Image of linear derivations and Mathieu-Zhao subspaces

Sakshi Gupta

(Joint work with Divya Ahuja and Surjeet Kour)

School on Commutative Algebra and Algebraic Geometry in Prime Characteristics, ICTP, Italy

Department of Mathematics
Indian Institute of Technology Delhi, India

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Basic facts about Derivations

- **Notations:**
  - $K :=$ a field of characteristic zero.
  - $K^* = K \setminus \{0\}$.
  - $K[X] := K[x_1, x_2, \ldots, x_n]$ is the polynomial algebra in n variables over K.
  - $A := K$-algebra.

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- **K-Derivation:** A K-linear map $d : A \to A$ s.t.

$$d(a+b) = d(a) + d(b),$$
$$d(ab) = bd(a) + ad(b) \quad \forall\ a, b \in A.$$

Basic facts about Derivations

- ***$K\mathcal{E}$*-Derivation:** A K-linear map $\delta : A \to A$ s.t.

$$\begin{aligned} \delta(a+b) &= \delta(a)+\delta(b), \\ \delta(ab) &= b\delta(a)+a\delta(b)-\delta(a)\delta(b) \quad \forall\ a,b \in A. \end{aligned}$$

Basic facts about Derivations

- ***$K\mathcal{E}$*-Derivation:** A K-linear map $\delta : A \to A$ s.t.

$$\delta(a+b) = \delta(a) + \delta(b),$$
$$\delta(ab) = b\delta(a) + a\delta(b) - \delta(a)\delta(b) \quad \forall\ a, b \in A.$$

Note: δ is a $K\mathcal{E}$-derivation of A if and only if $\delta = \mathrm{I} - \phi$, for some K-endomorphism ϕ of A, where I denotes the identity automorphism of A.

Basic facts about Derivations

- **Linear K-Derivation:** A K-derivation (or $K\mathcal{E}$-derivation) d of $K[X]$ is called linear if

$$d(x_i) = \sum_{j=1}^{n} a_{ij}x_j, \;\; i = 1, \ldots, n,$$

where $a_{ij} \in K$. The matrix $A = [a_{ij}]$ is called associated matrix of the derivation d.

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  If the corresponding matrix A is nilpotent, then the derivation d is called linear locally nilpotent derivation of $K[X]$.

Mathieu-Zhao subspaces

- First introduced by Zhao[1].
- By subspace of a K-algebra, we always mean a K-linear subspace.

[1] Zhao, Wenhua. (2010) in 2010. *Generalizations of the image conjecture and the Mathieu conjecture, Journal of Pure and Applied Algebra 214(7): 1200-1216.*

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Mathieu-Zhao subspaces

A K-subspace $\mathcal{M}$ of A is called a Mathieu-Zhao subspace of A if the following equivalent conditions holds:

1. If $f \in A$ is such that $f^m \in \mathcal{M}$ for all $m \geq 1$, then for every $g \in A$, we have $gf^m \in \mathcal{M}$ for all large m (i.e. there exists some $m_g \in \mathbb{N}$ such that $gf^m \in \mathcal{M}$ for all $m \geq m_g$).
2. If $f \in A$ is such that $f^m \in \mathcal{M}$ for large m, then for every $g \in A$, we have $gf^m \in \mathcal{M}$ for all large m.

[1] Zhao, Wenhua. (2010) in 2010. *Generalizations of the image conjecture and the Mathieu conjecture, Journal of Pure and Applied Algebra 214(7): 1200-1216.*

Mathieu-Zhao subspaces

- Mathieu-Zhao subspace is the natural generalization of notion of ideals in a ring.
- Every ideal is a Mathieu-Zhao subspace but not all Mathieu-Zhao subspaces are ideals.

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Example

For any $n \geq 1$ and integral domain R of characteristic zero, let $\mathcal{A} = M_{n\times n}(R)$ be the algebra of $n \times n$ matrices with entries in R and $\mathcal{M}$ the subspace of trace-zero matrices. The subspace $\mathcal{M}$ is a Mathieu subspace of $\mathcal{A}$ but certainly cannot be an ideal of $\mathcal{A}$.

The LFED and LNED Conjecture

Zhao proposed the following conjectures[2]:

The LFED Conjecture

Let d be a linear K-derivation (or $K\mathcal{E}$-derivation) of $K[X]$. Then the image of the derivation d is a Mathieu-Zhao subspace of $K[X]$.

[2] Zhao, Wenhua. (2018). *Some open problems on locally finite or locally nilpotent derivations and $\mathcal{E}$-derivations. Communications in Contemporary Mathematics 20(04): 1750056.*

[3] Van den Essen, Arno, Zhao, Wenhua. (2019). *On images of locally finite derivations and E-derivations. Journal of Pure and Applied Algebra 223(4):1689-1698*

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Let d be a linear locally nilpotent K-derivation (or $K\mathcal{E}$-derivation) of $K[X]$. Then d maps every ideal of $K[X]$ to a Mathieu-Zhao subspace of $K[X]$.

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The LNED Conjecture

Let d be a linear locally nilpotent K-derivation (or $K\mathcal{E}$-derivation) of $K[X]$. Then d maps every ideal of $K[X]$ to a Mathieu-Zhao subspace of $K[X]$.

- Zhao et.al[3] showed that if $\bar{K}$ is an algebraic closure of K and the LFED (or LNED) Conjecture holds for $\bar{A} = \bar{K} \otimes_K A$, then it also holds for A.

[2] Zhao, Wenhua. (2018). *Some open problems on locally finite or locally nilpotent derivations and $\mathcal{E}$-derivations. Communications in Contemporary Mathematics 20(04): 1750056.*

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Results

Definition: A set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where $\lambda_i \in K$ for $1 \leq i \leq n$ is said to be linearly independent over $\mathbb{N}_0$ if there exists no non-trivial linear combination over $\mathbb{N}_0$ that equals to zero.

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Theorem (–,Ahuja, Kour)

Let d be a linear K-derivation of $K[X]$. If the eigenvalues of the associated matrix of d are linearly independent over $\mathbb{N}_0$, then

- $\operatorname{Im} d$ *is an ideal of* $K[X]$.
- *Moreover,* $\operatorname{Im} d = (x_1, x_2, \ldots, x_n)$.

Results

- Consider $R = K[x_1, x_2, x_3, x_4]$, the polynomial algebra in four variables over K. Up to conjugation of matrices, there are five possible Jordan forms of a 4×4 matrix, i.e.

$$(A_1) = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} (A_2) = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} (A_3) = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

$$(A_4) = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} (A_5) = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}$$

for $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4 \in K$.

Results

• Let d be a linear K-derivation of R. Then d is conjugate to one of the following linear K-derivations of R.

1. $d_1(x_1) = \lambda_1 x_1,\ d_1(x_2) = \lambda_2 x_2,\ d_1(x_3) = \lambda_3 x_3,\ d_1(x_4) = \lambda_4 x_4;$
2. $d_2(x_1) = \lambda_1 x_1 + x_2,\ d_2(x_2) = \lambda_1 x_2,\ d_2(x_3) = \lambda_2 x_3,\ d_2(x_4) = \lambda_3 x_4;$
3. $d_3(x_1) = \lambda_1 x_1 + x_2,\ d_3(x_2) = \lambda_1 x_2 + x_3,\ d_3(x_3) = \lambda_1 x_3,\ d_3(x_4) = \lambda_2 x_4;$
4. $d_4(x_1) = \lambda_1 x_1 + x_2,\ d_4(x_2) = \lambda_1 x_2,\ d_4(x_3) = \lambda_2 x_3 + x_4,\ d_4(x_4) = \lambda_2 x_4;$
5. $d_5(x_1) = \lambda_1 x_1 + x_2,\ d_5(x_2) = \lambda_1 x_2 + x_3,\ d_5(x_3) = \lambda_1 x_3 + x_4,\ d_5(x_4) = \lambda_1 x_4;$
   where $\lambda_i \in K$ for $i = 1, 2, 3$ and 4.

[4] van den Essen, Arno, Wright, David, Zhao, Wenhua. (2011) *Images of locally finite derivations of polynomial algebras in two variables. Journal of Pure and Applied Algebra 215(9): 2130-2134.*

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3. $d_3(x_1) = \lambda_1 x_1 + x_2,\ d_3(x_2) = \lambda_1 x_2 + x_3,\ d_3(x_3) = \lambda_1 x_3,\ d_3(x_4) = \lambda_2 x_4;$
4. $d_4(x_1) = \lambda_1 x_1 + x_2,\ d_4(x_2) = \lambda_1 x_2,\ d_4(x_3) = \lambda_2 x_3 + x_4,\ d_4(x_4) = \lambda_2 x_4;$
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   where $\lambda_i \in K$ for $i = 1, 2, 3$ and 4.

Now we study the image of $d_i's$.

• It has been proved[4] that $\mathrm{Im} d_1$ is a Mathieu-Zhao subspace of R.

[4] van den Essen, Arno, Wright, David, Zhao, Wenhua. (2011) *Images of locally finite derivations of polynomial algebras in two variables. Journal of Pure and Applied Algebra 215(9): 2130-2134.*

Results

Theorem (–,Ahuja, Kour)

- 1. *If* $\lambda_1 \in K^*$ *and* $\lambda_2 = \lambda_3 = 0$, *then* $\mathrm{Im} d_2$ *is an ideal of* R *generated by* x_1 *and* x_2.
  2. *If* $\lambda_2 \in K^*$ *and* $\lambda_1 = \lambda_3 = 0$, *then* $\mathrm{Im} d_2$ *is an ideal of* R *generated by* x_2 *and* x_3.

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  2. *If* $\lambda_1 = \lambda_2 = 0$, *then* $\text{Im} d_4$ *is a Mathieu-Zhao subspace of* R.
- *If* $\lambda_1 \in K^*$, *then* $\text{Im} d_5$ *is an ideal of* R *generated by* x_1, x_2, x_3 *and* x_4.

Results

- Outline of the proof:
  - Varying weights on R are assigned that correspond to distinct derivations on R.

Derivation	**Weight**
d_2	$(\lambda_1, \lambda_1, 0, 0)$ $(0, 0, \lambda_2, 0)$
d_3	$(\lambda_1, 0, 0, 0)$ $(0, 0, 0, \lambda_2)$ $(1, 0, -1, 0)$
d_4	$(\lambda_1, \lambda_1, 0, 0)$ $(1, 0, 0, -1)$ $(0, -1, 1, 0)$
d_5	$(\lambda_1, \lambda_1, \lambda_1, \lambda_1)$

[5] Liu, Dayan, Sun, Xiaosong. (2020). *The factorial conjecture and images of locally nilpotent derivations. Bulletin of the Australian Mathematical Society.* 101(1): 71–79.

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d_3	$(\lambda_1, 0, 0, 0)$ $(0, 0, 0, \lambda_2)$ $(1, 0, -1, 0)$
d_4	$(\lambda_1, \lambda_1, 0, 0)$ $(1, 0, 0, -1)$ $(0, -1, 1, 0)$
d_5	$(\lambda_1, \lambda_1, \lambda_1, \lambda_1)$

  - The proof utilizes the established factorial conjecture[5] for homogeneous polynomials in two variables.

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Linear $K\mathcal{E}$-derivation

- Now, we discuss the image of a linear $K\mathcal{E}$-derivation of $R = K[x_1, x_2, x_3, x_4]$.

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- Let δ be a linear $K\mathcal{E}$-derivation. Then $\delta = \mathrm{I} - \phi$ for a linear K-endomorphism ϕ of R.

Proposition

Let ϕ be a linear K-endomorphism of R. Then ϕ is conjugate to one of the following K-endomorphisms of R.

1. $\phi_1(x_1) = \lambda_1 x_1$, $\phi_1(x_2) = \lambda_2 x_2$, $\phi_1(x_3) = \lambda_3 x_3$, $\phi_1(x_4) = \lambda_4 x_4$;
2. $\phi_2(x_1) = \lambda_1 x_1 + x_2$, $\phi_2(x_2) = \lambda_1 x_2$, $\phi_2(x_3) = \lambda_2 x_3$, $\phi_2(x_4) = \lambda_3 x_4$;
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   where $\lambda_i \in K$ for $i = 1, 2, 3$ and 4.

Linear $K\mathcal{E}$-derivation

- Let $\delta_i = \mathrm{I} - \phi_i$ for $1 \leq i \leq 5$.
- Then, upto conjugation, a linear $K\mathcal{E}$-derivation of R is conjugate to δ_i for some $1 \leq i \leq 5$.

[6] Van den Essen, A., Sun, X. (2018). *Monomial preserving derivations and Mathieu-Zhao subspaces. J. Pure Appl. Algebra. 222(10):3219–3223.*

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Linear $K\mathcal{E}$-derivation

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- Then, upto conjugation, a linear $K\mathcal{E}$-derivation of R is conjugate to δ_i for some $1 \leq i \leq 5$.
- Essen et.al in 2018[6] proved that $\mathrm{Im}\delta_1$ is a Mathieu-Zhao subspace.
- Further in 2022, Tian et.al[7] derived the result that $\mathrm{Im}\delta_2$ is a Mathieu-Zhao subspace.

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Results

We have established the following result on δ_i's.

Theorem (–,Ahuja, Kour)

- 1. *If* $\lambda_1^p \lambda_2^q \neq 1$ *for all* $p, q \in \mathbb{N}$*, then* $\text{Im}\delta_3$ *is a Mathieu-Zhao subspace of* R*.*
  2. *If there exists* $p, q \in \mathbb{N}$ *such that* $\lambda_1^p = 1$ *and* $\lambda_2^q = 1$*, then* $\text{Im}\delta_3$ *is a Mathieu-Zhao subspace of* R*.*

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- *If* $\lambda_1^p \neq 1$ *for all* $p \in \mathbb{N}$, *then* $\mathrm{Im}\delta_5$ *is a Mathieu-Zhao subspace of* R.

Results

- Outline of the proof:
  - Introduced new K-algebra endomorphisms corresponding to the given derivations.

Automorphism	New Matrix
ϕ_3	$[\phi_{\lambda_1,\lambda_2}] = \begin{bmatrix} \lambda_1 & \lambda_1 & -\frac{\lambda_1}{2} & 0 \\ 0 & \lambda_1 & -\lambda_1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}$
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  - Used the same weights as defined for d_i's to complete the proof.

Bibliography I

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Thank you