Upper bounds on two Hilbert coefficients

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(Joint work with J. Elias and L. T. Hoa)
ICTP, Trieste, Italy, 5/2023

- Motivation
- Main results

Motivation

- $(A, \mathfrak{m})$: Noetherian local ring, $I \subset A$: $\mathfrak{m}$-primary.
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- $H^1_{I,M}(n) = P^1_{I,M}(n)$ - for $n \gg 0$: the Hilbert-Samuel polynomial

$$P^1_{I,M}(n) = e_0(I, M)\binom{n+d}{d} - e_1(I, M)\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d(I, M),$$

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then the integers

$$e_0(I,M), ..., , e_d(I,M) \qquad (1)$$

are called the Hilbert coefficients of M with respect to I.

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- $$e_i(I, M) = \frac{Q^{(i)}_{I,M}(1)}{i!} \qquad (2),$$

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  for all $i \geq 0$, where where $Q^{(i)}_{I,M}$ denotes the i-th derivation of $Q_{I,M}$
  - $0 \leq i \leq d$, this value of $e_i(I, M)$ agrees with the one defined in (1).
  - using (2) we can talk about the Hilbert coefficients $e_i(I, M)$ with $i > d$.
- M: Cohen-Macaulay modules

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- Rossi-Valla (2010): Let b be a positive integer such that $IM \subseteq m^b M$

$$e_1(I, M) \leq \binom{e_0(I, M) - b + 1}{2}$$

If $d = 1$ and $e_0(I, M) \neq e_0(m^b, M)$

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- Rossi-Valla (2005) $e_1(I) \leq \binom{e_0(I)}{2} - \binom{\mu(I)-d}{2} - \ell(A/I) + 1$, where $\mu(I)$ denotes the number of generators of I.

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- Elias (2008) Let $I \subseteq \mathfrak{m}^b$ be an $\mathfrak{m}$-primary ideal of an one-dimensional Cohen-Macaulay ring A. Then

$$e_1(I) \leq (e_0(\mathfrak{m}) - 1)(e_0(I) - be_0(\mathfrak{m})) + e_1(\mathfrak{m}).$$

Problem 1

- Find better bounds than those obtained by Rossi-Valla (2010), Elias (2005), and give some conditions for achieving equality.
- Find better bounds than those obtained by Elias (2008) for any dimension d.

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- J. Elias, M.E. Rossi and G. Valla (1996). In the case $I = \mathfrak{m}$ and $M = A$, We can show that $e_2(\mathfrak{m}) < \frac{2}{3}e_0(\mathfrak{m})^3$.

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Problem 2

Find better bounds on $e_2(I, M)$ and give some conditions for achieving equality.

Main results

$R = \oplus_{i\geq 0} R_i$: Noetherian standard graded ring over a local Artinian ring $(R_0, \mathfrak{m}_0)$, $R_+ = \oplus_{i>0} R_i$.

Definition

Let E be a finitely generated graded R-module. We set

$$a_i(E) = \begin{cases} \max\{n \mid H^i_{R_+}(E)_n \neq 0\} & \text{if } H^i_{R_+}(E) \neq 0, \\ -\infty & \text{if } H^i_{R_+}(E) = 0, \end{cases}$$

The Castelnuovo-Mumford regularity of E.

$$\operatorname{reg}(E) := \max\{a_i(E) + i \mid i \geq 0\}.$$

- The associated graded module of M with respect to I is defined by

$$G_I(M) = \bigoplus_{n \geq 0} I_n M / I_{n+1} M.$$

- We use the following notations

$$pn(I, M) := pn(G_I(M)) = \min\{n | P_{I,M}(t) = H_{I,M}(t) \text{ for all } t \geq n\},$$

The first upper bounds on $e_1(I, M)$)

Proposition 1

Let M be a Cohen-Macaulay module and of dimension $d \geq 1$. Let b be a positive integer such that $IM \subseteq \mathfrak{m}^b M$. Then

$$e_1(I, M) \leq \binom{e_0(I, M) - b + 1}{2} + b - \ell(M/IM). \tag{1}$$

If $d = 1$ and the equality in (1) holds, then we have

(i) $a_0(G_I(M)) \leq 0$,

(ii) Either $\operatorname{reg}(G_I(M)) = pn(I, M) = e_0(I, M) - b$ or $e_0(I, M) \in \{b, b+1\}$,

(iii) $H_{I,M}(n) = b + n$ for all $1 \leq n \leq pn(I, M) - 1$.

Proposition 2

If $d = 1$ and $e_0(I, M) > e_0(\mathfrak{m}^b, M)$, then

$$e_1(I, M) \leq \binom{e_0(I, M) - b}{2} + b + 1 - \ell(M/IM). \qquad (2)$$

If the equality in (2) holds, then we have

(i') $a_0(G_I(M)) \leq 0$,

(ii') Either $\operatorname{reg}(G_I(M)) = pn(I, M) = e_0(I, M) - b - 1$ or $e_0(I, M) \in \{b + 1, b + 2\}$,

(iii') $H_{I,M}(n) = n + b + 1$ for all $1 \leq n \leq pn(I, M) - 1$.

Proposition 3

Let M be an one-dimensional Cohen-Macaulay A-module and I an $\mathfrak{m}$-primary ideal. Let b be the largest positive integer such that $IM \subseteq \mathfrak{m}^b M$. Assume that $e_0(I, M) \geq b + 2$. Then the following conditions are equivalent:

(i) $e_1(I, M) = \binom{e_0(I,M)-b+1}{2} + b - \ell(M/IM)$,

(ii) $HP_{I,M}(z) = \frac{\ell(M/IM)+(b+1-\ell(M/IM))z+\sum_{i=2}^{e_0(I,M)-b} z^i}{1-z}$,

(iii) $a_0(G_I(M)) \leq 0$ and $\mathrm{reg}(G_I(M)) = e_0(I, M) - b$,

(iv) $\mathrm{reg}(G_I(M)) = \binom{e_0(I,M)-b+2}{2} + b - e_1(I, M) - \ell(M/IM) - 1$.

If one of the above conditions is satisfied, then $b = 1$ and $e_0(I, M) = e_0(\mathfrak{m}, M)$.

Proposition 4

Let M be an one-dimensional Cohen-Macaulay A-module and I an $\mathfrak{m}$-primary ideal such that $I \subseteq \mathfrak{m}^b$, $e_0(I, M) > e_0(\mathfrak{m}^b, M)$ and $e_0(I, M) \geq b + 3$, where b is a positive integer. Then the following conditions are equivalent:

(i) $e_1 = \binom{e_0(I,M)-b}{2} + b + 1 - \ell(M/IM)$,

(ii) $HP_{I,M}(z) = \frac{\ell(M/IM)+(b+2-\ell(M/IM))z+\sum_{i=2}^{e_0(I,M)-b-1} z^i}{1-z}$,

(iii) $a_0(G_I(M)) \leq 0$ and $\operatorname{reg}(G_I(M)) = e_0(I, M) - b - 1$,

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The second upper bounds on $e_1(I, M))$

Elias (2008) Let $I \subseteq \mathfrak{m}^b$ be an $\mathfrak{m}$-primary ideal of an one-dimensional Cohen-Macaulay ring A. Then

$$e_1(I) \leq (e_0(\mathfrak{m}) - 1)(e_0(I) - be_0(\mathfrak{m})) + e_1(\mathfrak{m}).$$

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Modifying the bound in the above result, we can give a new bound on $e_1(I)$ for any dimension.

Theorem 5

Let A be a Cohen-Macaulay ring of dimension $d \geq 1$. Let $I \subseteq \mathfrak{m}^b$ be an $\mathfrak{m}$-primary ideal, where $b \geq 1$. Then

$$e_1(I) \leq \frac{1}{2b-1}\binom{e_0(I) - b + 1}{2} - \binom{\mu(m) - d}{2}.$$

Problem 2: (Upper bounds on $e_2(I, M)$)

Theorem 6

Let M be a Cohen-Macaulay module of $\dim(M) = d \geq 2$ over $(A, \mathfrak{m})$. Let I be an $\mathfrak{m}$-primary ideal such that $IM \subseteq \mathfrak{m}^b M$, where b is a positive integer. Then

$$e_2(I, M) \leq \binom{e_0(I, M) - b + 1}{3} \left(< \frac{1}{6} e_0(I, M)^3\right).$$

- By results of Kirby-Mehran (1982) we can show that $e_2(I, M) < \frac{1}{8} e_0(I, M)^4$.

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Theorem 7

Let M be a Cohen-Macaulay module of $\dim(M) = d \geq 2$ over $(A, \mathfrak{m})$ and I an $\mathfrak{m}$-primary ideal. Let b be the largest integer such that $IM \subseteq \mathfrak{m}^b M$. Assume that $e_0(I, M) \geq b + 2$. The following conditions are equivalent:

(i) $e_2(I, M) = \binom{e_0(I,M)-b+1}{3}$,

(ii) $HP_{I,M}(z) = \frac{\ell(M/IM)+(1+b-\ell(M/IM))z+\sum_{i=2}^{e_0(I,M)-b} z^i}{(1-z)^d}$,

(iii) $\operatorname{depth}(G_I(M)) \geq d-1$ and $e_1(I, M) = \binom{e_0(I,M)-b+1}{2} + b - \ell(M/IM)$,

(iv) $\operatorname{depth}(G_I(M)) \geq d-1$, $\operatorname{reg}(G_I(M)) = e_0(I) - b$ and $a_{d-1}(G_I(M)) \leq 1 - d$,

(v) $\operatorname{depth}(G_I(M)) \geq d-1$ and $\operatorname{reg}(G_I(M)) = \binom{e_0(I,M)-b+2}{2} + b - e_1(I, M) - \ell(M/IM) - 1$.

If one of the above conditions holds, then $b = 1$.

$M = A$: Cohen-Macaulay rings

For the case $M = A$, using the bound of Theorem 5, we can give a better bound in the case $b \geq 2$. We need some more preparation.

Definition

- The ideal $J \subseteq I$ is called an *M-reduction* of I if $I^{n+1}M = JI^nM$ for all $n \gg 0$.
- The number:
$$r_J(I, M) = \min\{n \geq 0 |\ I^{n+1}M = JI^nM\}$$
is called the *M-reduction number of I with respect to J*.
- An M-reduction of I is called *minimal* if it does not strictly contain another M-reduction of I.
- The number
$$r(I, M) := \min\{r_J(I, M) |\ J \text{ is a minimal } M\text{-reduction of } I\}$$
is called the *M-reduction number* of I.

Some relationships between the reduction number and Hilbert coefficients:

Lemma 8

Let M be an one-dimensional Cohen-Macaulay module and I an $\mathfrak{m}$-primary ideal such that $IM \subseteq \mathfrak{m}^b M$ for some positive integer b. Then

$$r(I, M) \leq e_0(I, M) - b.$$

Lemma 9

Let M be an one-dimensional Cohen-Macaulay module and I an $\mathfrak{m}$-primary ideal. Then

$$e_2(I, M) \leq (r'(I, M) - 1)e_1(I, M),$$

where we set $r'(I, M) := \max\{1, r(I, M)\}$.

Using the above two lemmas, we can give a new bound on $e_2(I, M)$.

Lemma 10

Let M be a Cohen-Macaulay module of dimension $d \geq 2$ and I an $\mathfrak{m}$-primary ideal such that $IM \subseteq \mathfrak{m}^b M$ for some positive integer b. Assume that $e_0(I, M) \geq b + 1$. Then

$$e_2(I, M) \leq (e_0(I, M) - b - 1)e_1(I, M).$$

Theorem 5

Let A be a Cohen-Macaulay ring of dimension $d \geq 1$. Let $I \subseteq \mathfrak{m}^b$ be an $\mathfrak{m}$-primary ideal, where $b \geq 1$. Then

$$e_1(I) \leq \frac{1}{2b-1}\binom{e_0(I)-b+1}{2} - \binom{\mu(m)-d}{2}.$$

Combining the above result with Theorem 5 we can give a better bound in the case $b \geq 2$.

Theorem 11

Let I be an $\mathfrak{m}$-primary ideal of a Cohen-Macaulay ring $(A, \mathfrak{m})$ of dimension $d \geq 2$ and such that $I \subseteq \mathfrak{m}^b$ for some positive integer b. Assume that $e_0(I, M) \geq b+1$. Then

$$e_2(I) \leq \frac{3}{2b-1}\binom{e_0(I)-b+1}{3} - (e_0(I)-b-1)\binom{\mu(\mathfrak{m})-d}{2}.$$

THANK TO YOUR ATTENTION!