Affine Semigroups of Maximal Projective Dimension

Om Prakash
(Joint work with Dr. Kriti Goel and Prof. Indranath Sengupta)

Indian Institute of Technology Gandhinagar

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Preliminaries

- By an affine semigroup S, we mean a finitely generated (additive) submonoid of $\mathbb{N}^d$ for some positive integer d.
- The cardinality of the minimal generating set of an affine semigroup S is known as the embedding dimension of S, and it is denoted by $e(S)$.
- Let S be minimally generated by $a_1, \ldots, a_n \in \mathbb{N}^d$. The semigroup ring $k[S] = k[t^{a_1}, \ldots, t^{a_n}]$ of S is a k-subalgebra of the polynomial ring $k[t_1, \ldots, t_d]$, where $t^{a_i} = t_1^{a_{i1}} \cdots t_d^{a_{id}}$ for $a_i = (a_{i1}, \ldots, a_{id})$ and for all $i = 1, \ldots, n$.
- Set $R = k[x_1, \ldots, x_n]$ and define a map $\pi : R \to k[S]$ given by $\pi(x_i) = t^{a_i}$ for all $i = 1, \ldots, n$. Set deg $x_i = a_i$ for all $i = 1, \ldots, n$.

Pseudo-Frobenius elements of submonoids of $\mathbb{N}^d$

- R is a multi-graded ring and that π is a degree preserving surjective k-algebra homomorphism. Let $I_S = \ker \pi$. Then I_S is a homogeneous ideal, generated by binomials, called the defining ideal of S.

- Consider the cone of S in $\mathbb{Q}^d_{\geq 0}$,
$$\text{cone}(S) := \{\textstyle\sum_{i=1}^n \lambda_i a_i \mid \lambda_i \in \mathbb{Q}_{\geq 0}, i = 1, \ldots, n\}$$
and set $\mathcal{H}(S) := (\text{cone}(S) \setminus S) \cap \mathbb{N}^d$.

Definition

An element $f \in \mathcal{H}(S)$ such that $f + s \in S$ for all $0 \neq s \in S$, is called pseudo- Frobenius elements of S. The set of pseudo-Frobenius elements of S is denoted by $\text{PF}(S)$.
$$\text{PF}(S) = \{f \in \mathcal{H}(S) \mid f + a_j \in S, \forall j \in [1, n]\}.$$
We call the cardinality of this set the Betti-type of S.

Pseudo-Frobenius elements of submonoids of $\mathbb{N}^d$

Remark

Pseudo-Frobenius elements may not exist. Indeed, let

$$S = \langle (2,0), (1,1), (0,2) \rangle.$$

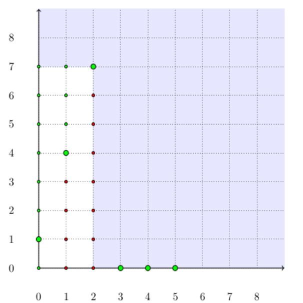
Then S is the subset of points in $\mathbb{N}^2$ whose sum of coordinates is even. Thus, we have that $\mathcal{H}(S) + S = \mathcal{H}(S)$. Therefore $\mathrm{PF}(S) = \emptyset$.

- If $\mathcal{H}(S)$ is finite then the set of pseudo-Frobenius elements is always non-empty.
- Consider the partial order $\preceq_S$ on $\mathbb{N}^d$, where for all $x, y \in \mathbb{N}^d$, $x \preceq_S y$ if $y - x \in S$. If $\mathcal{H}(S)$ is a non-empty finite set then $\mathrm{PF}(S) = \mathrm{Maximals}_{\preceq_S} \mathcal{H}(S)$.

Pseudo-Frobenius elements of submonoids of $\mathbb{N}^d$

Example

Let $S = \langle (0,1), (3,0), (4,0), (5,0), (1,4), (2,7) \rangle$.



- $\mathcal{H}(S)$= Set of all red points.
- $\mathrm{PF}(S) = \{(1,3), (2,6)\}$.

Maximal projective dimension semigroups

- S has **maximal projective dimension(MPD)** if $\text{pdim}_R\ k[S] = n-1$. Equivalently, $\text{depth}_R\ k[S] = 1$.
- (J. I Garcia-Garcia et. al., 2020) S is MPD if and only if $\text{PF}(S) \neq \emptyset$.
- (J. I Garcia-Garcia et. al., 2020) If S is a MPD-semigroup, then $a \in S$ is the S-degree of the $(n-2)$th minimal syzygy of $k[S]$ if and only if $a \in \{f + \sum_{i=1}^{n} a_i,\ f \in \text{PF}(S)\}$.
- The cardinality of the set of pseudo-Frobenius elements is equal to the last Betti number of $k[S]$ over R.

Example

Let $S = \langle a_1 = (2,11), a_2 = (3,0), a_3 = (5,9), a_4 = (7,4) \rangle$. Then we have a minimal free resolution of $k[S]$,

$$0 \to R(-(81,93)) \oplus R(-(94,82)) \to R^6 \to R^5 \to R \to k[S] \to 0.$$

Therefore, $\mathrm{pdim}_R\ k[S] = 3$. Hence, S is MPD. Thus, we have

$$\mathrm{PF}(S) = \{(81,93) - \sum_{i=1}^{4} a_i, (94,82) - \sum_{i=1}^{4} a_i\}.$$

Therefore, $\mathrm{PF}(S) = \{(64,89), (77,58)\}$.

Definition

Let $\prec$ be a term order on $\mathbb{N}^d$. Then $F(S)_\prec = \max_\prec \mathcal{H}(S)$, if it exists, is called a Frobenius element of S. Note that Frobenius elements of S may not exist. However, if $|\mathcal{H}(S)| < \infty$, then S has Frobenius elements.

Definition

Fix a term order $\prec$ such that $F(S)_\prec = \max_\prec \mathcal{H}(S)$ exists.

1. If $\mathrm{PF}(S) = \{F(S)_\prec\}$, then S is called a **$\prec$-symmetric semigroup**.
2. If $\mathrm{PF}(S) = \{F(S)_\prec, F(S)_\prec/2\}$, then S is called **$\prec$-pseudo-symmetric**.

$\prec$-symmetric semigroups

- If $\mathcal{H}(S)$ is a non-empty finite set, then S is said to be a $\mathcal{C}$-semigroup, where $\mathcal{C}$ denotes the cone of the semigroup. When S is a $\mathcal{C}$-semigroup, we give a characterization of $\prec$-symmetric and $\prec$-pseudo-symmetric semigroups.

Theorem (– , Goel, Sengupta)

Let S be a $\mathcal{C}$-semigroup and let $F(S)_{\prec}$ denote the Frobenius element of S with respect to an order $\prec$. Then S is a $\prec$-symmetric semigroup if and only if for each $g \in \text{cone}(S) \cap \mathbb{N}^d$ we have:

$$g \in S \iff F(S)_{\prec} - g \notin S.$$

Theorem (– , Goel, Sengupta)

Let S be a $\mathcal{C}$-semigroup and let $F(S)_{\prec}$ denote the Frobenius element of S with respect to an order $\prec$. Then S is a $\prec$-pseudo-symmetric semigroup if and only if $F(S)_{\prec}$ is even, and for each $g \in \text{cone}(S) \cap \mathbb{N}^d$ we have:

$$g \in S \iff F(S)_{\prec} - g \notin S \text{ and } g \neq F(S)_{\prec}/2.$$

- Let S be a $\mathcal{C}$-semigroup and $\prec$ be a monomial order satisfying that every monomial is preceded only by a finite number of monomials. Define the Frobenius number of S as
$$\mathcal{N}(F(S)_{\prec}) = |\mathcal{H}(S)| + |\{g \in S \mid g \prec F(S)_{\prec}\}|$$

Extended Wilf's conjecture. (J. I. Garcia-Garcia et. al., 2018) Let S be a $\mathcal{C}$-semigroup and $\prec$ be a monomial order satisfying that every monomial is preceded only by a finite number of monomials. Then

$$\mathcal{N}(F(S)_{\prec}) + 1 \leq e(S) \cdot |\{g \in S \mid g \prec F(S)_{\prec}\}|$$

- On cone(S), define a usual relation $\leq_c$ as follows:
$$g \leq_c f \text{ if } g_i \leq f_i \text{ for all } i \in [1, d].$$

Extended Wilf's conjecture

Theorem (– , Goel, Sengupta)

Let S be a $\mathcal{C}$-semigroup with full cone. Then

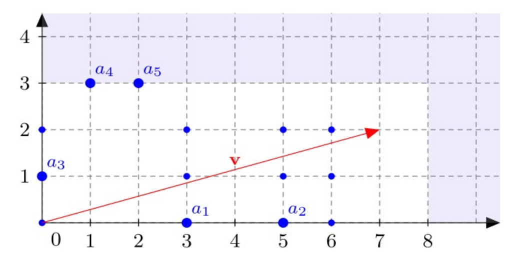
1. S is $\prec$-symmetric if and only if
$$|\mathcal{H}(S)| = |\{g \in S \mid g \leq_c F(S)_\prec\}|.$$
2. S is $\prec$-pseudo-symmetric if and only if $F(S)_\prec$ is even and
$$|\mathcal{H}(S) \setminus \{F(S)_\prec/2\}| = |\{g \in S \mid g \leq_c F(S)_\prec\}|$$

Theorem (– , Goel, Sengupta)

Let S be a $\mathcal{C}$-semigroup with full cone. If S is $\prec$-symmetric or $\prec$-pseudo-symmetric semigroup, then extended Wilf's conjecture holds.

Example

$S = \langle a_1 = (3,0), a_2 = (5,0), a_3 = (0,1), a_4 = (1,3), a_5 = (2,3) \rangle$. Let $\prec$ denote the degree lexicographic order. Then $F(S)_\prec = (7,2)$ and S is $\prec$-symmetric.



- $|\mathcal{H}(S)| = 12 = |\{g \in S \mid g \leq_c F(S)_\prec\}|$.
- $e(S) = 5,\ \mathcal{N}(F(S)_\prec) = 53$.

Gluing of MPD-semigroups

Definition

Let $G(S)$ be the group generated by S. Let A be the minimal generating system of S and $A = A_1 \cup A_2$ be a nontrivial partition of A . Let S_i be the submonoid of $\mathbb{N}^d$ generated by $A_i, i \in 1, 2$. Then $S = S_1 + S_2$. We say that S is the **gluing** of S_1 and S_2 by s if
(1) $s \in S_1 \cap S_2$ and,
(2) $G(S_1) \cap G(S_2) = s\mathbb{Z}$.

Theorem (– , Goel, Sengupta)

Let S be a gluing of S_1 and S_2. Then S is MPD if and only if S_1 and S_2 are MPD. Moreover,

$$\mathrm{PF}(S) = \{f + g + s \mid f \in \mathrm{PF}(S_1), g \in \mathrm{PF}(S_2)\}.$$

- We show by a class of MPD-semigroups of embedding dimension four that there is no upper bound on the Betti-type of MPD-semigroups in terms of embedding dimension.
- Let $a \geq 3$ be an odd natural number and $p \in \mathbb{Z}^+$. Define

$$S_{a,p} = \langle (a,0),(0,a^p),(a+2,2),(2,2+a^p)\rangle.$$

- Define the set

$$\Delta = \{(a^p(a+2)-(\ell+2)a-2, a^p(\ell+2)-2) \mid 0 \leq \ell < a^p-1\}.$$

Unboundedness of Betti-type

Proposition (– , Sengupta)

$S_{a,p}$ is an MPD-semigroup and $\Delta \subseteq \mathrm{PF}(S_{a,p})$.

Theorem (– , Sengupta)

For each $e \geq 4$, there exists a class of MPD-semigroups of embedding dimension e in $\mathbb{N}^2$, where there is no upper bound on the Betti-type in terms of the embedding dimension e.

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Thank you for your attention!