Symbolic Powers of Edge Ideals of Weighted Oriened Graphs

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Outline

Definition

Let R be a notherian ring, and I an ideal in R. The n-th symbolic power of I is defined to by

$$I^{(n)} = \bigcap_{P \in \mathrm{Ass}\,(R/I)} (I^n R_P \cap R).$$

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Let I be an ideal in a noetherian ring R.
(a) For all n ≥ 1, $I^n \subseteq I^{(n)}$.
(b) If $a \geq b$, then $I^{(a)} \subseteq I^{(b)}$.
(c) For all $a, b \geq 1$, $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$

Theorem (Nagata, Zariski)

Let $\mathbb{K}$ be a perfect field. Let I be a radical ideal in a polynomial ring $R = \mathbb{K}[x_1, \ldots, x_m]$. Then for all $n \in \mathbb{N}$, we have

$$I^{(n)} = \left(f \mid \frac{\partial^{|a|} f}{\partial a} \in I \text{ for all } a \in \mathbb{N}^m \text{ with } |a| \leq n-1 \right).$$

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Simple Graphs

Definition

A finite graph G is a pair $G = \big(V(G), E(G)\big)$ where V(G)=$\{x_1, \ldots, x_n\}$ is the set of vertices of G, and $E(G)$ is a collection of two element subsets of $V(G)$, usually called the edges of G.

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The edge ideal of a simple graph G is defined to be

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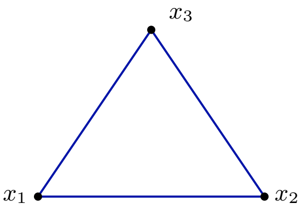

$$I(G) = (x_1x_2, x_2x_3, x_3x_1)$$

Weighted oriented graphs

Definition

A weighted oriented graph is a triplet $D = (V(D), E(D), w)$, where $V(D)$ is the vertex set, $E(D)$ is the edge set and w is a weight function $w : V(D) \longrightarrow \mathbb{N}^+$, where $\mathbb{N}^+ = \{1, 2, \ldots\}$.

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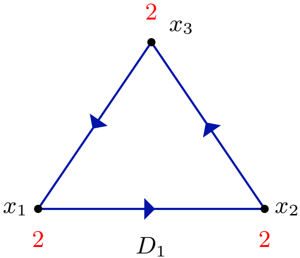
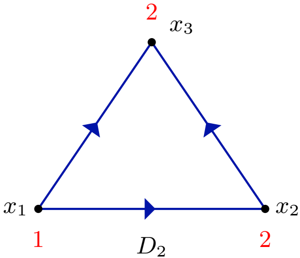


$I(D_1) = (x_1 x_2^2, x_2 x_3^2, x_3 x_1^2)$



$I(D_2) = (x_1 x_2^2, x_2 x_3^2, x_1 x_3^2)$

Symbolic powers of weighted oriented graphs

Theorem (C. Bocci et al.,2016)

Let G be a graph on vertices $\{x_1, \ldots, x_n\}$, $I = I(G) \subseteq k[x_1, \ldots, x_n]$ be the edge ideal of G and $V_1, \ldots, V_r$ be the minimal vertex covers of G. Let P_j be the monomial prime ideal generated by the variables in V_j. Then

$$I = P_1 \cap \cdots \cap P_r$$

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and

$$I^{(m)} = P_1^m \cap \cdots \cap P_r^m.$$

Theorem (Y. Pitones et al., 2019)

Let D be a weighted oriented graph and $C_1, \dots, C_s$ are the strong vertex covers of D, then the irredundant irreducible decomposition of $I(D)$ is

$$I(D) = I_{C_1} \cap \cdots \cap I_{C_s}$$

where each $I_{C_i} = (L_1^D(C_i) \cup \{x_j^{w(x_j)} \mid x_j \in L_2^D(C_i) \cup L_3^D(C_i)\})$, $\mathrm{rad}(I_{C_i}) = P_i = (C_i)$.

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Let $I \subset R = k[x_1, \dots, x_n]$ and $I = Q_1 \cap \cdots \cap Q_m$ be a primary decomposition of ideal I. For $P \in \operatorname{Ass}(R/I)$, we denote $Q_{\subseteq P}$ to be the intersection of all Q_i with $\sqrt{Q_i} \subseteq P$. If C is a strong vertex cover of a weighted oriented graph D, then $(C) \in \operatorname{Ass}(R/I(D))$. We denote $I_{\subseteq C}$ as $I_{\subseteq (C)}$.

Lemma (S. Cooper et al.,2017)

Let I be the edge ideal of a weighted oriented graph D and $C_1, \dots, C_r$ are the maximal strong vertex covers of D. Then

$$I^{(s)} = (I_{\subseteq C_1})^s \cap \cdots \cap (I_{\subseteq C_r})^s.$$

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Comparing symbolic powers of edge ideals of weighted oriented graphs

Question: When is $I^{(s)} = I^s$ for all $s \geq 1$?

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Theorem (A.Simis et al.,1994)

For an ideal $I = I(G)$, we have $I^{(s)} = I^s$ for all $s \geq 1$ if and only if G is bipartite.

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Lemma (M. Mandal and D.K. Pradhan, 2021)

Let D *be a weighted oriented graph. If* $V(D)$ *is a strong vertex cover of* D, *then* $I(D)^{(s)} = I(D)^s$ *for all* $s \geq 2$.

Comparing symbolic powers of weighted oriented graphs

In [2], if a simple graph contains an induced odd cycle $C_{2n+1} = (x_1, \ldots, x_{2n+1})$, the authors have shown that the $(n+1)-$th ordinary and symbolic power of its edge ideal are different.

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Proposition

Let D be a weighted oriented graph. Let D' be an induced odd cycle with underlying graph $C_{2n+1} = (x_1, \dots, x_{2n+1})$ where $V(C_{2n+1}) \nsubseteq N_D^+(V^+(D))$ and it satisfies the condition "$V(C_{2n+1}) \setminus N_D^+(V^+(D))$ contains one vertex which is not source in D', otherwise, it contains a vertex which is source in D' with trivial weight in D". Then $I(D)^{(n+1)} \neq I(D)^{n+1}$.

Weighted oriented graphs with induced odd cycles

Theorem

Let D be a weighted oriented graph such that each edge of D lies in some induced odd cycle of it. Then $V(D)$ is a strong vertex cover of D if and only if $I(D)^{(s)} = I(D)^s$ for all $s \geq 2$.

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1. odd cycle
2. complete graph
3. clique sum of finite number of odd cycles and complete graphs
4. complete $m-$partite graph

Symbolic powers of weighted oriented unicyclic graphs

Theorem

Let D be a weighted naturally oriented unicyclic graph with a unique odd cycle $C_{2n+1} = (x_1, \ldots, x_{2n+1})$. Then $I(D)^{(s)} = I(D)^s$ for all $s \geq 2$ if and only if $w(x) \geq 2$ when $\deg_D(x) \geq 2$ for all $x \in V(D)$.

Symbolic powers of weighted oriented even cycles

Corollary

Let D be a weighted naturally oriented even cycle whose underlying graph is $C_n = (x_1, \ldots, x_n)$, where $n \neq 4$ and at least one vertex of D has non-trivial weight. Then $I(D)^{(s)} = I(D)^s$ for all $s \geq 2$ if and only if all vertices of D have non-trivial weights.

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Proposition

Let D be a weighted naturally oriented even cycle whose underlying graph is $C_4 = (x_1, x_2, x_3, x_4)$ and at least one vertex of D has non-trivial weight. Then $I(D)^{(s)} = I(D)^s$ for all $s \geq 2$ if and only if D satisfies one of the following conditions:

1. *all vertices of D have non-trivial weights,*
2. *one vertex of D has non-trivial weight,*
3. *only two non-consecutive vertices of D have non-trivial weights.*

Comparing symbolic powers of weighted oriented graphs

Notation

Let D be a weighted oriented graph, where $U \subseteq V^+(D)$ be the set of vertices which are sinks and $w_j = w(x_j)$ if $x_j \in V^+(D)$. Let D' be the weighted oriented graph obtained from D after replacing w_j by $w_j = 1$ if $x_j \in U$. Let $V(D) = V(D') = V = \{x_1, \ldots, x_n\}$. Let $R = k[x_1, \ldots, x_n] = \bigoplus_{d=0}^{\infty} R_d$ be the standard graded polynomial ring. Consider the map

$$\Phi : R \longrightarrow R \text{ where } x_j \longrightarrow x_j \text{ if } x_j \notin U \text{ and } x_j \longrightarrow x_j^{w_j} \text{ if } x_j \in U.$$

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Theorem

Let I and $\tilde{I}$ be the edge ideals of D and D', respectively. Then $\Phi(\tilde{I}^s) = I^s$ and $\Phi(\tilde{I}^{(s)}) = I^{(s)}$ for all $s \geq 1$.

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Corollary

Let D *be a weighted oriented graph where the vertices of* $V^+(D)$ *are sinks and its underlying graph is* G. *Then* G *is bipartite if and only if* $I(D)^{(s)} = I(D)^s$ *for all* $s \geq 2$.

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Theorem

Let D *be a weighted oriented star graph whose underlying graph is* S_n *for some* $n \geq 2$. *Then* $I(D)^{(s)} = I(D)^s$ *for all* $s \geq 2$.

Outline

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Thank you