $r_J(\mathfrak{m}) = 3 \leqslant e_1(\mathfrak{m}) - e_0(\mathfrak{m}) + \lambda(R/\mathfrak{m}) + 3 = 7$. Note that the bound $\frac{de_0(\mathfrak{m})}{o(\mathfrak{m})} - 2d + 1 = 3.8 - 6 + 1 = 19$ given by Vasconcelos is larger than our bound.

Bounds in dimension three

- Theorem (Mandal,_) Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geqslant 3$ and I an $\mathfrak{m}$ primary ideal with $\operatorname{depth} G(I) \geqslant d-3$. Then

$$r_J(I) \leqslant e_1(I) - e_0(I) + \lambda(R/I) + 1 + (e_2(I) - 1)e_2(I) - e_3(I). \tag{3}$$

- Corollary Let $(R, \mathfrak{m})$ be a three dimensional Cohen-Macaulay local ring and I an $\mathfrak{m}$-primary ideal. Then the following statements hold.

Bounds in dimension three

- Theorem (Mandal,_) Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geqslant 3$ and I an $\mathfrak{m}$ primary ideal with $\operatorname{depth} G(I) \geqslant d-3$. Then

$$r_J(I) \leqslant e_1(I) - e_0(I) + \lambda(R/I) + 1 + (e_2(I) - 1)e_2(I) - e_3(I). \tag{3}$$

- Corollary Let $(R,\mathfrak{m})$ be a three dimensional Cohen-Macaulay local ring and I an $\mathfrak{m}$-primary ideal. Then the following statements hold.
  1. If $e_2(I) = 0$ or 1 then $r(I) \leqslant e_1(I) - e_0(I) + \lambda(R/I) + 1 - e_3(I)$.
  2. If $e_2(I) = 1$ and I is asymptotically normal then $r(I) \leqslant e_1(I) - e_0(I) + \lambda(R/I) + 1$.
  3. If $e_2(I) = 2$ then $r(I) \leqslant e_1(I) - e_0(I) + \lambda(R/I) + 2 - e_3(I)$.

- $e_2(I)(e_2(I) - 1) - e_3(I) \geqslant 0$ for integrally closed I.

Bounds in dimension three

- In particular, when $e_2(I) = 0$ and $e_3(I) = 0$., then Rossi's bound holds.
- (Tony P., 2017) In dimension three, $e_2(I) = 0 = e_3(I)$ implies that the Ratliff-Rush filtration of I behaves well modulo a superficial element.
- Suppose I is integrally closed. Then $e_2(I) = 0$ implies $G(I)$ is Cohen-Macaulay.
- Theorem (_,Yadav) Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geqslant 3$ and I an $\mathfrak{m}$-primary ideal. Suppose the Ratliff-Rush filtration of I behaves well modulo a superficial sequence $x_1, \ldots, x_{d-2} \in I$. Then
$$e_3(I) \geqslant e_2(I) - e_1(I) + e_0(I) - \ell(R/I).$$
- Marley proved the above inequality assuming $\operatorname{depth} G(I) \geqslant d - 1$.

Bounds in dimension three

Example 3

(Corso-Polini-Rossi) Let $R = k[[X, Y, Z, U, V, W]]/(Z^2, ZU, ZV, UV, YZ - U^3, XZ - V^3)$ be a three dimensional Cohen-Macaulay local ring. Let x, y, z, u, v, w denote the corresponding images of X, Y, Z, U, V, W in R and $m = (x, y, z, u, v, w)$. Then $G(\mathfrak{m})$ has depth 1 and

$$H(\mathfrak{m}, t) = \frac{1 + 3t + 3t^3 - t^4}{(1-t)^3}$$

which gives $e_2(\mathfrak{m}) = 3, e_1(\mathfrak{m}) = 8, e_0(\mathfrak{m}) = 6, \ell(R/\mathfrak{m}) = 1, e_3(\mathfrak{m}) = -1$. Thus $e_2(\mathfrak{m}) - e_1(\mathfrak{m}) + e_0(\mathfrak{m}) - \ell(R/\mathfrak{m}) = 0$ and $e_3(\mathfrak{m}) = -1$. Therefore the Ratliff-Rush filtration of $\mathfrak{m}$ does not behave well modulo superficial element.

Bounds in dimension three

- Theorem (_,Yadav) Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geqslant 3$, I an integrally closed $\mathfrak{m}$-primary ideal and J a minimal reduction of I. Then

$$e_3(I) \leqslant \frac{(r_J(I) - 1)}{2}\big(e_2(I) - e_1(I) + e_0(I) - \ell(R/I)\big); \tag{4}$$

$$e_3(I) \leqslant \frac{\big(e_1(I) - e_0(I) + \ell(R/I)\big)}{2}\big(e_2(I) - e_1(I) + e_0(I) - \ell(R/I)\big) \text{ and} \tag{5}$$

$$e_3(I) \leqslant \frac{\big(e_2(I) - 1\big)}{2}\big(e_2(I) - e_1(I) + e_0(I) - \ell(R/I)\big). \tag{6}$$

Further, suppose $d = 3$ and equality holds in any one of (4), (5) or (6). Then the Ratliff-Rush filtration of I behaves well modulo a superficial element. Conversely, suppose the Ratliff-Rush filtration of I behaves well modulo a superficial sequence $x_1, \ldots, x_{d-2} \in I$. Then (i) equality holds in (4) provided $r_J(I) \leqslant 3$; (ii) equality holds in (5) provided $e_1(I) - e_0(I) + \ell(R/I) \leqslant 2$ and (iii) equality holds in (6) provided $e_2(I) \leqslant 3$.

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- Theorem (_,Yadav) Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geqslant 3$, I an integrally closed $\mathfrak{m}$-primary ideal and $J \subseteq I$ a minimal reduction of I. Suppose $\operatorname{depth} G(I) \geqslant d-3$. Then

$$r_J(I) \leqslant e_1(I) - e_0(I) + \ell(R/I) + 1 + e_2(I)(e_2(I) - e_1(I) + e_0(I) - \ell(R/I)) - e_3(I).$$

- Rossi's bound remains mystery.

Extension in Non Cohen-Macaulay case

- Theorem (Mandal,_) Let $(R, \mathfrak{m})$ be a one dimensional Buchsbaum local ring and I an $\mathfrak{m}$-primary ideal. Let J be a minimal reduction of I then

$$r_J(I) \leqslant e_1(I) - e_1(J) - e_0(I) + \lambda(R/I) + 2.$$

- Theorem (Mandal,_) Let $(R, \mathfrak{m})$ be a two dimensional Buchsbaum local ring and I an $\mathfrak{m}$-primary ideal. Let J be a minimal reduction of I and $\operatorname{depth} G(I^t) > 0$ for some $t \geqslant 1$ then

$$r_J(I) \leqslant e_1(I) - e_1(J) - e_0(I) + \lambda(R/I) + t + 1$$

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Thank You