<u>Setup for last talk:</u> $(R, \underset{\substack{\cup \\ p}}{\mathfrak{m}}, k)$ complete local domain dim d

w/ k perfect.

"char p" char $R = p$

"mixed char" char $R = 0$

<u>BCM Regular:</u> [Ma - Schwede]

B a BCM R^+-alg

R is BCM regular wrt B "B-reg"

if normal $\mathbb{Q}$-Gor and

$R \longrightarrow B$ splits.

$$I_B(R) := \operatorname{im}\Big(\operatorname{Hom}_R(B,R) \longrightarrow R\Big)$$
$$\varphi \longmapsto \varphi(1)$$

"B-test ideal", also well-defined $\forall$ solid R-algebras

($I_B(R) = R \iff R$ B-reg)

R is BCM regular if (normal $\mathbb{Q}$-Gor and) B-reg $\forall$ BCM R^+-alg B

$$I_{\mathcal{B}}(R) = \bigcap_{\text{BCM } R^+\text{-alg } B} I_B(R) \underset{\text{domination}}{=} I_B(R) \text{ for } \forall B \text{ "suff large"}$$

<u>BCM Rational</u>: [Ma - Schwede]

B a BCM R^+-alg

R is BCM-rat wrt B "B-rat"

if CM and $H^{\dim R}_{\mathfrak{m}} R \hookrightarrow H^{\dim R}_{\mathfrak{m}} B$

or equiv

$$\operatorname{Hom}_R(B, \omega_R) \twoheadrightarrow \omega_R$$

$$I_B(\omega_R) := \mathrm{im}\left(\mathrm{Hom}_R(B, \omega_R) \longrightarrow \omega_R \right)$$
$$\varphi \longmapsto \varphi(1)$$

"B-test mod", also well-defined $\forall$ solid R-algebras

$\left(I_B(\omega_R) = \omega_R \iff R \text{ } B\text{-rat.} \right)$

R is BCM-rat if

(CM and) B-rat

$\forall$ BCM R^+-alg B

$$I_{\mathcal{B}}(\omega_R) = \bigcap_{\text{BCM } R^+\text{-alg } B} I_B(\omega_R) \underset{\text{domination}}{=} I_B(\omega_R) \text{ for } \forall B \text{ "suff large"}$$

<u>Easy</u>: BCM-reg $\Longrightarrow$ BCM-rat

$\Longleftarrow$ + Gor

BCM-reg $\overset{?}{\Longrightarrow}$ R^+-reg $\Longleftrightarrow$ splinter + $\mathbb{Q}$-Gor

Open/conj: converse?

$[HH, M, \overset{sm}{S}, T, L, BMPSTWW, \ldots]$

Thm $\operatorname{char} R = p$

$$\text{BCM-reg} \iff \begin{matrix} \mathbb{Q}\text{-Gor} \\ + \\ (\text{str}) \text{ F-reg} \end{matrix}$$

$$\tau_{\mathcal{B}}(R) = \tau(R) = \bigcap_{I \subseteq R} (I : I^*)$$

$$= \tau_{R^+}(R) = \bigcap_{\substack{R \subseteq S \\ \text{finite}}} \tau_S(R) = \tau_S(R) \quad \forall\, S \text{ suff large}$$

$$\text{BCM-rat} \iff \text{F-rat}$$

$$\tau_{\mathcal{B}}(\omega_R) = \tau_{R^+}(\omega_R) = \tau(\omega_R)$$

$$= \left(H^{\dim R}_{\mathfrak{m}} R \big/ 0^*_{H^d_{\mathfrak{m}} R} \right)^\vee$$

$$= \bigcap_{\substack{R \subseteq S \\ \text{finite}}} \tau_S(\omega_R) = \tau_S(\omega_R) \quad \forall\, S \text{ suff large}$$

(R $\mathbb{Q}$-Gor CM)

char $= p$		mixed char		char $= 0$
F-reg	$\longleftrightarrow$	BCM reg	$\xRightarrow{[\frac{1}{p}]}$	KLT
$\tau(R)$	$\longleftrightarrow$	$\tau_{\mathcal{B}}(R)$	$\overset{[\frac{1}{p}]}{\underset{?}{\Longrightarrow}}$	$\mathcal{J}(K)$
F-rat	$\longleftrightarrow$	BCM rat	$\xRightarrow{[\frac{1}{p}]}$	rat
$\tau(\omega_R)$	$\longleftrightarrow$	$\tau_{\mathcal{B}}(\omega_R)$	$\overset{[\frac{1}{p}]}{\underset{?}{\Longrightarrow}}$	$\mathcal{J}(\omega_R)$

<u>Thm</u> [BMPSTWW] (in progress)

$$\tau_{\mathcal{B}}(\omega_R)[\tfrac{1}{p}] = \mathcal{J}(\omega_{R[\frac{1}{p}]})$$

Some Mixed Char Examples of BCM-reg.

- $R = \dfrac{\mathbb{Z}_p[\![y_2, \dots, y_n]\!]}{(p^m + y_2^m + \dots + y_n^m)}$ $\quad p \gg 0$, $m < n$ $\quad$ (mod p is F-reg) ← [MSTWW]

- $R = \dfrac{\mathbb{Z}_p[\![x, z]\!]}{(x^2 + p^2 z + z^3)}$, $p > 5$

  (RDP w/ mixed char $(0, p > 5)$ ← [CRMPST], 2 dim'l klt pairs w/ std coeff w/ mixed char $(0, p > 5)$ ← [BMPSTWW])

- $\mathbb{Q}$-Gor direct summands of regular rings

  analog of result of Gabber–Romero

- log regular rings:

  $\cong \dfrac{W(k)[\![M]\!]}{(p - f)}$ $\quad M$ strongly convex saturated normal monoid

  $f \in I_M + (p^2) \subseteq W(k)[\![M]\!]$

  ideal of monomials

Additional
Setup : Earlier setup plus

Cohen-Gabber $\Rightarrow$ $\exists$
$$(A, m_A, k) \hookrightarrow (R, m, k)$$
module finite
gen separable

w/

char p : $A = k[[x_1, \ldots, x_d]]$

mixed char : $A = W(k)[[x_2, \ldots, x_d]]$
$x_1 = p$

$$A_\infty := A[x_1^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]^\wedge_p \leftarrow \text{perfd}$$

($= A_{perf}$ in char p)

$$R^{A_\infty}_{perfd} := (R \otimes_A A_\infty)_{perfd}$$

($= R_{perf}$ in char p)

Normalized Length: (Faltings)

M a m_A-power torsion A_∞-mod

$$\rightsquigarrow \lambda_\infty(M)$$

① M finitely presented
$\Rightarrow$ M defined /
$$A_e := A[x_1^{1/p^e}, \ldots, x_d^{1/p^e}] \quad \text{some } e$$

i.e. $$M = M_e \otimes_{A_e} A_\infty$$

$$\lambda_\infty(M) := \frac{1}{p^{ed}} \, l_{A_e}(M_e)$$

② M fin generated

$$\Rightarrow \lambda_\infty(M) := \inf \{ \lambda_\infty(M') \mid M' \twoheadrightarrow M, \; M' \text{ fin presented} \}$$

③ M arbitrary

$$\Rightarrow \lambda_\infty(M) = \sup \{ \lambda_\infty(M') \mid M' \subseteq M \text{ fin gen} \}$$

One checks: well-defined, additive on ses

<u>Examples</u>: ① $I \subseteq A$, $l_A(A/I) < \infty$

$$\Longrightarrow \lambda_\infty(A_\infty / I A_\infty) = l_A(A/I).$$

② $0 \neq f \in A$, char $A = p$, $A_\infty = A_{perf}$

$$\lambda_\infty \left(A_{perf} / (f^{1/p^\infty}, m_A) \right) = 0$$

f^{1/p^∞}-almost zero

$$\wedge|$$

$$\lambda_\infty \left(A_{perf} / (f^{1/p^e}, m_A) \right) = \ell_{A_e} \left(A_e / (f^{1/p^e}, m_A) \right) / p^{ed}$$

$$= \ell_A \left(A / (f, m_A^{[p^e]}) \right) / p^{ed} \underset{e \gg 0}{\approx} \frac{e_{HK}(A/f)}{p^e}$$

[CLMST] $\longrightarrow 0$ as $e \to \infty$.

Thm Setup as above w/ char $R = p$

① $I \subseteq R$ w/ $\ell_R(R/I) < \infty$

$$\Longrightarrow \lambda_\infty \left(R_{perf} / I R_{perf} \right) = e_{HK}(I).$$

② $I_\infty := \{ x \in R_{perf} \mid R \xrightarrow{1 \mapsto x} R_{perf}$

$$\Longrightarrow \lambda_\infty \left(R_{perf} / I_\infty R_{perf} \right) = s(R).$$

Caution: R_{perf} not fg / A_{perf}

① <u>proof idea</u>: Consider ses

$$0 \longrightarrow R^{1/p^e} \otimes_{A^{1/p^e}} A_{perf} \longrightarrow R_{perf} \longrightarrow C_e \longrightarrow 0$$

uses separability of R/A

$$\begin{array}{ccccc} \dfrac{R^{1/p^e} \otimes_{A^{1/p^e}} A_{perf}}{I(R^{1/p^e} \otimes_{A^{1/p^e}} A_{perf})} & \twoheadrightarrow & \overbrace{\dfrac{R^{1/p^e} \otimes_{A^{1/p^e}} A_{perf}}{(IR_{perf}) \cap (R^{1/p^e} \otimes_{A^{1/p^e}} A_{perf})}}^{*_e} & \subseteq & R_{perf}/IR_{perf} \\ \downarrow & & \downarrow & & \\ \lim_{e\to\infty} \lambda_\infty(-) & = & \lim_{e\to\infty} \lambda_\infty(*_e) & = & \lambda_\infty\left(\dfrac{R_{perf}}{IR_{perf}}\right) \\ \| & & \uparrow & & \\ e_{HK}(I) & & & & \end{array}$$

$\geq$: clear term by term.

$\leq$: $\exists\ g$ s.t. $g^{1/p^e} \in \mathrm{Ann}_{R_{perf}} C_e\ \forall e$

$$g^{1/p^e} R_{perf} \subseteq R^{1/p^e} \otimes_{A^{1/p^e}} A_\infty \subseteq R_{perf}$$

so $\lambda_\infty(*_e) \geq \lambda_\infty\left(\dfrac{R^{1/p^e} \otimes_{A^{1/p^e}} A_{perf}}{I(R^{1/p^e} \otimes_{A^{1/p^e}} A_{perf}) : g^{1/p^e}} \right)$

$= \frac{1}{p^{ed}} l_R\left(\dfrac{R}{I^{[p^e]} : g} \right) = \frac{1}{p^{ed}}\left(l_R(R/I^{[p^e]}) - l_R(R/(I^{[p^e]}, g)) \right)$

$\xrightarrow[e \to \infty]{} e_{HK}(I) - 0 = e_{HK}(I).$

(2) proof idea:

$I_e := \{ r \in R : R \xrightarrow{\cdot r^{1/p^e}} R^{1/p^e} \text{ not pure (split)} \}$

$a_e = \lambda(R/I_e) \underset{\substack{\uparrow \\ k = k^p}}{=} \lambda(R^{1/p^e}/I_e^{1/p^e})$

$R^A_{perfd} = R_{perf} = \bigcup_{e \geq 0} R^{1/p^e}$

$\cup\!\!\!|$

$I_\infty := \{ x \in R_{perf} : R \xrightarrow{\cdot x} R_{perf} \text{ not pure} \}$

$\underset{\text{easy}}{=} \bigcup_{e \geq 0} I_e^{1/p^e} \qquad (I_e^{1/p^e} = I_\infty \cap R^{1/p^e})$

$$g^{1/p^e} R_{perf} \subseteq R^{1/p^e} \otimes_{A^{1/p^e}} A_\infty \subseteq R_{perf}$$

$$\lambda_\infty(R_{perf}/I_\infty) = \sup_e \lambda_\infty\left(\frac{R^{1/p^e} \otimes_{A^{1/p^e}} A_\infty}{I_\infty \cap (R^{1/p^e} \otimes_{A^{1/p^e}} A_\infty)}\right)$$ "almost f.g. / A_∞"

and use

$$I_e^{1/p^e} \otimes_{A^{1/p^e}} A_\infty \subseteq I_\infty \cap (R^{1/p^e} \otimes_{A^{1/p^e}} A_\infty) \subseteq (I_e : g)^{1/p^e} \otimes_{A^{1/p^e}} A_\infty$$

[CLMST] ✓

Perfectoid Hilbert-Kunz & signature:

$I \subseteq R$ w/ $\ell_R(R/I) < \infty$

$$\rightsquigarrow \quad e_{HK}^{\underline{x}}(I) = \lambda_\infty\left(\frac{\hat{R}^A_{perfd}}{I\hat{R}^A_{perfd}}\right)$$

$$I_\infty := \{x \in \hat{R}^A_{perfd} : R \xrightarrow{\cdot x} \hat{R}^A_{perfd} \text{ not pure (split)}\}$$

$$s_{perfd}^{\underline{x}}(R) := \lambda_\infty(\hat{R}^A_{perfd}/I_\infty)$$

Q: Independence of $\underline{x}$?

Some key results:

- $e^{\chi}_{perfd}(R) := e^{\chi}_{perfd}(m) \geq 1$ w/ $=$ iff R reg

- $s^{\chi}_{perfd}(R) \leq 1$ w/ $=$ iff R regular

- $s^{\chi}_{perfd}(R) = \inf\limits_{I \subsetneq J} \dfrac{e^{\chi}_{perfd}(I) - e^{\chi}_{perfd}(J)}{\lambda(J/I)}$

- $I \subseteq J \subseteq R$ finite colength

  $\Rightarrow e^{\chi}_{perfd}(I) \geq e^{\chi}_{perfd}(J)$

  w/ $=$ iff $I^{epf} = J^{epf}$

  full extended plus closure

  $x \in I^{epf} \iff \exists\ 0 \neq c \in R$

  [Heitmann - Ma] w/ $c^{1/p^e} x \in (I, p^n)R^+$

  $\forall\ e, n > 0$

- $y_1, \ldots, y_d$ sop $\Rightarrow e^{\chi}_{perfd}(\underline{y}) = \chi(\underline{y})$

  (not easy)

(open: $\frac{1}{d!} e(I) \overset{?}{\leq} e^{\chi}_{perfd}(I) \leq e(I)$)

uses $I^{[p^e]} \subseteq I^{p^e}$ in char p — take min reduction

- If R $\mathbb{Q}$-Gor,

$s^{\underline{x}}_{\text{perfd}}(R) > 0 \iff R$ BCM reg

"$\Leftarrow$": Say R Gorenstein
$\Delta \in R$ gens socle mod $(\underline{x})$

$$s^{\underline{x}}_{\text{perfd}}(R) = \lambda_\infty(R^A_{\text{perfd}}/I_\infty) = 0$$

$\therefore$ v-almost zero

so $1 \overset{a}{\in} I_\infty = (\underline{x})R^A_{\text{perfd}} :_{R^A_{\text{perfd}}} \Delta$

or $\Delta \overset{a}{\in} (x)R^A_{\text{perfd}}$

almost BCM [BS]

$\xRightarrow[\text{Gabber}]{}$ $\exists$ BCM B w/ $\Delta \in (x)B$,
so $R \longrightarrow B$ not pure,
and R not BCM reg.

- $$s^{\underline{x}}_{\text{perf, relrat}}(R) = \inf_{\substack{(\underline{y}) \subsetneq I \\ \text{param}}} \frac{e^{\underline{x}}_{\text{perfd}}(\underline{y}) - e^{\underline{x}}_{\text{perfd}}(I)}{\ell(I/(\underline{y}))}$$

positive $\iff R$ BCM rational

- $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{n}, \ell)$ finite split quasi-étale domain ext'n

$$s^{\chi}_{perfd}(S) \cdot [\ell : k] = s^{\chi}_{perfd}(R) \cdot \mathrm{rank}_R(S)$$

($\Rightarrow$ applications to local fundamental groups, finite torsion in the class group)

Q: Compat w/ localization? semicontinuity?