(Non) linearity of regularity of Tor over complete intersections

(Joint work with Marc Chardin and Navid Nemati)

School on Commutative Algebra and Algebraic Geometry in Prime Characteristics

by

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1. Motivation of our research work.
2. $\operatorname{reg}\left(\operatorname{Ext}_A^{2i}(M,N)\right)$ and $\operatorname{reg}\left(\operatorname{Ext}_A^{2i+1}(M,N)\right)$ have linear behavior for $i \gg 0$
3. Similar results hold true for Tor when $\dim\left(\operatorname{Tor}_i^A(M,N)\right) \leqslant 1$ for $i \gg 0$
4. Two examples showing that the behavior of the regularity of Tor modules could be pretty hectic when the latter condition is not satisfied.
5. Finally, if time permits, then we will discuss the proofs.

Importance of Castelnuovo–Mumford regularity

- **Castelnuovo–Mumford regularity** is a kind of universal bound for important invariants of graded modules such as
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More precisely, for all $n > \mathrm{reg}(M)$, the Hilbert function value of M at n coincides with Hilbert polynomial value at n. It follows from another definition, where $Q = Q_0[x_1, \ldots, x_d]$ is a Noetherian standard $\mathbb{N}$-graded ring.

$$\mathrm{reg}(M) := \max\left\{\mathrm{end}\left(H^i_{Q_+}(M)\right) + i \;\middle|\; 0 \leqslant i \leqslant \dim(M)\right\}.$$

Some existing results in the literature related to our work

1. (Eisenbud-Huneke-Ulrich, 2006) If $\dim(\operatorname{Tor}_1^Q(M,N)) \leqslant 1$, then

$$\operatorname{reg}\left(\operatorname{Tor}_i^Q(M,N)\right) - i \leqslant \operatorname{reg}(M) + \operatorname{reg}(N) \quad \text{for every } 0 \leqslant i \leqslant d.$$

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2. (Chardin) If $\dim(\operatorname{Tor}_1^Q(M,N)) \leqslant 1$, then

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3. (Chardin - Divaani-Aazar, 2008) If $\dim(M \otimes_Q N) \leqslant 1$, then

$$\max_{0 \leqslant i \leqslant d} \left\{\operatorname{reg}\left(\operatorname{Ext}_Q^i(M,N)\right) + i\right\} = \operatorname{reg}(N) - \operatorname{indeg}(M),$$

where $\operatorname{indeg}(M) := \inf\{n \in \mathbb{Z} : M_n \neq 0\}$.

Theorem (Chardin)

Suppose S is a standard graded ring over a field, but S is not a polynomial ring. Let $d := \min\{\dim(M), \dim(N)\}$. If $\dim\left(\operatorname{Tor}_i^S(M,N)\right) \leqslant 1 \;\forall\, i \geqslant i_0$, then

$$\operatorname{reg}\left(\operatorname{Tor}_i^S(M,N)\right) \leqslant i + \operatorname{reg}(M) + \operatorname{reg}(N) + \left\lfloor \frac{i+d}{2} \right\rfloor (\operatorname{reg}(S) - 1) \;\forall\, i \geqslant i_0.$$

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Theorem (– , Puthenpurakal, 2019)

Set $A := Q/(\mathbf{f})$, where $Q = K[X_1, \ldots, X_d]$, and $\mathbf{f} = f_1, \ldots, f_c$ is a homogeneous Q-regular sequence. Then

1. $\operatorname{reg}\left(\operatorname{Ext}_A^i(M, I^n N)\right) \leqslant \rho_N(I) \cdot n - w \cdot \left\lfloor \frac{i}{2} \right\rfloor + e$ for all $i, n \geqslant 0$,
2. $\operatorname{reg}\left(\operatorname{Ext}_A^i(M, N/I^n N)\right) \leqslant \rho_N(I) \cdot n - w \cdot \left\lfloor \frac{i}{2} \right\rfloor + e'$ for all $i, n \geqslant 0$,

where $e, e' \in \mathbb{Z}$, $w := \min\{\deg(f_j) : 1 \leqslant j \leqslant c\}$, and
$\rho_N(I)$ is an invariant defined in terms of reduction ideals of I with respect to N.

The main questions we will study in this talk

Over graded complete intersection rings:

Question

For $\ell \in \{0,1\}$, do there exist $a_\ell, a'_\ell \in \mathbb{Z}_{>0}$ and $e_\ell, e'_\ell \in \mathbb{Z} \cup \{-\infty\}$ such that

1. (i) $\operatorname{reg}\left(\operatorname{Ext}_A^{2i+\ell}(M,N)\right) = -a_\ell \cdot i + e_\ell$ for all $i \gg 0$?
2. (ii) $\operatorname{reg}\left(\operatorname{Tor}^A_{2i+\ell}(M,N)\right) = a'_\ell \cdot i + e'_\ell$ for all $i \gg 0$?

Our main results.

We (jointly with Chardin and Nemati) proved that:

1. the answer to (i) is positive, even in a more general situation, while
2. the answer to (ii) is negative. We found examples for that.
3. However, if $\dim\left(\operatorname{Tor}_i^A(M,N)\right) \leqslant 1$ for all $i \gg 0$, the second question does have a positive answer.

□

The main positive results

Theorem (Chardin, – , Nemati, 2022)

Let Q be a standard graded Noetherian algebra, $A := Q/(\mathbf{f})$, where $\mathbf{f} := f_1, \ldots, f_c$ is a homogeneous Q-regular sequence. Let M and N be finitely generated graded A-modules such that $\operatorname{Ext}^i_Q(M,N) = 0$ for all $i \gg 0$. Then

1. for every $\ell \in \{0,1\}$, there exist $a_\ell \in \{\deg(f_j) : 1 \leqslant j \leqslant c\}$ and $e_\ell \in \mathbb{Z} \cup \{-\infty\}$ such that

$$\operatorname{reg}\left(\operatorname{Ext}^{2i+\ell}_A(M,N)\right) = -a_\ell \cdot i + e_\ell \quad \text{for all } i \gg 0.$$

2. if further Q is *local or the epimorphic image of a Gorenstein ring, M has finite projective dimension over Q and

$$\dim\left(\operatorname{Tor}^A_i(M,N)\right) \leqslant 1 \quad \text{for all } i \gg 0,$$

then, for every $\ell \in \{0,1\}$, there exist $a'_\ell \in \{\deg(f_j) : 1 \leqslant j \leqslant c\}$ and $e'_\ell \in \mathbb{Z} \cup \{-\infty\}$ such that

$$\operatorname{reg}\left(\operatorname{Tor}^A_{2i+\ell}(M,N)\right) = a'_\ell \cdot i + e'_\ell \quad \text{for all } i \gg 0.$$

Example 1 (the leading term of $\operatorname{reg}\left(\operatorname{Tor}_i^A(M,N)\right)$ can be arbitrarily large)

Example (Chardin, – , Nemati, 2022)

Let $Q := K[Y,Z,V,W]$ be a polynomial ring with usual grading over a field K, and $A := Q/(Y^2,Z^2)$. Write $A = K[y,z,v,w]$. Fix an integer $m \geqslant 1$. Set

$$M := \operatorname{Coker}\left(\begin{bmatrix} y & z & 0 & 0 \\ -v^m & -w^m & y & z \end{bmatrix} : \begin{matrix} A(-m)^2 \\ \bigoplus \\ A(-1)^2 \end{matrix} \longrightarrow \begin{matrix} A(-m+1) \\ \bigoplus \\ A \end{matrix} \right)$$

and $N := A/(y,z)$. Then, for every $i \geqslant 1$, we have

1. $\operatorname{indeg}\left(\operatorname{Ext}_A^i(M,N)\right) = -i-m+1$ and $\operatorname{reg}\left(\operatorname{Ext}_A^i(M,N)\right) = -i$.
2. $\operatorname{indeg}\left(\operatorname{Tor}_i^A(M,N)\right) = i$ and $\operatorname{reg}\left(\operatorname{Tor}_i^A(M,N)\right) = (m+1)i + (2m-2)$.

Remark

1. In this example, $\dim\left(\operatorname{Tor}_i^A(M,N)\right) = 2$ for all $i \gg 0$.
2. $\operatorname{reg}\left(\operatorname{Tor}_i^A(M,N)\right)$ is eventually linear, but the leading term depends on M.
3. It shows that the finiteness result for $\operatorname{Tor}_*^A(M,N)$ that we proved under the condition that $\operatorname{Tor}_{\ll 0}^A(M,N) \leqslant 1$ can fail if this hypothesis is removed.

Example (Chardin, – , Nemati, 2022)

Let $Q := K[X,Y,Z,U,V,W]$ be a standard graded polynomial ring over a field K **of characteristic** 2, and $A := Q/(X^2,Y^2,Z^2)$. Write $A = K[x,y,z,u,v,w]$. Set

$$M := \operatorname{Coker}\left(\begin{bmatrix} x & y & z & 0 & 0 & 0 \\ u & v & w & x & y & z \end{bmatrix} : A(-1)^6 \longrightarrow A^2\right) \quad \text{and} \quad N := A/(x,y,z).$$

Then, for every $n \geqslant 1$, we have

1. $\operatorname{indeg}\left(\operatorname{Ext}_A^n(M,N)\right) = \operatorname{reg}\left(\operatorname{Ext}_A^n(M,N)\right) = -n$.
2. $\operatorname{indeg}\left(\operatorname{Tor}_n^A(M,N)\right) = n$ and $\operatorname{reg}\left(\operatorname{Tor}_n^A(M,N)\right) = n + f(n)$, where

$$f(n) := \begin{cases} 2^{l+1} - 2 & \text{if } n = 2^l - 1 \\ 2^{l+1} - 1 & \text{if } 2^l \leqslant n \leqslant 2^{l+1} - 2 \end{cases} \quad \text{for all integers } l \geqslant 1.$$

Remark

1. In this example, the following sets are dense in $[2,3]$:

$$\{\operatorname{reg}(\operatorname{Tor}_{2n}^A(M,N))/2n : n \geqslant 1\} \quad \text{and} \quad \{\operatorname{reg}(\operatorname{Tor}_{2n+1}^A(M,N))/2n+1 : n \geqslant 1\}.$$

Sketch of the proof of the main result

Hypothesis

1. The ring Q is a standard graded Noetherian algebra.
2. $A = Q/(\mathbf{f})$, where $\mathbf{f} := f_1, \ldots, f_c$ is a homogeneous Q-regular sequence.
3. Set $w_j := \deg(f_j)$.
4. M and N are finitely generated graded A-modules such that $\operatorname{Ext}^i_Q(M,N) = 0$ for all $i \gg 0$.

Remark

We have studied the graded modules

1. $\operatorname{Ext}^\star_A(M,N) := \bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^i_A(M,N), \quad \operatorname{Tor}^A_\star(M,N) := \bigoplus_{i\in\mathbb{Z}} \operatorname{Tor}^A_{-i}(M,N)$,
2. $H^l_{A_+}(\operatorname{Tor}^A_\star(M,N))^\vee := \bigoplus_{i\in\mathbb{Z}} H^l_{A_+}(\operatorname{Tor}^A_i(M,N))^\vee$

over the graded ring $T := A[y_1, \ldots, y_c]$ with $\deg(y_j) = 2$ for $1 \leqslant j \leqslant c$.
These y_j are induced by the Eisenbud operators.

Finite generation of $\mathrm{Ext}_A^\star(M,N)$ and $H^l_{A_+}\left(\mathrm{Tor}_\star^A(M,N)\right)^\vee$

Theorem (Gulliksen)

The graded module $\mathrm{Ext}_A^\star(M,N)$ is finitely generated over $A[y_1,\ldots,y_c]$ provided $\mathrm{Ext}_Q^i(M,N)=0$ for all $i \gg 0$.

- Hence the bigraded module $\mathrm{Ext}_A^\star(M,N) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}_A^i(M,N)$ is also finitely generated over $T = K[x_1,\ldots,x_d,y_1,\ldots,y_c]$.

Theorem (Chardin, – , Nemati, 2022)

If $\dim\left(\mathrm{Tor}_i^A(M,N)\right) \leqslant 1$ for all $i \gg 0$, then

$$H^0_{A_+}\Big(\mathrm{Tor}_\star^A(M,N)\Big)^\vee := \bigoplus_{i\geqslant 0} H^0_{A_+}\Big(\mathrm{Tor}_i^A(M,N)\Big)^\vee \text{ and}$$

$$H^1_{A_+}\Big(\mathrm{Tor}_\star^A(M,N)\Big)^\vee := \bigoplus_{i\geqslant 0} H^1_{A_+}\Big(\mathrm{Tor}_i^A(M,N)\Big)^\vee$$

are finitely generated over $A[y_1,\ldots,y_c] = K[x_1,\ldots,x_d,y_1,\ldots,y_c]$.

- Hence the linearity of regularity of Ext and Tor follows from a theorem due to Bagheri-Chardin-Hà.

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Thank you!