A density function associated with a Noetherian filtration of homogeneous ideals

Sudeshna Roy

Tata Institute of Fundamental Research, Mumbai, India

Joint work with Suprajo Das (IIT Bombay) and Vijaylaxmi Trivedi (TIFR, Mumbai)

School on Commutative Algebra and Algebraic Geometry in Prime Characteristics

ICTP, Italy

May 3, 2023

Setup

$A = \bigoplus_{n\geq 0} A_n = k[h_1, \ldots, h_r]$: a standard graded over a field k.

$R = \bigoplus_{(m,n)\in\mathbb{N}^2} R_{m,n} = A[g_1, \ldots, g_s]$ is a bigraded k-algebra with $\deg h_i = (1, 0)$ and $\deg g_j = (d_j, e_j)$.

Then there is an A-algebra homomorphism

$$S := k[X_1, \ldots, X_r, Y_1, \ldots, Y_s] \xrightarrow{\varphi} R$$

such that $X_i \mapsto h_i$ and $Y_j \mapsto g_j$ for $i = 0, \ldots, r$ and $j = 1, \ldots, s$.

So there exists a bigraded minimal free resolution of R:

$$0 \to \bigoplus_{j=1}^{\eta_t} S(-a_{tj}, -b_{tj})^{\beta_{tj}} \to \cdots \to \bigoplus_{j=1}^{\eta_1} S(-a_{1j}, -b_{1j})^{\beta_{1j}} \to S \to R \to 0.$$

The bigraded Hilbert series of S is

$$H_S(x,y) = \sum_{(m,n)\in\mathbb{N}^2} \ell(S_{m,n})x^m y^n = \frac{1}{(1-x)^r(1-x^{d_1}y^{e_1})\cdots(1-x^{d_s}y^{e_s})}.$$

So $H_R(x,y) = P(x,y)\cdot H_S(x,y)$, where $P(x,y) = \sum_{i=0}^{t}(-1)^i\big(\sum_{j=1}^{\eta_i}\beta_{ij}x^{a_{ij}}y^{b_{ij}}\big)$. Hence $\ell(R_{m,n}) = \sum_{i=0}^{t}(-1)^i\big(\sum_{j=1}^{\eta_i}\beta_{ij}\ell(S_{m-a_{ij},n-b_{ij}})\big)$.

- Note $\ell(S_{m,n})$ is

$$\phi_M(m,n) = \#\left\{(\lambda_1,\dots,\lambda_{r+s})\in\mathbb{N}^{r+s} \;\middle|\; \overbrace{\begin{bmatrix}1 & \cdots & 1 & d_1 & \cdots & d_s\\ 0 & \cdots & 0 & e_1 & \cdots & e_s\end{bmatrix}}^{M}\cdot\begin{bmatrix}\lambda_1\\ \vdots\\ \lambda_{r+s}\end{bmatrix} = \begin{bmatrix}m\\ n\end{bmatrix}\right\},$$

where the function $\phi_M\colon \mathbb{N}^2\to\mathbb{N}$ is called the *vector partition function*.

Specific examples of R

- An $\mathbb{N}$-graded filtration $\mathcal{F} = \{I_n\}_{n\geq 0}$ of ideals in A is a collection of ideals which satisfies the conditions:

  (1) $I_0 = A$, (2) $I_{n+1} \subseteq I_n \ \forall n$ and (3) $I_n I_m \subseteq I_{n+m} \ \forall n, m \geq 0$.

Then $R := \mathcal{R}(\mathcal{F}) = \oplus_{(m,n)\in\mathbb{N}^2}(I_n)_m t^n$ is a bigraded k-algebra if I_n'are homog.

- We say $\mathcal{F}$ is a *Noetherian* filtration if $\mathcal{R}(\mathcal{F})$ is Noetherian.

Examples.

(i) the I-adic filtration $\{I^n\}_{n\geq 0}$,

(ii) the integral closure filtration $\{\overline{I^n}\}_{n\geq 0}$,

(iii) the tight closure filtration $\{(I^n)^*\}_{n\geq 0}$,

(iv) $\{I^n :_R J^\infty\}_{n\geq 0}$, where $I, J \subseteq K[X_1, \ldots, X_r]$ are monomial ideals.

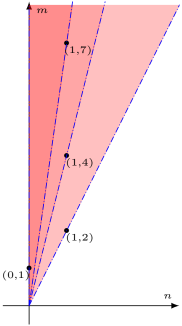
Vector partition functions

$M = \left[\mathbf{v_1} \cdots \mathbf{v_{n_1}}\right]$, an $m_1 \times n_1$-matrix with columns $\mathbf{v_i} \in \mathbb{N}^{m_1}$ with $m_1 \leq n_1$.

- $\mathrm{pos}(M) := \left\{ \sum_{i=1}^{n_1} \lambda_i \mathbf{v_i} \in \mathbb{R}^{m_1} \mid \lambda_1, \ldots, \lambda_{n_1} \in \mathbb{R}_{\geq 0} \right\}$.
- For $\sigma \subset [n_1]$, define $M_\sigma = [\mathbf{v_i} \mid i \in \sigma]$.
- σ is a *basis* if $\#\sigma = \mathrm{rank}(M_\sigma) = m_1$.
- The *chamber complex* is the polyhedral subdivision of the $\mathrm{pos}(M)$ which is defined as the common refinement of cones $\mathrm{pos}(M_\sigma)$ where σ's are bases.
- For a chamber $\mathfrak{C}$, let $\Delta(\mathfrak{C}) = \{\sigma \subset [n_1] \mid \mathfrak{C} \subseteq \mathrm{pos}(M_\sigma)\}$.
- For $\sigma \in \Delta(\mathfrak{C})$, set $G_\sigma := \mathbb{Z}^{m_1}/M_\sigma\mathbb{Z}$. We say σ is *non-trivial* if $G_\sigma \neq \{0\}$.
- Denote the image of $\mathbf{u} \in \mathbb{Z}^{m_1}$ in G_σ by $[\mathbf{u}]_\sigma$.

Let $M = \begin{bmatrix} 1 & 1 & 2 & 4 & 7 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$.

- $\operatorname{pos}(M) = \operatorname{pos}\Bigg(\underbrace{\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}}_{M_{\{3,4\}}}\Bigg) \cup \operatorname{pos}\left(\begin{bmatrix} 4 & 7 \\ 1 & 1 \end{bmatrix}\right) \cup \operatorname{pos}\left(\begin{bmatrix} 0 & 7 \\ 1 & 1 \end{bmatrix}\right)$
- $\operatorname{pos}\left(\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}\right) \subset \operatorname{pos}\left(\begin{bmatrix} 2 & 7 \\ 1 & 1 \end{bmatrix}\right) \implies \{3,5\} \in \Delta(\operatorname{pos}(M_{\{3,4\}}))$
- $G_{\{3,4\}} = \frac{\mathbb{Z}^2}{\mathbb{Z}(2,1)+\mathbb{Z}(4,1)}$. Notice $G_{\{1,5\}} = 0$.



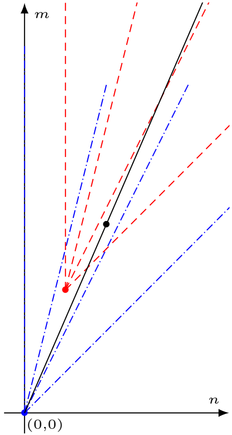
Theorem (Strumfels, 1995). For each chamber $\mathfrak{C}$, $\exists$ a polynomial $P_{\mathfrak{C}}$ of degree $n_1 - m_1$ and for each non-trivial $\sigma \in \Delta(\mathfrak{C})$, $\exists$ a polynomial $Q_\sigma^{\mathfrak{C}}$ of degree $\#\sigma - m_1$ and a function $\Omega_\sigma : G_\sigma \setminus \{0\} \to \mathbb{Q}$ s. t.

$$\phi_M(\mathbf{u}) = P_{\mathfrak{C}}(\mathbf{u}) + \sum_{\sigma \in \Delta(\mathfrak{C}),\ [\mathbf{u}]_\sigma \neq 0} \Omega_\sigma\left([\mathbf{u}]_\sigma\right) \cdot Q_\sigma^{\mathfrak{C}}(\mathbf{u}) \quad \text{for all } \mathbf{u} \in \mathfrak{C} \cap \mathbb{Z}^{m_1}.$$

- Let $M = \begin{bmatrix} 1 & \cdots & 1 & d_1 & \cdots & d_s \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{bmatrix}$.
- Set

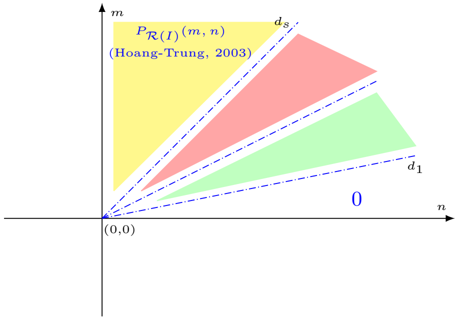
$$\mathfrak{C}_s = \begin{cases} \text{pos}\left(\begin{bmatrix} d_j & d_{j+1} \\ 0 & 1 \end{bmatrix}\right) \text{ if } 1 \leq j \leq s-1 \\ \text{pos}\left(\begin{bmatrix} 1 & d_s \\ 0 & 1 \end{bmatrix}\right) \text{ if } j = s \end{cases}.$$

- Then $\text{pos}(M) = \cup_{j=1}^{s} \mathfrak{C}_j$.
- We define the restricted chamber of $\mathfrak{C}_j$ as $\mathfrak{R}C_{j_0} = \bigcap(\mathfrak{C}_{j_0} + (\alpha, \beta))$.



Theorem (Das, - and Trivedi, 2023).
Suppose depth $A \geq 1$ and dim $A = d$. Let $I \subseteq A$ be a homog ideal minimally generated in degrees $d_1 < \cdots < d_s$. Let $\{\mathfrak{C}_j\}_{1 \leq j \leq s}$ be the corresponding cones in $\mathbb{R}^2$. Then for every $\mathfrak{C}_{j_0}$, $\exists$ a polynomial $P_{j_0}(X, Y)$ of total degree $r_0 \leq d-1$ and a quasi polynomial $Q_{j_0}[X, Y]$ of total degree $< r_0$ such that

$$\ell(I_n)_m = P_{j_0}(m, n) + Q_{j_0}(m, n) \quad \text{for every } (m, n) \in \Re C_{j_0} \cap \mathbb{N}^2.$$



Main Theorem

Theorem (Das, - and Trivedi, 2023).
The function $f_{A,I} : [0, \infty) \setminus \{d_j\}_{1 \leq j \leq s} \to \mathbb{R}_{\geq 0}$ given by

$$x \mapsto \lim_{n \to \infty} \frac{\ell(I^n)_{\lfloor xn \rfloor}}{n^{d-1}}$$

is a well-defined continuous function. Moreover,

$$f_{A,I}(x) = \begin{cases} P_j(x) & \text{for } \; x \in (d_j, d_{j+1}) \; \text{ with } \; j \leq s-1, \\ P_s(x) & \text{for } \; x \in (d_s, \infty), \end{cases}$$

where $P_j(x)$ is a polynomial of degree $\leq d-1$ $\forall j$ and $\deg P_s = d-1$.

References

- N. D. Hoang and N. V. Trung, *Hilbert polynomials of non-standard bigraded algebras*, Mathematische Zeitschrift, Vol. 245 (2003), 309–334.
- B. Sturmfels, *On Vector Partition Functions*, Journal of Combinatorial Theory, Vol 72 (1995), No. 2, 302–309.
- V. Trivedi, *Hilbert-Kunz density function and Hilbert-Kunz multiplicity*, Transactions of the American Mathematical Society (2018), Vol. 370, No. 12, 8403–8428.