A density function for a Noetherian filtration of homogeneous ideals

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Abstract

We investigate the behaviour of the Hilbert functions of bigraded algebras over a field, generated by elements of bidegrees $(1,0),(d_1,e_1),\dots,(d_s,e_s)$, where d_i's (resp. e_i's) are non-negative (resp. positive) integers for $i=1,\dots,s$. It is well known that for a standard bigraded algebra R, the Hilbert function $H_R(m,n)$ is represented by a polynomial for all $m,n \gg 0$. However, N. D. Hoang and N. V. Trung showed that if all e_i's are one then there exist integers m_0,n_0 such that $H_R(m,n)$ is equal to a polynomial in m,n for $m \geq dn+n_0$ and $n \geq n_0$, where $d=\max\{d_1,\dots,d_s\}$. We concentrate on how $H_R(m,n)$ behaves in the complementary region. Thereby we define a density function for a Noetherian filtration of homogeneous ideals in a standard graded algebra. Our main ingredient is the structure theorem for vector partition functions due to B. Sturmfels.

Setup

- $A=\bigoplus_{n\geq 0} A_n = k[h_1,\dots,h_r]$: a standard graded algebra over a field k.
- $R=\bigoplus_{(m,n)\in\mathbb{N}^2} R_{m,n} = A[g_1,\dots,g_s]$: a bigraded k-algebra with $\deg h_i=(1,0)$ and $\deg g_j=(d_j,e_j)$ for $i=1,\dots,r$ and $j=1,\dots,s$.
- There is an A-algebra homomorphism

$$\varphi : S := k[X_1,\dots,X_r,Y_1,\dots,Y_s] \longrightarrow R$$

such that $\varphi(X_i)=h_i$ and $\varphi(Y_j)=g_j$. Consider S as bigraded with $\deg X_i=(1,0)$ and $\deg Y_j=(d_j,e_j)$ for all i,j. So there is a finite bigraded minimal +free resolution of R:

$$0 \to \bigoplus_{j=1}^{\eta_t} S(-a_{tj},-b_{tj})^{\beta_{tj}} \to \cdots \to \bigoplus_{j=1}^{\eta_1} S(-a_{1j},-b_{1j})^{\beta_{1j}} \to S \to R \to 0.$$

- The bigraded Hilbert series of S is

$$H_S(x,y) = \sum_{(m,n)\in\mathbb{N}^2} \dim_k S_{m,n}\, x^m y^n = \frac{1}{(1-x)^r(1-x^{d_1}y^{e_1})\cdots(1-x^{d_s}y^{e_s})}.$$

Thus $H_R(x,y)=P(x,y)\cdot H_S(x,y)$, where $P(x,y)=\sum_{i=0}^{t}(-1)^i\big(\sum_{j=1}^{\eta_i}\beta_{ij}x^{a_{ij}}y^{b_{ij}}\big)\in\mathbb{Z}[x,y]$. Hence

$$\dim_k R_{m,n} = \sum_{i=0}^{t}(-1)^i\Bigg(\sum_{j=1}^{\eta_i}\beta_{ij}\dim_k S_{m-a_{ij},n-b_{ij}}\Bigg).$$

Note. For each pair $(m,n)\in\mathbb{N}^2$,

$$\dim_k S_{m,n} = \phi_M(m,n) = \#\left\{(\lambda_1,\dots,\lambda_{r+s})\in\mathbb{N}^{r+s} \;\middle|\; \overbrace{\begin{bmatrix}1 & \cdots & 1 & d_1 & \cdots & d_s\\ 0 & \cdots & 0 & e_1 & \cdots & e_s\end{bmatrix}}^{M}\cdot\begin{bmatrix}\lambda_1\\ \vdots\\ \lambda_{r+s}\end{bmatrix}=\begin{bmatrix}m\\ n\end{bmatrix}\right\},$$

where the function $\phi_M\colon \mathbb{N}^2\to\mathbb{N}$ is called the vector partition function.

Specific examples of R

- An $\mathbb{N}$-graded filtration $\mathcal{F}=\{I_n\}_{n\geq 0}$ of ideals in A is a collection of ideals which satisfies the conditions:

(1) $I_0=A$, (2) $I_{n+1}\subseteq I_n\ \forall n$ and (3) $I_nI_m\subseteq I_{n+m}\ \forall n,m\geq 0$.

Then $R:=\mathcal{R}(\mathcal{F})=\oplus_{(m,n)\in\mathbb{N}^2}(I_n)_m t^n$ is a bigraded k-algebra if all I_n's are homogeneous.

- We say $\mathcal{F}$ is a Noetherian filtration if $\mathcal{R}(\mathcal{F})$ is Noetherian.

Examples. (i) the I-adic filtration $\{I^n\}_{n\geq 0}$,
(ii) the integral closure filtration $\{\overline{I^n}\}_{n\geq 0}$,
(iii) the tight closure filtration $\{(I^n)^*\}_{n\geq 0}$,
(iv) $\{I^n :_R J^\infty\}_{n\geq 0}$, where $I,J\subseteq K[X_1,\dots,X_r]$ are monomial ideals.

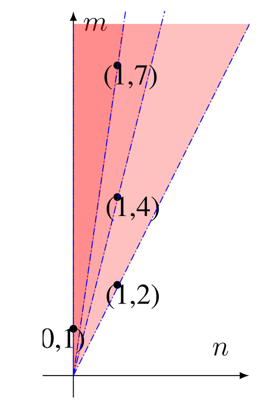
Vector partition functions

$M=[\mathbf{v_1}\cdots\mathbf{v_{n_1}}]$: an $m_1\times n_1$-matrix with columns $\mathbf{v_i}\in\mathbb{N}^{m_1}$ with $m_1\leq n_1$.

- $\operatorname{pos}(M) := \left\{\sum_{i=1}^{n_1}\lambda_i\mathbf{v_i}\in\mathbb{R}^{m_1} \mid \lambda_1,\dots,\lambda_{n_1}\in\mathbb{R}_{\geq 0}\right\}$.
- For $\sigma\subset[n_1]$, define $M_\sigma=[\mathbf{v_i}\mid i\in\sigma]$.
- σ is a *basis* if $\#\sigma=\operatorname{rank}(M_\sigma)=m_1$.
- The chamber complex is the polyhedral subdivision of the $\operatorname{pos}(M)$ which is defined as the common refinement of cones $\operatorname{pos}(M_\sigma)$ where σ's are bases.
- For a chamber $\mathfrak{C}$, let $\Delta(\mathfrak{C})=\{\sigma\subset[n_1]\mid \mathfrak{C}\subseteq\operatorname{pos}(M_\sigma)\}$.
- For $\sigma\in\Delta(\mathfrak{C})$, set $G_\sigma:=\mathbb{Z}^{m_1}/M_\sigma\mathbb{Z}$. We say σ is *non-trivial* if $G_\sigma\neq\{0\}$.
- Denote the image of $\mathbf{u}\in\mathbb{Z}^{m_1}$ in G_σ by $[\mathbf{u}]_\sigma$.

Example. Let $M=\begin{bmatrix}1&1&2&4&7\\0&0&1&1&1\end{bmatrix}$.

- $\operatorname{pos}(M)=\operatorname{pos}\left(\underbrace{\begin{bmatrix}2&4\\1&1\end{bmatrix}}_{M_{\{3,4\}}}\right)\cup\operatorname{pos}\left(\begin{bmatrix}4&7\\1&1\end{bmatrix}\right)\cup\operatorname{pos}\left(\begin{bmatrix}0&7\\1&1\end{bmatrix}\right)$.
- $\operatorname{pos}\left(\begin{bmatrix}2&4\\1&1\end{bmatrix}\right)\subset\operatorname{pos}\left(\begin{bmatrix}2&7\\1&1\end{bmatrix}\right)\implies\{3,5\}\in\Delta(\operatorname{pos}(M_{\{3,4\}}))$.
- $G_{\{3,4\}}=\frac{\mathbb{Z}^2}{\mathbb{Z}(2,1)+\mathbb{Z}(4,1)}$. Notice $G_{\{1,5\}}=0$.



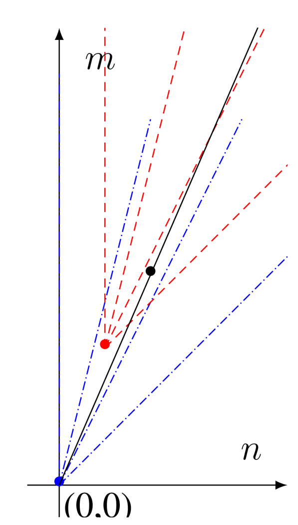
Theorem (Strumfels, 1995). For each chamber $\mathfrak{C}$, $\exists$ a polynomial $P_\mathfrak{C}\in\mathbb{Q}[X_1,\dots,X_{m_1}]$ of of degree n_1-m_1 and for each non-trivial $\sigma\in\Delta(\mathfrak{C})$, $\exists$ a polynomial $Q_\sigma^\mathfrak{C}\in\mathbb{Q}[X_1,\dots,X_{m_1}]$ of degree $\#\sigma-m_1$ and a function $\Omega_\sigma\colon G_\sigma\setminus\{0\}\to\mathbb{Q}$ such that

$$\phi_M(\mathbf{u}) = P_\mathfrak{C}(\mathbf{u}) + \sum_{\sigma\in\Delta(\mathfrak{C}),\,[\mathbf{u}]_\sigma\neq 0}\Omega_\sigma([\mathbf{u}]_\sigma)\cdot Q_\sigma^\mathfrak{C}(\mathbf{u}) \quad \text{for all } \mathbf{u}\in\mathfrak{C}\cap\mathbb{Z}^{m_1}.$$

An observation

- Take M as in the Setup. Without loss of generality we assume that $\frac{d_1}{e_1}<\cdots<\frac{d_s}{e_s}$.
- Set

$$\mathfrak{C}_s=\begin{cases}\operatorname{pos}\left(\begin{bmatrix}d_j & d_{j+1}\\ e_j & e_{j+1}\end{bmatrix}\right) & \text{if } 1\leq j\leq s-1\\[1ex] \operatorname{pos}\left(\begin{bmatrix}1 & d_s\\ 0 & e_s\end{bmatrix}\right) & \text{if } j=s\end{cases}.$$



- Then $\operatorname{pos}(M)=\cup_{j=1}^{s}\mathfrak{C}_j$.

➢ Fix j_0. We define the restricted chamber of $\mathfrak{C}_{j_0}$ as

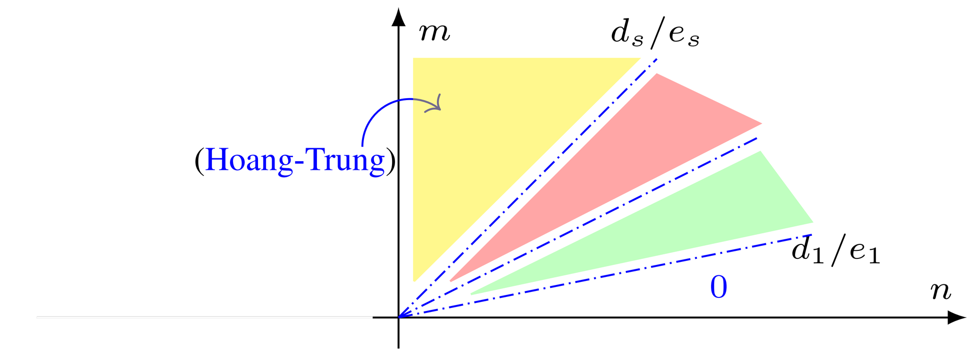
$$\mathfrak{R}C_{j_0}=\bigcap(\mathfrak{C}_{j_0}+(a_{ij},b_{ij})).$$

Lemma. Fix $x\in(\frac{d_{j_0}}{e_{j_0}},\frac{d_{j_0+1}}{e_{j_0+1}})$, where we define $\frac{d_{j+1}}{e_{j+1}}=\infty$ if the vector $(d_{j+1},e_{j+1})=(1,0)$. For a given $(\alpha,\beta)\in\mathbb{R}^2$, $\exists$ an integer n_1 such that $(\lfloor xn\rfloor+\alpha,n+\beta)\in\mathfrak{R}C_{j_0}\ \forall n\geq n_1$.

Main results

Theorem (Das, - and Trivedi, 2023).
Suppose that $\operatorname{depth} A\geq 1$ and $\dim A=d$. Let $\mathcal{F}=\{I_n\}_{n\geq 0}$ be a Noetherian filtration of homogeneous ideals in A. Consider the induced bigraded structure on $\mathcal{R}(\mathcal{F})$. Let $\{\mathfrak{C}_j\}_{1\leq j\leq s}$ be the corresponding cones in $\mathbb{R}^2$. Then for every $\mathfrak{C}_{j_0}$, there exist a polynomial $P_{j_0}(X,Y)\in\mathbb{Q}[X,Y]$ of total degree $r_0\leq d-1$ and a quasi polynomial $Q_{j_0}[X,Y]$ of total degree $<r_0$ such that

$$\dim_k(I_n)_m = P_{j_0}(m,n)+Q_{j_0}(m,n) \quad \text{for every } (m,n)\in\mathfrak{R}C_{j_0}\cap\mathbb{N}^2.$$



Remark. Let $I\subseteq A$ be a homogeneous ideal minimally generated in degrees $d_1<\cdots<d_s$. Then by Hoang-Trung (2003), $\exists$ integers m_0,n_0 such that for $m\geq d_sn+m_0$ and $n\geq n_0$, the Hilbert function $H_{\mathcal{R}(I)}(m,n)=\dim_k(I^n)_m$ is equal to a polynomial $P_{\mathcal{R}(I)}(m,n)$ of total degree $d-1$. Observe that our result generalizes this statement when $\mathcal{F}$ is the I-adic filtration.

Theorem (Das, - and Trivedi, 2023).
Let $\mathcal{F}=\{I_n\}_{n\geq 0}$ be a Noetherian filtration of homogneous ideals in A. Then the function $f_{A,\mathcal{F}}:[0,\infty)\setminus\{\frac{d_j}{e_j}\}_{1\leq j\leq s}\to\mathbb{R}_{\geq 0}$ given by

$$x\mapsto\lim_{n\to\infty}\frac{\dim_k(I_n)_{\lfloor xn\rfloor}}{n^{d-1}}$$

is a well-defined continuous function. Moreover,

$$f_{A,\mathcal{F}}(x)=\begin{cases}P_j(x) & \text{for } x\in\left(\frac{d_j}{e_j},\frac{d_{j+1}}{e_{j+1}}\right) \text{ with } j\leq s-1,\\ P_s(x) & \text{for } x\in\left(\frac{d_s}{e_s},\infty\right),\end{cases}$$

where $P_j(x)$ is a polynomial of degree $\leq d-1\ \forall j$ and $\deg P_s=d-1$.

➢ We further show that if all generators with their degrees lying on slopes $<\frac{d_i}{e_i}$ are nilpotents and there is a non-nilpotent generator of degree (d_i,e_i) then P_j is a zero polynomial if $j<i$ and a non-zero polynomial otherwise.

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