Examples of perfectoids:

① $\mathbb{Z}_p[p^{1/p^\infty}]^{\wedge p}$, $\mathbb{Z}_p[p^{1/p^\infty}, x_2^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]^{\wedge p}$

② k perfect,

$W(k)[[x_2, \ldots, x_d]][p^{1/p^\infty}, x_2^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]^{\wedge p}$

③ (R, m, k) complete local domain

$\operatorname{char} k = p$

perfect

$R^{+^{\wedge p}}$ = perfectoid

(check kernel of Frob gen by $p^{1/p}$)

④ A perfectoid, $f_1^{1/p^\infty}, \ldots, f_n^{1/p^\infty} \in A$

then $A/(f_1^{1/p^\infty}, \ldots, f_n^{1/p^\infty})^{\wedge p}$ perfectoid

Thm [Bhatt - Scholze]

$$\underset{\text{perfectoid}}{R} \longrightarrow \underset{\text{p-complete}}{S}$$

either
① finitely presented
or
② integral
up to p-completion

$\exists !$ $S_{\mathrm{perfd}} :=$ "smallest" perfectoid / S

$$\begin{tikzcd} & S_{\mathrm{perfd}} \arrow[d, dotted, "\exists !"] \\ S \arrow[ur] \arrow[r] & A \end{tikzcd}$$

any perfd

Rmks:

- $R \twoheadrightarrow S \Rightarrow$ $R = R_{\mathrm{perfd}} \twoheadrightarrow S_{\mathrm{perfd}}$
- indep of R
- S_{perfd} p-tf if S is [MSTWW]
- more generally, without ① or ②, [BS] formulate $S_{\mathrm{perfd}} \in D^{\geq 0}(S)$ in the derived setting

[BIM]

Proof sketch in mixed char: ($\Rightarrow$)

Assume $(R, \underset{p}{\overset{\vee}{\mathfrak{m}}}, k)$ reg local, $k = \bar{k}$

$$R \longrightarrow \hat{R} = \begin{cases} k[\![x_1, \dots, x_d]\!] & \text{(1)} \\ W(k)[\![x_2, \dots, x_d]\!] & \text{(2)} \\ \dfrac{W(k)[\![x_1, \dots, x_d]\!]}{(p-f)} & \text{(3)} \\ f \in (x_1, \dots, x_d)^2 . & \end{cases}$$

all 3 cases, construct

$$\hat{R} \longrightarrow \underset{\text{perfectoid}}{A} \qquad \text{fflat}$$

e.g. for (3)

$$A = \frac{W(k)[\![x_1, \dots, x_d]\!][x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]^{\wedge p}}{(p-f)}$$

For ($\Leftarrow$) $R \longrightarrow \underset{\text{perfectoid}}{A}$ ffflat

to mimic earlier argument, only

need $\mathrm{fd}_A\, A/\sqrt{\mathfrak{m}A} < \infty$.

Apply Lemma below w/

$p, f_2, \ldots, f_d$ a sop for R. ✓

<u>Lemma:</u> A perfectoid

$I = \sqrt{(p, f_2, \ldots, f_n)}$

$\Longrightarrow \mathrm{fd}_A\, A/I \leq n$

<u>Pf:</u> $\sqrt{(p)} = (\varpi^{1/p^\infty})$ flat

$\bar{A} = A/\sqrt{(p)} \Longrightarrow \mathrm{fd}_A\, \bar{A} \leq 1$

$\bar{A}$ perfect $\Rightarrow A/I = \bar{A}/\bar{I} = \bar{A}/(f_2^{1/p^\infty}, \dots, f_n^{1/p^\infty})$

$\Rightarrow \mathrm{fd}_{\bar{A}}\, A/I \leq n-1$

$\Rightarrow \mathrm{fd}_A\, A/I \leq n$ by spectral sequence

$E^2_{p,q} =$

$\mathrm{Tor}^{\bar{A}}_p(\mathrm{Tor}^A_q(-, \bar{A}), \bar{A}/\bar{I}) \Rightarrow \mathrm{Tor}^A_{p+q}(-, A/I)$ ✓

<u>Thm</u>: [André] Every Noeth complete local domain (R, m, k) admits a (perfectoid) big CM ("BCM") alg B (i.e. $\forall$ s.o.p.s are a reg seq)

<u>Thms</u>: [HH, Bhatt] $p \in m$

$\Rightarrow R^{+\wedge p}$ perfectoid BCM

Theorem [HH, André, Gabber, Bhatt]

$(R, \mathfrak{m}, k) \longrightarrow (S, \mathfrak{n}, \ell)$

$\mathfrak{m} \ni p$ local map of complete local domains

$\Longrightarrow \exists$

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & S \arrow[d] \\ R^+ \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & S^+ \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

"BCM R^+-alg"

w/ B perfectoid R^+-alg BCM over R

C perfd BCM S^+-alg

and can even take

$B = \widehat{R^+}$, $C = \widehat{S^+}$.

<u>Note:</u> M (big) R-mod, $M \neq \mathfrak{m} M$

$$\begin{tikzcd} M \text{ BCM} \arrow[d, Rightarrow] \\ \text{some s.o.p. a reg seq on } M \arrow[d, Rightarrow] \\ H^i_{\mathfrak{m}} M = 0 \quad i < \dim R \arrow[uu, bend left, Rightarrow, "\text{if } M = \hat{M}"] \end{tikzcd}$$

Gabber, Bhatt,
Scholze – Cesnavicius
...

(Refined) André's Flatness Lemma:

A perfectoid, $g \in A$. Then $\exists$

$A \longrightarrow A'$ fflat mod p^n $\forall n \geq 1$
perfectoid

that g has a compat system g^{1/p^n} of pth roots in A'.

Sketch of André's Thm:

For simplicity, $k = \bar{k}$ and
$\operatorname{char} R = 0 \neq p = \operatorname{char} k$.

Write $R = S/Q$ w/ $S = W(k)[\![z_1, \dots, z_n]\!]$

$p \notin Q$

$c = \operatorname{ht} Q$

$\underset{\text{prime avoid}}{\Longrightarrow}$ $\exists$ $f_1, \dots, f_c \in Q$

$p, f_1, \dots, f_c$ reg seq in S

extend to $p, f_1, \dots, f_c, x_2, \dots, x_d$

full sop on S $(c+d = \dim S)$

$\Longrightarrow$ $p, x_2, \dots, x_d$ sop for

$S/(f_1, \dots, f_c)$ & S/Q

Pick $g \notin Q$ w/ $gQ \subseteq (f_1, \dots, f_c)$

$$S_\infty := S[p^{1/p^\infty}, z_1^{1/p^\infty}, \dots, z_n^{1/p^\infty}]^{\wedge p}$$

By the flatness lemma, $\uparrow$ perfectoid, fflat / S

$$\underset{\text{perfd}}{S_\infty \longrightarrow S'_\infty} \quad \text{fflat mod } p^n \ \forall_{n>0}$$

so that $g, f_1, \dots, f_c$ have p-power roots e.g.

check S_∞ p-tors free $+$ S'_∞ p-sep $\Longrightarrow$ S'_∞ also p tors free

$p, f_1, \dots, f_c, x_2, \dots, x_d$ res seq in S, S_∞

$p\ f_1^{1/p^e}, \dots, f_c^{1/p^e}, x_2, \dots, x_d$ so also S'_∞ reg seq on S'_∞ $\forall$ e

$\Longrightarrow$ $p, x_2, \dots, x_d$ reg seq on

$\uparrow$ $T := S'_\infty / (f_1^{1/p^\infty}, \dots, f_c^{1/p^\infty})^{\wedge p}$

a, b M-reg implies a M/bM-reg

g^{1/p^∞}-almost isom

"$g^{-1/p^\infty}T$"

$$T \longrightarrow T' := \mathrm{Hom}_T\left((g^{1/p^\infty}), T\right)$$

check this is a commutative ring

$\Rightarrow$ $p, x_2, \ldots, x_d$ g^{1/p^∞}-almost regular sequence

and $T'/(p, x_2, \ldots, x_d)T'$ not g^{1/p^∞}-almost zero.

and $gQ \subseteq (f_1, \ldots, f_c)$

$$\Rightarrow (g^{1/p^\infty})Q \subseteq \sqrt{(f_1, \ldots, f_c)}$$

$$= (f_1^{1/p^\infty}, \ldots, f_c^{1/p^\infty})$$

$\Rightarrow$
$$\begin{array}{ccccc} S & \longrightarrow & T & \longrightarrow & T' \\ Q & & \longmapsto & & 0 \end{array}$$
, so T' is an k-algebra.

Gabber's Trick

Final step: $B = W^{-1} \prod_{\mathbb{N}} T'$

w/ W mult seq gen by

$(g, g^{1/p}, g^{1/p^2}, \dots)$

Have ① $p, x_2, \dots, x_d$

reg sequence on B

② $B/_{m}B \neq 0$ (check)

replace B by $B^{\wedge m}$ to

get ((almost) perfectoid) BCM. ✓

<u>Remarks:</u> (R, m, k) complete local

$\{B_\lambda\}$ set of BCM R-algs or R^+-alg

$\underset{[MS]}{\Longrightarrow}$ can find BCM R-alg B

dominating all B_λ

+ compat w/ $\varinjlim B_\lambda$ if a directed system

($\Rightarrow$ existence of BCM R^+-algs)

Corollary: (Direct Summand Thm)

(R, m, k) regular local $\Longrightarrow$ R is a splinter ring, ie $\forall\ R \subseteq S$ mod fin domain extn splits: $\exists\ \varphi \in \mathrm{Hom}_R(S,R)$ w/ $\varphi|_R = id_R$

$\left(\Leftrightarrow R \to R^+ \text{ is pure} \right)$

Notes: • splinter $\Longrightarrow$ normal,

$\not\Longleftarrow$ in general but holds if $\mathbb{Q} \subseteq R$

• For (R, m, k) local

$\hat{R}$ splinter $\Longrightarrow$ R splinter $\left(\Leftrightarrow R^h \text{ splinter} \right)$

$\not\Longleftarrow$ in general but holds if $R \to \hat{R}$ regular

If $p \in m$,

splinter $\Longrightarrow$ CM

↑ use $R^{+\wedge p}$ BCM

Proof of corollary:

WLOG R complete, say $R \subseteq S$ mod finite domain extn also complete local $x_1, \dots, x_d \in R$ regular sop

$\Rightarrow$ also sop for S.

$S \longrightarrow B$ BCM S-alg

$\Rightarrow R \longrightarrow B$ fflat

$\mathrm{Tor}_i^R(k, B) = H_i(\underline{x}, B) = 0 \quad i > 0$

$\Rightarrow R \underset{\varphi}{\overset{\longleftarrow}{\longrightarrow}} B$ splits

uses R complete $\quad \Rightarrow R \underset{\varphi|_S}{\overset{\longleftarrow}{\subseteq}} S$ splits ✓