Tight Hilbert Polynomial and F-rational local rings

joint work with

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May 4, 2023

I-admissible filtrations

- Let $(R, \mathfrak{m})$ be a Noetherian local ring and I be an $\mathfrak{m}$-primary ideal.

- A sequence of ideals $\mathcal{F} = \{I_n\}_{n\in\mathbb{Z}}$ is called an I**-filtration** if $\forall\, m, n \in \mathbb{Z}$,
(1) $I_n = R; n \leq 0$, (2) $I_n \supseteq I_{n+1}$, (3) $I_m I_n \subseteq I_{m+n}$ and (4) $I^n \subseteq I_n$.

- The I-filtration $\mathcal{F}$ is called I**-admissible** if there exists $r \in \mathbb{N}$ such that $I_n \subseteq I^{n-r}$ for all $n \in \mathbb{Z}$.

- The **integral closure** $\overline{I}$ of I is the set of all $x \in R$ such that $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ where $a_i \in I^i$ for $1 \leq i \leq n$.

- Theorem **(Rees)**: A Noetherian local ring $(R, \mathfrak{m})$ is analytically unramified if and only if the normal filtration, i.e. $\{\overline{I^n}\}_{n\in\mathbb{Z}}$ of any $\mathfrak{m}$-primary ideal I is I-admissible.

Hilbert-Samuel polynomial

- Let $\mathcal{F} = \{I_n\}_{n \in \mathbb{Z}}$ be an I-admissible filtration and $d = \dim R$.

- The function $H_{\mathcal{F}}(n) := \ell(R/I_n)$ is called the **Hilbert-Samuel function** of $\mathcal{F}$.

- $H_{\mathcal{F}}(n)$ is a polynomial function, i.e. there exists a polynomial $P_{\mathcal{F}}(x) \in \mathbb{Q}[x]$ of degree d, called the **Hilbert-Samuel polynomial** of $\mathcal{F}$, such that $H_{\mathcal{F}}(n) = P_{\mathcal{F}}(n)$, for large n, where

$$P_{\mathcal{F}}(n) = e_0(\mathcal{F})\binom{n+d-1}{d} - e_1(\mathcal{F})\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(\mathcal{F}).$$

- The coefficients $e_i(\mathcal{F}), 0 \le i \le d$ are called as **Hilbert coefficients** and $e_0(\mathcal{F})$ is called the **multiplicity** of $\mathcal{F}$.

Tight closure of ideals

- For a Noetherian ring R of prime characteristic p and an ideal I in R, we use the following notations:
  - R° := complement of the union of the minimal primes of R,
  - $q := p^e$, with $e \in \mathbb{N} = \{0, 1, 2 \ldots\}$,
  - The e-th Frobenius power of I, $I^{[q]} := (r^q | r \in I)$.
- The **tight closure** of I is defined as
  $I^* = \{x \in R|$ there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all $q \gg 0\}$.
- I is said to be tightly closed if $I = I^*$.
- $I \subseteq I^* \subseteq \bar{I}$.
- Briançon-Skoda Theorem: Let R be a Noetherian ring of prime characteristic p and I be an ideal of R generated by n elements. Then

$$\overline{I^{n+r}} \subseteq (I^{r+1})^* \text{ for all } r \in \mathbb{N}.$$

- If I is principal, then $\bar{I} = I^*$.

Test elements and tight closure

- An element $c \in R^{\circ}$ such that for all I and for all $x \in I^*$, $cx^q \in I^{[q]}$ for all $q = p^e$ where, $e \in \mathbb{N}$ is called a **test element** for R.

- The ideal $\tau(R)$ generated by all the test elements is called the **test ideal** of R.

- The **parameter test ideal** of R, denoted by $\tau_{par}(R)$, is the ideal generated by $c \in R^{\circ}$ such that $cI^* \subset I$ for all parameter ideals I of R. An element of $\tau_{par}(R) \cap R^{\circ}$ is called a **parameter test element**.

- Theorem [1]: Let R be a reduced algebra of finite type over an excellent local ring of characteristic p. Let $c \in R^{\circ}$ be such that R_c is regular. Then some power of c is a test element for R.

[1] Hochster and Huneke, F-regularity, test elements, and smooth base change, Trans. Amer. Math. Soc., (1994).

The tight Hilbert polynomial

- The authors[2] introduced tight Hilbert polynomial in 2020.

- Let R be a d-dimensional analytically unramified local ring with prime characteristic p, I be an $\mathfrak{m}$-primary ideal and $\mathcal{T} = \{(I^n)^*\}_{n\in\mathbb{Z}}$. Then
  - $\mathcal{T}$ is an I-admissible filtration.
  - The **tight Hilbert polynomial** of I is given by

$$P_I^*(n) = e_0^*(I)\binom{n+d-1}{d} - e_1^*(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d^*(I)$$

  where $e_j^*(I) \in \mathbb{Z}$.

  - $e_0^*(I) = e(I)$, the multiplicity of I and the coefficients $e_j^*(I)$ for $j = 0, 1, \ldots, d$ are called the **tight Hilbert coefficients of** I.

[2] Goel, Mukundan, and Verma, Tight closure of powers of ideals and tight Hilbert polynomials, Math. Proc. Cambridge Philos. Soc., (2020).

Detecting properties of rings via Hilbert coefficients

- Theorem [3]: Let R be a Cohen-Macaulay local ring and I be an $\mathfrak{m}$-primary ideal. Then $e_1(I) = 0 \iff I$ is a complete intersection.

- Theorem[4]: A Noetherian ring $(R, \mathfrak{m})$ is regular if and only if R is unmixed and $e(\mathfrak{m}) = 1$.

- Vasconcelos Conjecture[5]: For any ideal Q generated by a system of parameters, $e_1(Q) < 0 \iff R$ is not Cohen-Macaulay.

- Theorem[6]: Let $(R, \mathfrak{m})$ be a local ring then $e_1(I) \leq 0$ for any parameter ideal I.

- Theorem[7]: A formally unmixed local ring is Cohen-Macaulay if and only if $e_1(Q) = 0$ for some parameter ideal Q.

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[3] Northcott, A note on the coefficients of the abstract Hilbert function, J. London Math. Soc., (1960).
[4] M. Nagata, Local Rings, Interscience Publishers, (1961).
[5] Vasconcelos, The Chern coefficients of local rings, Michigan Math. J., (2008).
[6] Mandal, Singh, Verma, Some conjectures about the Chern numbers of filtrations, J. Algebra, (2011).
[7] Ghezzi, Goto, Hong, Ozeki, Phuong, and Vasconcelos, Cohen-Macaulayness versus the vanishing of the first Hilbert coefficient of parameter ideals, J. Lond. Math. Soc., (2010).

F-rational rings

- If $x_1, x_2, \dots, x_h$ are elements of a Noetherian ring R and $\text{ht}(x_1, x_2, \dots, x_h) = h$ then we say that $x_1, \dots, x_h$ are **parameters** and the ideal $(x_1, x_2, \dots, x_h)$ is called a **parameter ideal.**

- A ring R is called **F-rational**[8] if the parameter ideals are tightly closed.

- Examples **(Bruns, Herzog)**: Let k be a field of characetristic p and $S = k[X, Y, Z]$.
- Let $R = S/(X^2 - Y^3 - Z^7)$ and $I = (y, z)$.
  Since I is an ideal generated by system of parameters and $x \in I^*$, we get $I^* \neq I$.
  Thus R is not F-rational.
- Let $R = S/(X^2 - Y^3 - Z^5)$ and $I = (y, z)$.
  Then $x \notin I^*$ if and only if $p > 7$.
  Hence R is F-rational if and only if $p > 7$.

- Theorem **(Goel, Mukundan, Verma)**: Let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring with prime characteristic p. Then R is F-rational $\Leftrightarrow$ $e_1^*(I) = 0$, for some I generated by system of parameters.

[8] Fedder, Watanabe, A characterization of F-regularity in terms of F-purity, Math. Sci. Res. Inst. Publ. (1989).

Vanishing of $e_1^*(Q)$ and F-rational local rings

- Question **(Huneke)**: Let $(R, \mathfrak{m})$ be analytically unramified unmixed local Noetherian ring and Q be an ideal generated by a system of parameters. Is it true that $e_1^*(I) = 0 \iff R$ is F-rational?

  - Let $\dim R = 1$ and $I = (a)$ be $\mathfrak{m}$-primary.
  - R is Cohen-Macaulay and since $e_1^*(I) = 0$, R is F-rational.
  - Let $(b) \subseteq \mathfrak{m}$ be a minimal reduction of $\mathfrak{m}$. From B-S Theorem, it follows that $\overline{(b)} = (b)^*$.
  - As R is F-rational, $(b)^* = (b)$. Thus $(b) = \overline{(b)} = \mathfrak{m}$. Hence R is a regular local ring.

- $\dim R = 2$: Let k be a field of prime characteristic $p \geq 3$ and $R = k[[x^4, x^3y, xy^3, y^4]]$. Let Q be any $\mathfrak{m}$-primary parameter ideal of R then $e_1^*(Q) = 0$ but R is not F-rational.

- Definition: Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring of characteristic p. Then

$$0^*_{H^d_{\mathfrak{m}}(R)} = \{\eta \in H^d_{\mathfrak{m}}(R) : \exists\, c \in R^\circ \text{ such that } cF^e(\eta) = 0 \text{ for all } e \gg 0\}.$$

Computing the tight Hilbert coefficients

Theorem: Let $(R, \mathfrak{m})$ be an excellent reduced equidimensional local ring of prime characteristic p and dimension $d \geq 2$. Let $x_1, x_2, \ldots, x_d$ be parameter test elements and $Q = (x_1, x_2, \ldots, x_d)$. Then

- $$e_1^*(Q) = e(Q) - \ell(R/Q^*) + e_1(Q),$$
  $$e_j^*(Q) = e_j(Q) + e_{j-1}(Q) \text{ for all } 2 \leq j \leq d,$$

- $$e_1^*(Q) = \sum_{i=2}^{d-1} \binom{d-2}{i-2} \ell(H_{\mathfrak{m}}^i(R)) + \ell(0^*_{H_{\mathfrak{m}}^d(R)}),$$

- For $i = 2, \ldots, d$,
  $$e_i^*(Q) = (-1)^{i-1} \left[\sum_{j=0}^{d-i} \binom{d-i-1}{j-2} \ell(H_{\mathfrak{m}}^j(R)) + \ell(H_{\mathfrak{m}}^{d-i+1}(R)) \right].$$

On equality $e_1^*(Q) = e_1(Q)$ and F-rationality

- Theorem **(Morales, Trung and Villamayor)**: Let $(R, \mathfrak{m})$ be an analytically unramified excellent local domain and I be an $\mathfrak{m}$-primary parameter ideal. If $\overline{e}_1(I) = e_1(I)$ then R is regular and $\overline{I^n} = I^n$ for all n.

- Theorem: Let $(R, \mathfrak{m})$ be an excellent reduced equidimensional local ring of prime characteristic p and dimension $d \geq 2$. Let $x_1, x_2, \ldots, x_d$ be parameter test elements and $Q = (x_1, x_2, \ldots, x_d)$. Then

$$R \text{ is F-rational } \Leftrightarrow e_1^*(Q) = e_1(Q) \Leftrightarrow e_1^*(Q) = 0 \text{ and depth } R \geq 2.$$

- Necessity of the depth condition: Let $S = \mathbb{F}_p[[X, Y, Z, W]]$ and $R = \frac{S}{I \cap J}$, where $I = (X, Y)$ and $J = (Z, W)$. Let $a = x + z$, $b = y + w$ and $Q = (a, b)$ then $e_1^*(Q) = 0$ but R is not F-rational.

References

- Winfried Bruns and Jurgen Herzog, *Cohen-Macaulay rings,* Cambridge University Press, Cambridge, 1993.
- Kriti Goel, Jugal K. Verma, and Vivek Mukundan, *Tight closure of powers of ideals and tight Hilbert polynomials,* Math. Proc. Cambridge Philos. Soc., 2020.
- Melvin Hochster and Craig Huneke, *Infinite integral extensions and big Cohen-Macaulay algebras,* Ann. of Math., 1992.
- Linquan Ma and Pham H. Quy, *A Buchsbaum theory for tight closure,* Trans. Amer. Math. Soc., 2021.
- David Rees, *A note on analytically unramified local rings,* J. London Math. Soc., 1961.
- Karen Smith, *Tight closure of parameter ideals,* Invent. Math., 1994.