On regularity of symbolic and ordinary powers of weighted oriented graphs

by

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joint work with **Ramakrishna Nanduri**

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Outline

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Definitions and notations

Definition (Symbolic power)

Let R be a Noetherian ring. The k-th symbolic power of $I \subset R$ is defined as

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Definition (Regularity)

Let R be a standard graded polynomial ring over a field and m be its maximal homogeneous ideal. Suppose M is a finitely generated graded R-module and suppose

$$0 \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{p,j}(M)} \cdots \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{\mathbf{0},j}(M)} \to M \to 0$$

is its minimal free resolution. Then the regularity of M is given by

$$\begin{aligned} \operatorname{reg} M &= \max\{j - i \mid \beta_{i,j}(M) \neq 0\}, \\ &= \max\{j + i \mid H^i_m(M)_j \neq 0\}. \end{aligned}$$

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Definition (Weighted oriented graph)

A *weighted oriented graph* is a graph $D = (V(D), E(D), w)$, where $V(D)$ is the vertex set of D, $E(D) = \{(x, y) \mid \text{ there is an edge from vertex } x \text{ to vertex } y\}$ is the *edge set* of D, and $w : V(D) \to \mathbb{N}$ is a map, called weight function.

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- $V^+ = \{x \in V(D) \mid w(x) > 1\}$.
- If V^+ are sinks then $I(D)$ has only minimal associated primes.

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1. Definitions and notations

2. Regularity comparison of symbolic and ordinary powers of weighted oriented graphs

Minh's Conjecture

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Let G be a simple graph. Then $\operatorname{reg}(I(G)^{(k)}) = \operatorname{reg}(I(G)^k)$ for all $k \geq 1$.

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Question

Let D be a weighted oriented graph. Is $\text{reg}(I(D)^{(k)}) = \text{reg}(I(D)^k)$ for all $k \geq 1$?

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Let D be a weighted oriented graph. Is $\operatorname{reg}(I(D)^{(k)}) = \operatorname{reg}(I(D)^k)$ for all $k \geq 1$?

- **Mandal and Pradhan (2021)** proved that if D is a weighted oriented odd cycle such that V^+ are sinks then $\operatorname{reg}(I(D)^{(k)}) \leq \operatorname{reg}(I(D)^k)$ for all $k \geq 1$.

Regularity comparison of symbolic and ordinary powers

Setting

Let D be a weighted oriented graph having at least one induced subgraph which is a directed line with edges $(y, x), (x, z) \in E(D)$ and $w(x) > 1$.

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Theorem (–, Nanduri)

Let D be a weighted oriented graph D as in the above setting. Then

$$\operatorname{reg}(I(D)^{(k)}) \leq \operatorname{reg}(I(D)^k) \text{ for all } k \geq 2.$$

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Corollary (–, Nanduri)

Let D be a weighted oriented graph whose underlying graph is bipartite. Then

$$\operatorname{reg}(I(D)^{(k)}) \leq \operatorname{reg}(I(D)^k) \text{ for all } k \geq 2.$$

Example

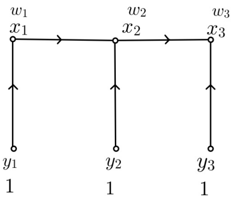

Figure 1: Cohen-Macaulay weighted oriented tree

Example

1. Let $w_1 = 6, w_2 = 4, w_3 = 7$. Then by Macaulay2 we have,

$$\begin{aligned} \operatorname{reg}(I(D)^{(2)}) = 23 &\quad < \quad \operatorname{reg}(I(D)^2) = 24 \\ \operatorname{reg}(I(D)^{(3)}) = 30 &\quad < \quad \operatorname{reg}(I(D)^3) = 32. \end{aligned}$$

2. Let $w_1 = 6, w_2 = 4, w_3 = 3$. Then by Macaulay2 we have,

$$\begin{aligned} \operatorname{reg}(I(D)^{(2)}) = 19 &\quad = \quad \operatorname{reg}(I(D)^2) = 19 \\ \operatorname{reg}(I(D)^{(3)}) = 26 &\quad = \quad \operatorname{reg}(I(D)^3) = 26. \end{aligned}$$

Regularity comparison of small symbolic and ordinary powers

Theorem (–, Nanduri)

Let D be any weighted oriented graph such that V^+ are sinks. Then

$$\operatorname{reg}(I(D)^{(k)}) \leq \operatorname{reg}(I(D)^k) \text{ for } k = 2, 3.$$

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Notation

- $\mathcal{F} :=$ the family of $\{x_i, x_j, x_r\}$ such that the induced subgraph on $\{x_i, x_j, x_r\}$ in G is a triangle.
- $N[H] := \bigcup_{x \in V(H)} N[x]$, for any subgraph H of D.

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Theorem (–, Nanduri)

Let D be a weighted oriented graph such that V^+ are sinks and underlying graph G has no induced triangles or the triangles are at most at a distance 2 from every vertex. Then

$$\operatorname{reg}(I(D)^2) \leq \max\left\{\operatorname{reg}(I(D)^{(2)}), \sum_{x \in N[T]} w(x) - |N[T]| + 1 + \sum_{x \in T} w(x) \mid T \in \mathcal{F}\right\}.$$

Sharp Upper Bound

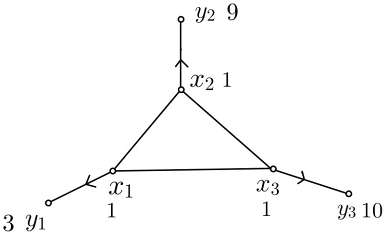

Figure 2: Cohen-Macaulay weighted oriented graph

Example

For above graph D, $I(D) = (x_1x_2, x_2x_3, x_3x_1, x_1y_1^3, x_2y_2^9, x_3y_3^{10})$. Using Macaulay2,

$\operatorname{reg}(I(D)^2) = 23 = \sum_{i=1}^{3} w(x_i) + \sum_{i=1}^{3} w(y_i) - 6 + 1 + \sum_{i=1}^{3} w(x_i)$. Note that

$\operatorname{reg}(I(D)^{(2)}) = 22 < \operatorname{reg}(I(D)^2)$.

Equality of regularity of second symbolic and ordinary powers

Corollary, (–, Nanduri)

Let D be a weighted oriented graph such that V^+ are sinks and underlying graph G has no induced triangles or the triangles are at most at a distance 2 from every vertex. Suppose for all $T \in \mathcal{F}$, $|N[T] \cap V^+| \leq 1$. Then $\text{reg}(I(D)^{(2)}) = \text{reg}(I(D)^2)$.

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Corollary (–, Nanduri)

Let D be a weighted oriented gap-free graph such that V^+ are sinks. Suppose for all $T \in \mathcal{F}$, $|N[T] \cap V^+| \leq 1$. Then $\text{reg}(I(D)^{(2)}) = \text{reg}(I(D)^2)$.

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Thank you

Degree Complex

For $\mathbf{a} = (a_1, \ldots, a_n) \in^n$, set $x^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ and $G_{\mathbf{a}} = \{i \in [n] \mid a_i < 0\}$, where $[n] = \{1, \ldots, n\}$. For every subset $F \subseteq [n]$, let $R_F = R[x_i^{-1} \mid i \in F]$.

Degree Complex

The degree complex of I with respect to $\mathbf{a}$ is defined as

$$\Delta_{\mathbf{a}}(I) = \{F \setminus G_{\mathbf{a}} \mid G_{\mathbf{a}} \subseteq F, x^{\mathbf{a}} \notin IR_F\}.$$

Link of F in Δ

Let Δ be a simplicial complex and F be a face in Δ. Then the Link of F in Δ is denoted by $lk_\Delta F$ and defined as

$$lk_\Delta F = \{G \in \Delta \mid F \cup G \in \Delta, F \cap G = \emptyset\}.$$

Lemma (Minh, Nam, Phong, Thuy and Vu (2022))

Let I be a monomial ideal in R. Then

$$\operatorname{reg}(R/I) = \max\{|\mathbf{a}| + i \mid \mathbf{a} \in \mathbb{N}^n, i \geq 0, \tilde{H}_{i-1}(lk_{\Delta_{\mathbf{a}}(I)} F; K) \neq 0 \text{ for some } F \in \Delta_{\mathbf{a}}(I) \text{ with } F \cap \operatorname{supp} \mathbf{a} = \emptyset\}.$$