Extreme events in deterministic dynamics with application to rogue waves

Giovanni Dematteis
(with Tobias Grafke, Miguel Onorato, and Eric Vanden-Eijnden)

ICTP School/Workshop on Wave Dynamics:
Turbulent vs Integrable Effects

Trieste, Aug 28, 2023

Outline

- Introduction
- Large Deviation Theory: fundamentals
- Computation of instantons in wave systems
- Experimental instantons in a wave flume
- Rogue waves in a real ocean?
- Outlook: future directions and applications

Outline

- **Introduction**
- Large Deviation Theory: fundamentals
- Computation of instantons in wave systems
- Experimental instantons in a wave flume
- Rogue waves in a real ocean?
- Outlook: future directions and applications

What are rogue waves?

Photograph of the Stolt Surf tanker as it met a rogue wave in 1977. The tanker deck, that was submerged, is 25 m above sea level (from the New York Times).

only anecdotal evidence until a few decades ago

In art

The Great Wave off Kanagawa, Katsushika Hokusai (c. 1830)

In history

(A chronology of freak wave encounters. PC Liu - Geofizika, 2007)

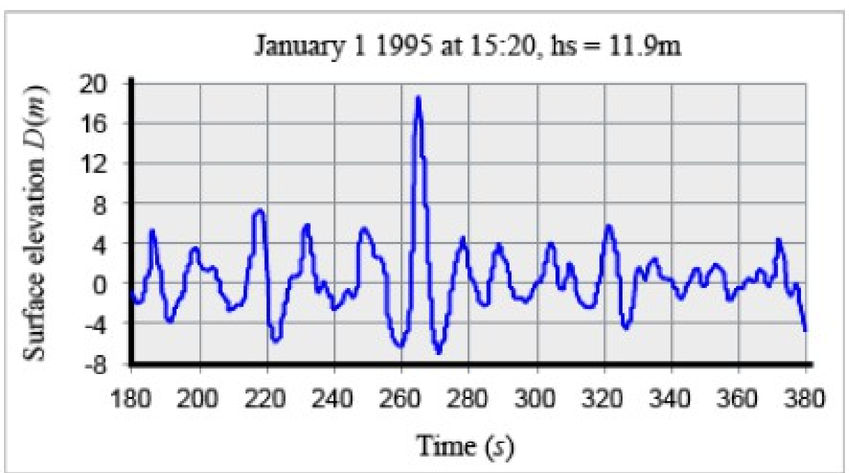
- Starting from “mythological” accounts about Columbus' fleet and Henry VIII's favourite ship Mary Rose which sank on July 19, 1545 (possibly due to a rogue wave and a sudden breeze causing her to capsize), there are tens of documented episodes
- Until recently often dismissed as seafarer lore

Rogue Waves

Definition:
Waves whose height from crest to trough exceeds twice the significant wave height H_s
(4 times the standard deviation of the surface elevation)

Photo by M. Onorato, Southern Ocean. July 3, 2017

Unpredictable and dangerous:
represent a serious threat to boats and naval structures



Time series of the Draupner wave, recorded in the North Sea (Jan 1, 1995)

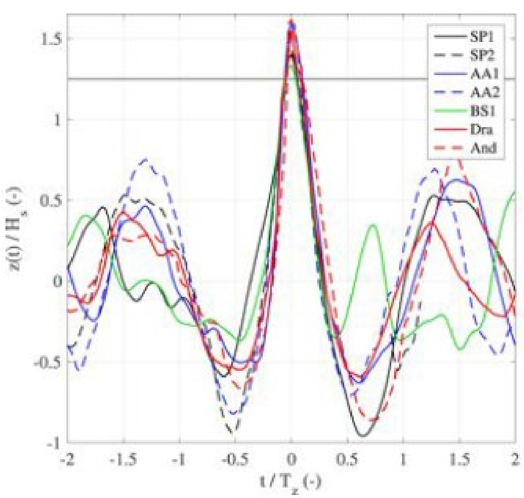
2006-2010: 78 rogue waves (all with either damage or human losses).
I Nikolkina, I Didenkulova, nat. hazards earth. syst. sci. (2011)

Mechanisms of emergence:
Two plausible scenarios have emerged over the years, **linear superposition** and **nonlinear focusing**

Two limiting theories for extreme waves

Linear (dispersive) theory

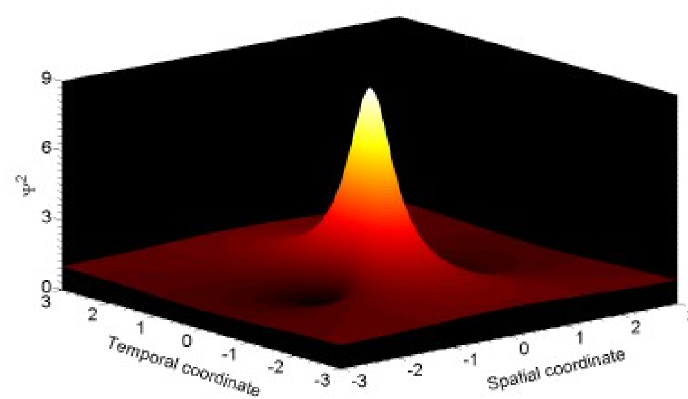
- **“Quasi-determinism” for Random Gaussian fields**: Lindgren (’70s), Boccotti (’80s)
- **Larger waves** have the shape of the **covariance** of the wave field
- **Gaussian PDF tails**: about 1/3000 waves is “rogue”



Comparing some of the available rogue wave records, from *Benetazzo A. et al., Scientific Reports 7.1 (2017): 8276.*

Semi-classical theory of NLS

- Benjamin-Feir modulational instability: Onorato, Chabchoub, etc. (2000's on)
- Theorem in Bertola & Tovbis, CPAM 66.5 (2013): large bumps tend nonlinearly to a **local Peregrine soliton**
- **Non-gaussianity** and increased probability of tail events



Peregrine soliton evolution

Looking for suitable theory

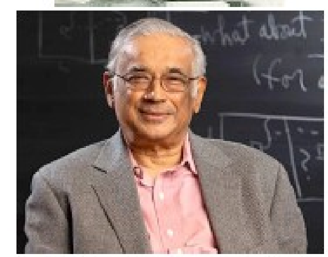
- Intrinsic randomness: need for a **statistical approach**

- incorporate the **effective dynamics** (e.g. dispersive vs nonlinear effects)

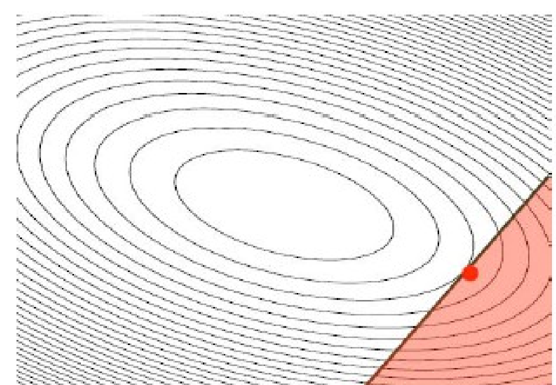
- Must be suitable for the description of **events in the distribution tails**, beyond the Central Limit Theorem descripton

Large Deviation Theory (LDT)

- Cramer (pioneering results, ‘30s). Varadhan, Gärtner, Ellis, etc. (formalization, from the ’60s on)
- Branch of **probability** theory that deals with the exponential decay of the probability of the tail events (**rare/extreme events**) — while the **central limit theorem** concerns **typical events**



- Motto: rare events are **predictable** in that they occur with high probability by the least unlikely scenario for them to happen
- **Rate function**: estimate of the probability of the tail events by knowledge of the most likely conditional realizations: **INSTANTONS**
—
deterministic optimization problem tractable in high dimensions



Some notable applications

- Statistical mechanics

  The entire theory can be naturally formalized in terms of LDT

  Touchette U, The large deviation approach to statistical mechanics, Phys. Rep. 2009

- Stochastic processes

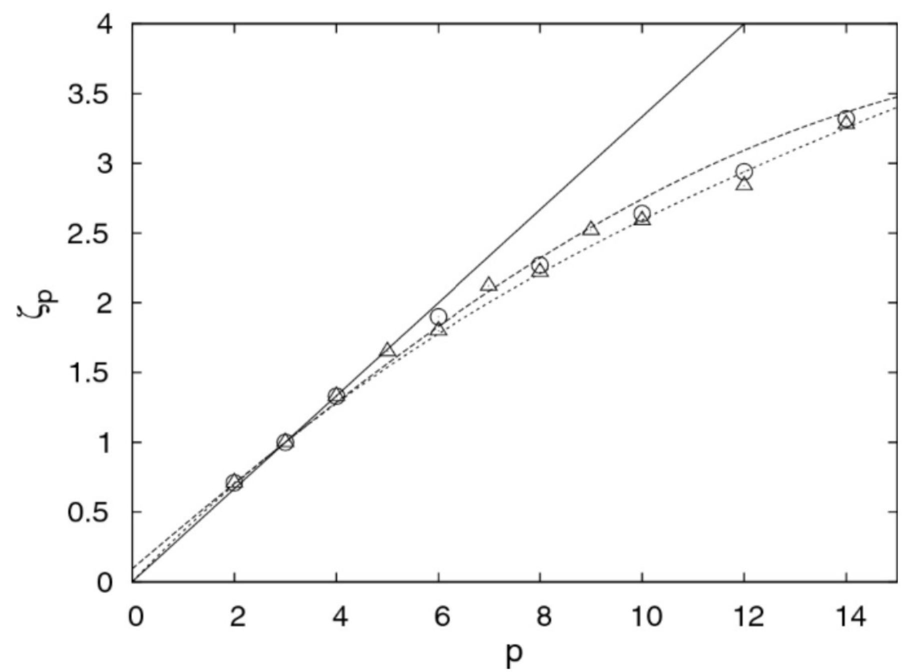
  Freidlin-Wentzell LDT for stochastic differential equations in the limit of small noise

  Freidlin MI, and Wentzell AD, Random perturbations of dynamical systems, Springer 1998

- *Turbulence "multifractality"*

  Frisch U, and Parisi G, Fully developed turbulence and intermittency, 1980

  Benzi R, and Vulpiani A, Multifractal approach to fully developed turbulence, 2022



Outline

- Introduction
- **Large Deviation Theory: fundamentals**
- Computation of instantons in wave systems
- Experimental instantons in a wave flume
- Rogue waves in a real ocean?
- Outlook: future directions and applications

Central Limit Theorem (CLT)

$$S_N = \sum_{n=1}^{N} X_n, \quad \text{with } X_n \text{ i.i.d. RVs:} \quad \mathbb{E}(X_n) = \mu, \quad \mathrm{var}(X_n) = \sigma^2$$

$$\text{Define observable:} \quad Z_n = \frac{S_n - N\mu}{\sqrt{N}\sigma}$$

CLT: $Z_N \xrightarrow{D} Z \sim \mathcal{N}(0,1), \quad \text{as } N \to \infty$

Sketch of proof. Characteristic function:

$$\varphi_{Z_N}(T) = \mathbb{E}e^{itZ_N} = \prod_{n=1}^{N} \mathbb{E}e^{i\frac{t}{\sqrt{N}}Y_n}$$

[MacLaurin exp. as $N \gg 1$]

$$= \prod_{n=1}^{N}(1 + \frac{t^2}{2N} + O(N^{-\frac{3}{2}})) \simeq (e^{-\frac{t^2}{2N}})^N = e^{-\frac{t^2}{2}}$$

Characteristic function of $\mathcal{N}(0,1)$

where $Y_n = \frac{X_n - \mu}{\sigma}, \quad \mathbb{E}(Y_n) = 0, \quad \mathrm{var}(Y_n) = 1$

Comments on CLT

- If X_n Gaussian RV $\Rightarrow$ CLT is exact (all moments in higher-order terms are vanishing)
- In general, expansion good up to $O(1)$ fluctuation of Y_n, i.e. $O(\sqrt{N}\sigma)$ fluctuation of S_N (1 std)
- Try to evaluate e.g. $\mathbb{P}(S_N \geq N\mu + N\sigma)$, fluctuations from the mean much larger than the 1std typical fluctuation. CLT provides NO ANSWER!
- First result in LDT was born to answer specifically this question: **Cramér's Theorem**

Cramér Theorem

Define $\quad P(z) := \mathbb{P}(\frac{s_N}{N} - \mu \geq z) \qquad [1 - \mathrm{CDF}, \quad \mathrm{PDF} = -\frac{dP(z)}{dz}]$

$$z \sim \frac{\sigma}{\sqrt{N}} \text{ typical fluctuation (CLT)}$$

Cramér Theorem: $\quad P(z) \asymp e^{-NI(z)}$

$$I(z) = \sup_{\lambda}(\lambda z - S(\lambda)) \quad [I(z): \text{ rate function}]$$

where $\quad S(\lambda) := \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} e^{\lambda S_N} \qquad$ [scaled cumulant generating function]

Here, $\quad f_N \asymp g_N \Leftrightarrow \lim_{N\to\infty} \frac{\log f_N}{\log g_N} = 1 \qquad [\log -\text{equivalence}]$

Given the measure of X_n, the procedure is:

- Calculate $S(\lambda)$ from definition
- Calculate $I(z)$ from Legendre transform of $S(\lambda)$
- obtain LDT approximation of $P(z)$, for all values of z also in the tail, i.e. leading exponential decay, up to power-law correction not seen by "$\asymp$"

Large Deviation Principle

Assume we already know tail distribution: $f(z) \asymp e^{-NI(z)}$

$$e^{NS(\lambda)} = e^{N \log \mathbb{E} e^{\lambda z}} = \mathbb{E} e^{N\lambda z} \sim \int e^{N\lambda z} e^{-NI(z)} dz$$

$$\sim e^{N \sup_z (\lambda z - I(z))}, \quad \text{as } N \gg 1 \qquad \text{[Laplace method]}$$

$$\Rightarrow S(\lambda) = \sup_z (\lambda z - I(z)) \qquad \text{[Legendre transform]}$$

If $S(\lambda)$ exists, then it is convex. Legendre transform can be inverted to obtain:

$$I(z) = \sup_\lambda (\lambda z - S(\lambda)), \quad P(z) \asymp e^{-NI(z)} \text{ as } N \gg 1 \quad \text{[Large Deviation Principle (LDP)]}$$

Theorems to prove the existence of an LDP:

- **Gärtner-Ellis:** Given $S(\lambda)$, guarantees that $I(z)$ exists
- **Varadhan:** Given $I(z)$, guarantees that $S(\lambda)$ exists
- **Contraction Principle:** Given $I(z)$ with an LDP for Z, guarantees the existence of LDP for $Y = g(z)$ with rate function $I'(y) = \inf_{z:y=g(z)} I(z)$

Dominating point "LDP"

Assume we already know tail distribution: $f(z) \asymp e^{-I(z)}$, for large z

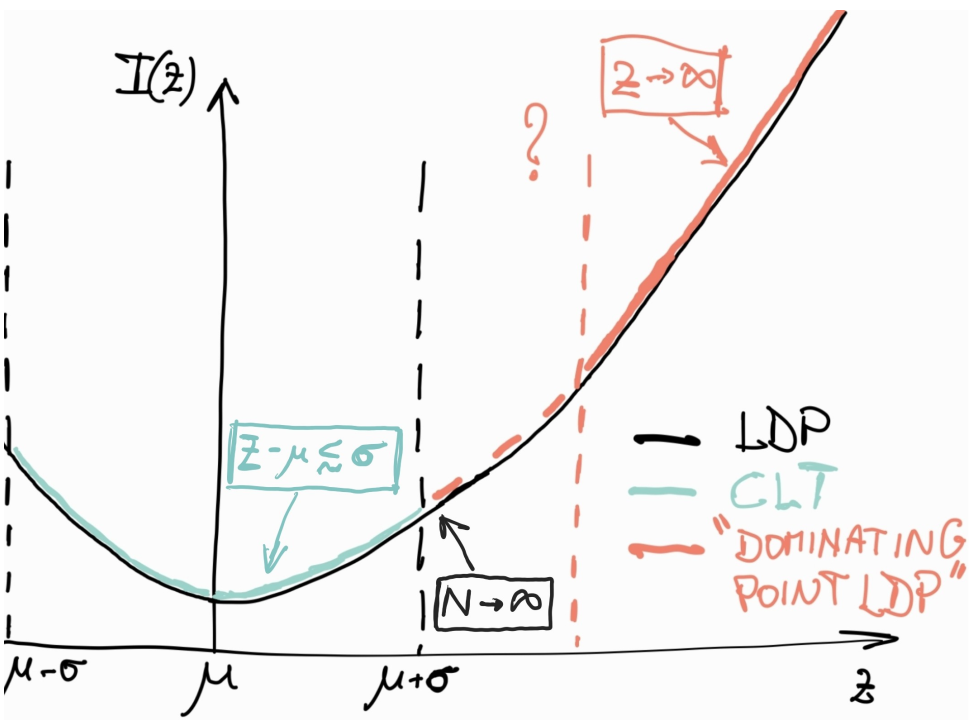
$$e^{S(\lambda)} = e^{\log \mathbb{E} e^{\lambda z}} = \mathbb{E} e^{\lambda z} \sim \int e^{\lambda z} e^{-I(z)} dz$$

$$\sim e^{\sup_z (\lambda z - I(z))}, \quad \text{as } \lambda \gg 1 \qquad \text{[Laplace method]}$$

$$\Rightarrow S(\lambda) = \sup_z (\lambda z - I(z)) \qquad \text{[Legendre transform]}$$

If $S(\lambda)$ exists, then it is convex. Legendre transform can be inverted to obtain:

$$I(z) = \sup_\lambda (\lambda z - S(\lambda)), \quad P(z) \asymp e^{-I(z)} \text{ as } z \gg 1 \qquad \text{[Large Deviation Principle (LDP)]}$$



- "Dominating-point" regime starts for "large-enough" z
- In the tail, same rate function as large N LDP, but different limit: no uniform convergence for any z
- Works of Ney, Iltis and others ('80s and '90s)
- *DG, Grafke T, and Vanden-Eijnden E, SIAM/ASA Journal of Uncertainty Quantification, 2019*

Outline

- Introduction
- Large Deviation Theory: fundamentals
- **Computation of instantons in wave systems**
- Experimental instantons in a wave flume
- Rogue waves in a real ocean?
- Outlook: future directions and applications

LDT for deterministic dynamics with random initial data

- Field u with **deterministic evolution equation** $\partial_t u = b(u)$
- **Random initial condition**:

$$u_0 \quad \text{with measure} \quad d\mu(u_0) \propto \exp(-I(u_0))du_0, u_0 \in \Omega \subset \mathbb{R}^N$$

- "**Observable**" function on field at final time T: $f(u(T))$. Being the evolution deterministic, will depend on the initial field via $F_T(u_0) = f(u(T))$ (deterministic mapping, in general nonlinear and very complicated!)

Consider the **tail distribution** of $F_T(u_0)$ (e.g. can estimate via Monte-Carlo)

$$P(z) = \mathbb{P}(f(u(T)) \geq z) = \mathbb{P}(F_T(u_0) \geq z), \quad \text{for large } z$$

Define the **rate function**

$$I(u_0^\star(z)) = \min_{\Omega(z)} I(u_0)\,,\ \Omega(z) = \{u_0 \in \Omega : F_T(u_0) \geq z\}$$

Can prove (**Theorem (LDP)**): $\quad P(z) \asymp \exp(-I(u_0^\star(z)))$

Note: will use θ or u_0 indistinctly for the initial conditions

In practice: constrained optimization

high-dimensional constrained minimization to find $u_0^\star(z)$

- Equivalent to the **unconstrained minimization** of the **cost function**

$$\min_{u_0 \in \Omega} E_T(\lambda, u_0), \quad E_T = I_T(u_0) - \lambda F_T(u_0)$$

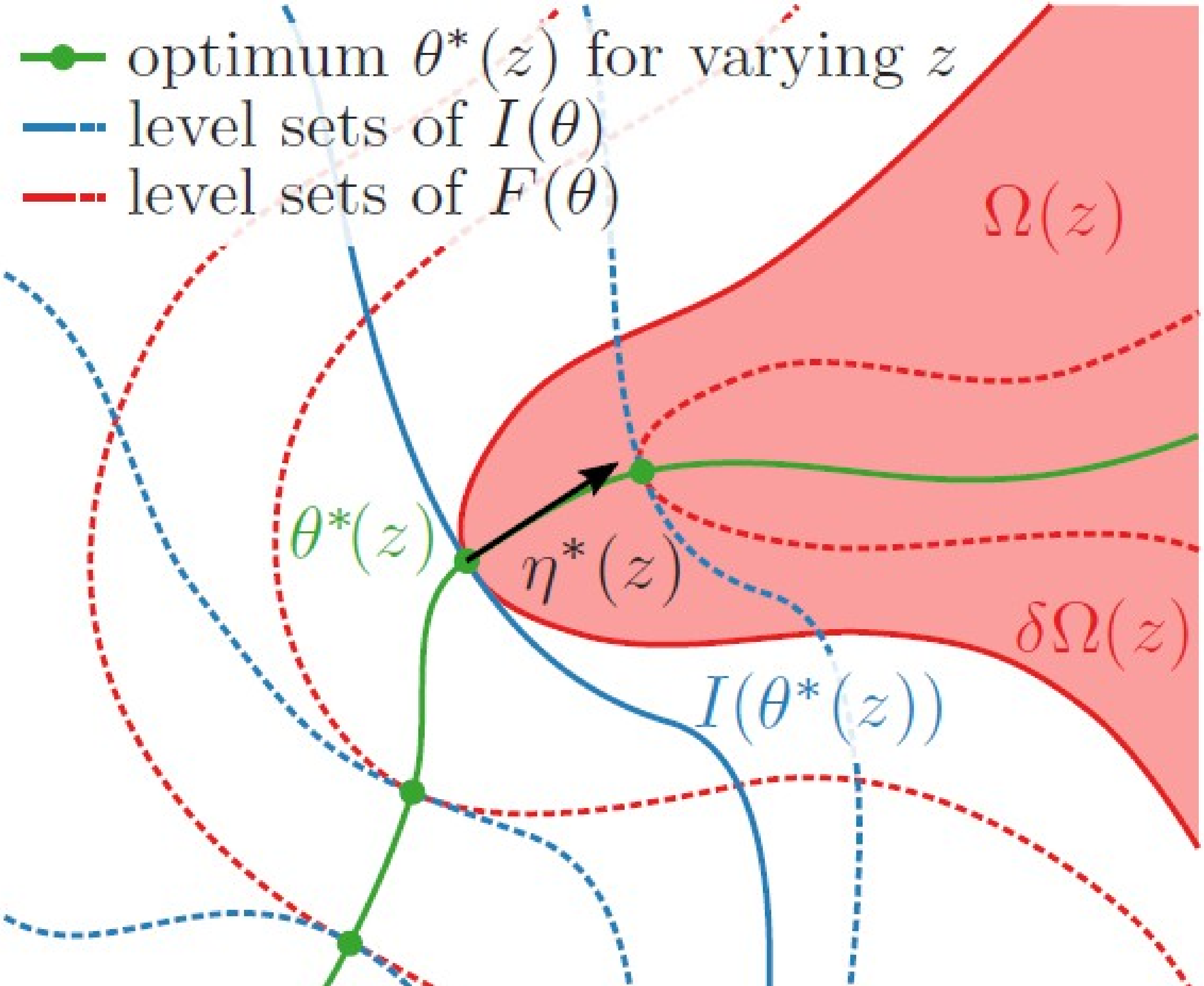
where λ is a Lagrange multiplier used to implement the constraint $F_T(u_0) \geq z$

- Optimizer found by condition

$$\nabla_{u_0} E_T = 0 \Rightarrow u_0^\star(\lambda) \Rightarrow u_0^\star(z), \quad \text{using} \quad \lambda(z) : F_T(u_0^\star(\lambda)) = z$$

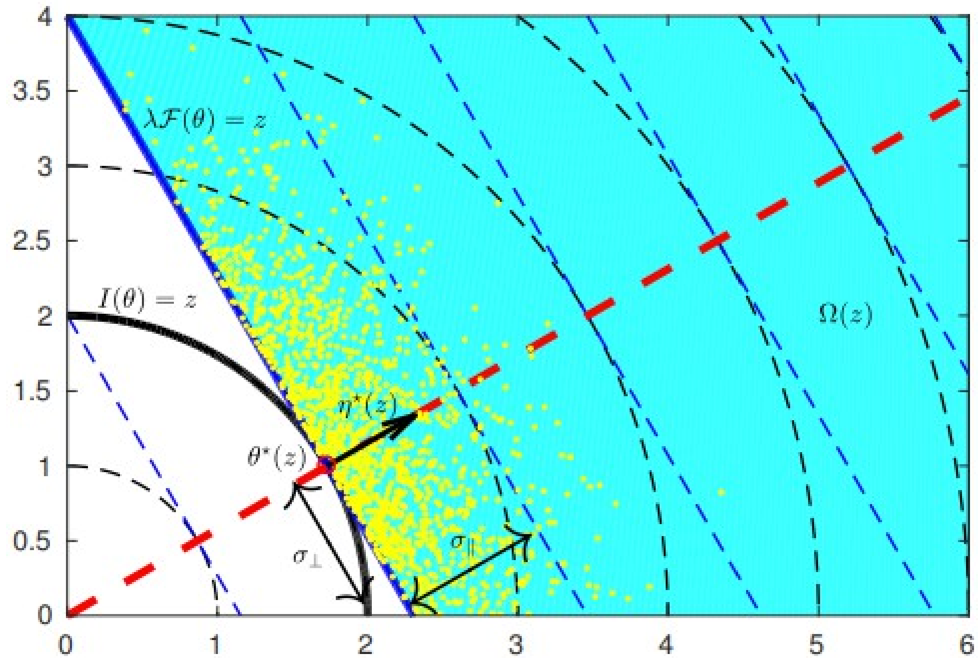
Note: we will call u_0* the **instanton** of the problem, borrowing the terminology from field theory (instanton = minimizer of action in path-integral formalism). See also works by Falkovich and others in the '90s in the theory of turbulence

Schematic picture of the LDP



Example: normal distribution with linear observable

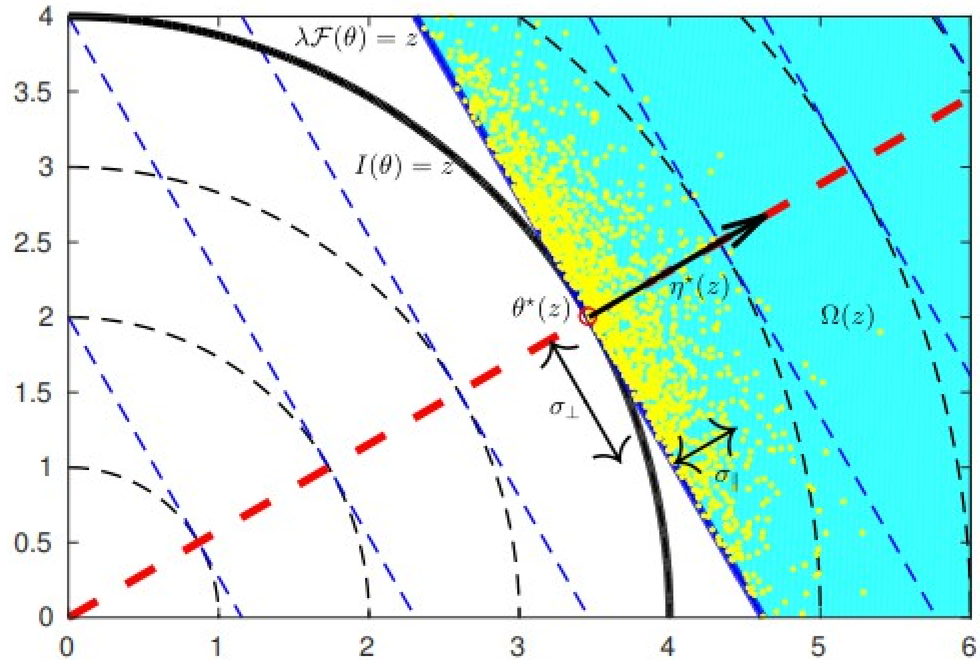
- $(X, Y) \sim \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$, $\theta = (x, y) \in \mathbb{R}^2$, i.e. $I(\theta) = \frac{x^2+y^2}{2}$
- Mapping $F(\theta) = b \cdot (x, y)$, $b = (\frac{\sqrt{3}}{2}, \frac{1}{2})$
- $\theta^\star(z) = zb$, $\eta^\star(z) = \nabla I(\theta)|_{\theta^\star(z)} = zb$
- LDP: $P(z) \asymp \exp(-\frac{z^2}{2})$ Exact: $P(z) \sim \frac{1}{z} \exp(-\frac{z^2}{2})$



Variance of the conditional event $F(\vartheta) \geq z$ decreases ($\to 0$ as $z \to \infty$) in the direction parallel to $\eta^\star(z)$: $\sigma_{\|} = O(|\eta^\star(z)|^{-1})$

Example: normal distribution with linear observable

- $(X,Y) \sim \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$, $\theta = (x,y) \in \mathbb{R}^2$, i.e. $I(\theta) = \frac{x^2+y^2}{2}$
- Mapping $F(\theta) = b \cdot (x,y)$, $b = (\frac{\sqrt{3}}{2}, \frac{1}{2})$
- $\theta^\star(z) = zb$, $\eta^\star(z) = \nabla I(\theta)|_{\theta^\star(z)} = zb$
- LDP: $P(z) \asymp \exp(-\frac{z^2}{2})$ Exact: $P(z) \sim \frac{1}{z}\exp(-\frac{z^2}{2})$



Variance of the conditional event $F(\vartheta) \geq z$ decreases ($\to 0$ as $z \to \infty$) in the direction parallel to $\eta^\star(z)$:
$\sigma_{\|} = O(|\eta^\star(z)|^{-1})$

- **Probability concentration in parallel direction: LDP**
- **Degeneracy in perpendicular directions: sub-exponential prefactor**

Wavefield w/ Gaussian independent Fourier modes, T=0

Wavefield envelope at $t = 0$: $u_0 : \left[-\frac{L}{2}, \frac{L}{2}\right] \to \mathbb{C}$, relates to surface elevation η via:

$\eta_0(x) = u_0(x) e^{i(k_0 x)}$, (first order of the Stokes series, sol. of Euler)

Periodic domain: $u(x) = \sum_{j \in \mathbb{Z}} \hat{u}_j e^{ik_j x}$, $k_j = \frac{2\pi}{L} j$

Gaussian statistics: $\hat{u}_j = a_j + ib_j$, with i.i.d. RVs $a_j, b_j \sim \mathcal{N}(0, n_j)$

n_j is the wave spectrum, relating to the space covariance $C(x)$ via:

$C(r) = \frac{1}{2}\mathbb{E}(u(x)\bar{u}(x+r)) = \sum_{j \in \mathbb{Z}} n_j e^{ik_j r}$, under spatial homogeneity

Probability exponent for LDT method:

$$\theta = \{a_j, b_j\} \qquad I(\theta) = \frac{1}{2}\frac{a_j^2 + b_j^2}{n_j}$$

Choose observable to define extreme wave:

$F(\theta) = |u_0(x = 0)|$ (because of translational invariance, x=0 general)

Analytical optimization

$$I(\theta) = \frac{1}{2}\frac{a_j^2 + b_j^2}{n_j},$$

$$F(\theta) = |u_0(x=0)|$$

$$E^\lambda(\theta) = I(\theta) - \lambda F(\theta)$$

$$\nabla_\theta E^\lambda(\theta) = 0 \Rightarrow \theta^\star(\lambda)$$

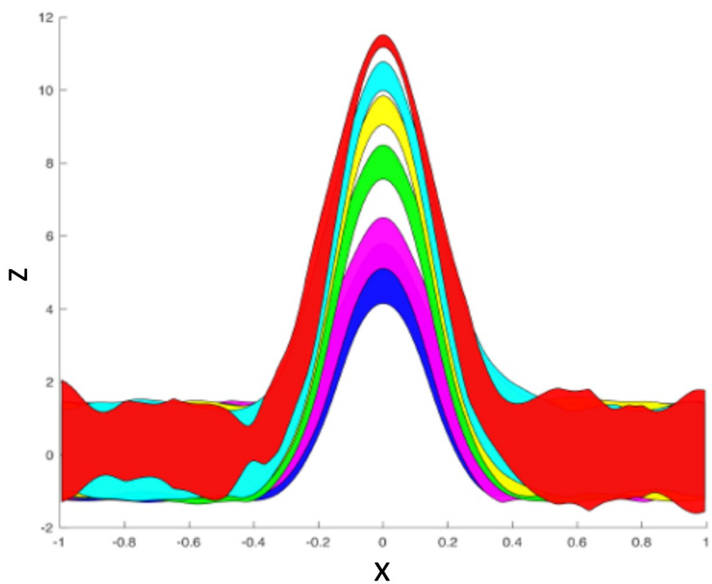
$$F(\theta^\star(\lambda)) = z \Rightarrow \lambda(z)$$

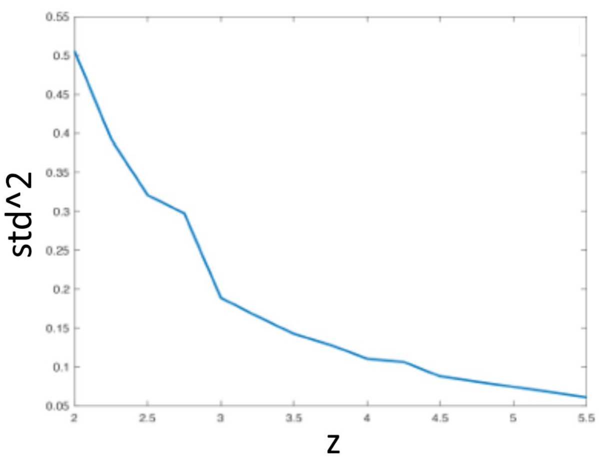
$$\theta^\star(\lambda(z)) \Rightarrow u_0^\star(z)$$

Compare with Boccotti-Lindgren linear theory:

Shape of extreme waves: covariance / Gaussian tail decay

Fun exercise: Montecarlo + post-processing (select maxima of size larger than z, translate max in x=0, and study collapse onto optimizer (instanton))





Introduce a dynamics

- Deep water, unidirectional spectrum, narrow-band around k_0
- Governing dynamics reduces to 1D Nonlinear Schroedinger Equation (NLS)

Zakharov, J. Appl. Mech. Theor. Phys. (1968)

$$i\left(\partial_t u + c_g \partial_x u\right) - \frac{\omega_0}{8k_0^2}\partial_x^2 u - \frac{1}{2}\omega_0 k_0^2 |u|^2 u = 0$$

$$\omega_0 = \sqrt{gk}, \quad c_g = \frac{d\omega}{dk}$$

Linear case

- Moving reference frame
- Small amplitude field
- u and x in units of $8^{-1/2}k_0^{-1}$ and t in units of ω_0^{-1}

$$i\left(\partial_t u + \cancel{c_g \partial_x u}\right) - \cancel{\frac{\omega_0}{8k_0^2}}\partial_x^2 u - \cancel{\frac{1}{2}\omega_0 k_0^2 |u|^2 u} = 0$$

$$\partial_t u = -i\partial_x^2 u$$

Now modes DO NOT INTERACT, preserving Gaussian statistics
Optimization at time T = Optimization at time 0

Instanton evolution, linear case

- Consider a large box
- Gaussian-shaped spectrum for simplicity
- Look for max at T=0: $u_0^*(z)$ given by covariance
- Use free-particle kernel, find instanton evolution

$$L \gg 1 \quad \Rightarrow \quad \frac{2\pi}{L}\sum_j \to \int_{\mathbb{R}} dk$$

$$n_k \propto e^{-\frac{k^2}{2\sigma_k^2}} \Rightarrow \frac{C(x)}{C(0)} = e^{-\frac{x^2}{2\sigma_X(t=0)^2}}$$

$$u(x,t) = \int_{-\infty}^{+\infty} dx'\ u(x',0)K(x,t;x',0)\,, \quad K(x,t;x',0) = \frac{1}{\sqrt{-4\pi i t}} e^{-i\frac{(x-x')^2}{4t}}$$

Initial condition: $\quad u(x,0) = z e^{i\varphi} e^{-\frac{x^2}{2\sigma_X(t=0)^2}}$

Instanton and Uncertainty Principle

$$u(x,t) = z\, e^{i\varphi} \sqrt{\frac{\sigma_x^2}{\sigma_x^2 - 2it}}\, e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2 + 4t^2\sigma_x^{-2}}}\, e^{-\frac{itx}{\sigma_x^4 + 4t^2}}$$

- $t = 0 : \quad u \to u_0$
- $|\sigma_x(t)| = \sqrt{\sigma_x^2 + 4t^2\sigma_x^{-2}}$
  min at $t = 0$ (maximal focusing), symmetric w.r.t. $t = 0$
- $|\sigma_x(t)| \geq \sigma_x = 1/\sigma_k \quad \Rightarrow \quad |\sigma_x(t)|\sigma_k \geq 1,$
  $= 1$ for the instanton at the point of maximal focusing

Instanton saturates the bound of the Uncertainty Principle at the maximal focusing point

(not physically possible to focus a wave packet further)

Nonlinear case: Numerical optimization by gradient descent

- Now use a **nonlinear dynamics**: $F_T(u_0) = f(u(T))$ is very complicated, **need for a numerical method** to perform the optimization
- Gradient descent in the cost-function landscape

$$\nabla_{u_0} E = \nabla_{u_0} I(u_0) - \lambda J^T(T, u_0) \nabla f(u(T, u_0)).$$

- The Jacobian $J(T, u_0) = \nabla_{u_0} u(T, u_0)$ evolves according to

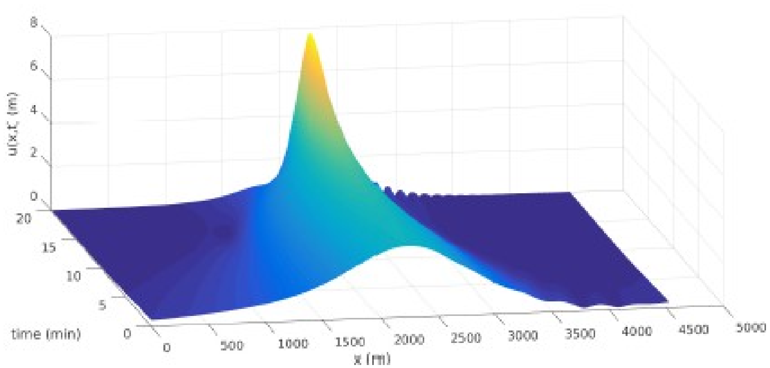
$$\partial_t J(t, u_0) = \nabla b(u(t, u_0)) J(t, u_0), \quad J(t=0, u_0) = Id$$

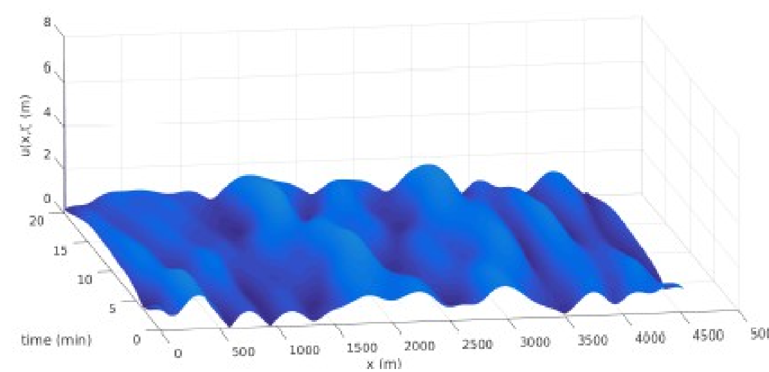
- Varying λ, we span all the values taken by z and **compute the optimizers** $u_0^\star(z)$ and the **rate function** $I(u_0^\star(z))$ by which we **estimate the probability tail** for large z (LDP)

Instanton computation

- NLS with periodic boundary conditions discretized on a grid of 2048 points
- Spectrum of initial data parametrized by 93 Fourier modes: **optimization in 185 dimensions**
- Solution evolved with pseudo-spectral exponential time differencing Runge Kutta of second order (ETDRK2) scheme
- Optimization through gradient descent with adaptive step (line search) and preconditioning of the gradient

Instantons vs standard events,
i.e. evolution from optimized/typical initial condition





The only operation performed here is to take an initial condition from optimization (left) or at random (right), and evolve the field

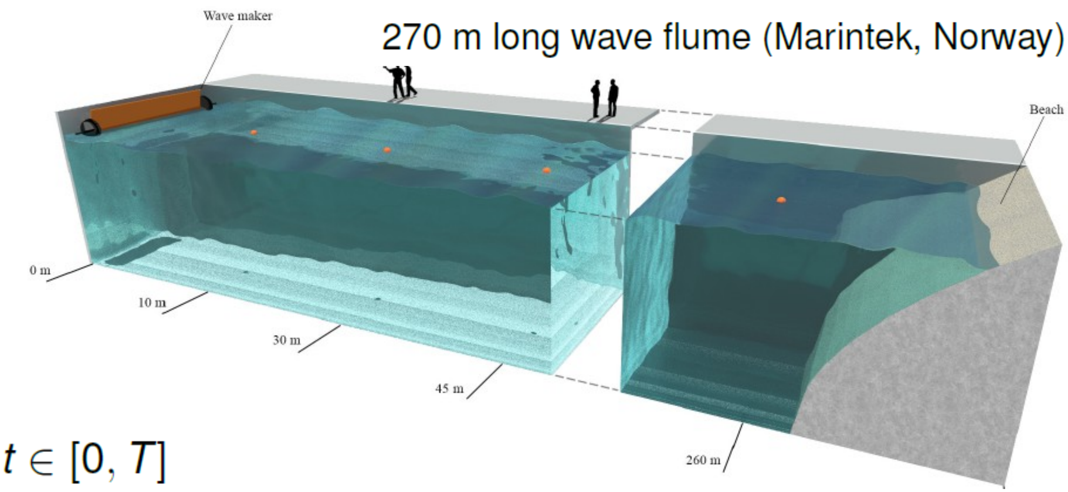
Outline

- Introduction
- Large Deviation Theory: fundamentals
- Computation of instantons in wave systems
- **Experimental instantons in a wave flume**
- Rogue waves in a real ocean?
- Outlook: future directions and applications

Experimental setup: the wave flume

Artificial wave tanks are a great test bed to mimick realistic "rescaled" sea states

270 m long wave flume (Marintek, Norway)



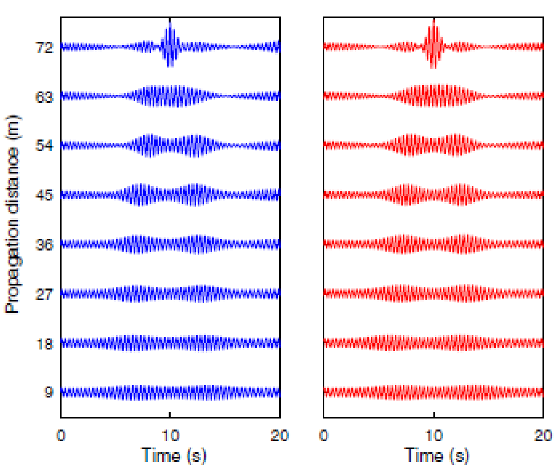
- Narrow-banded states: **<u>NLS equation</u>** is the governing equation at leading order:

$$\partial_x\psi + 2\frac{k_0}{\omega_0}\partial_t\psi + i\frac{k_0}{\omega_0^2}\partial_t^2\psi + 2ik_0^3|\psi|^2\psi = 0 \qquad t \in [0, T]$$

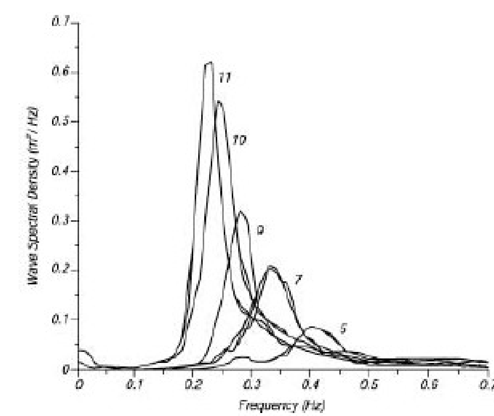
ψ: complex envelope, relating to the real surface elevation η via the Stokes series

$$\eta = |\psi|\cos(\theta) + \tfrac{1}{2}k_0|\psi|^2\cos(2\theta) + \ldots$$

where $\theta = k_0 x - \omega_0 t + \varphi$ (φ is a phase)



Generation of a rogue wave in a water tank and fit of the data using NLS, assessing the validity of the model. From Chabchoub et al, Phys. Rev. X (2012)

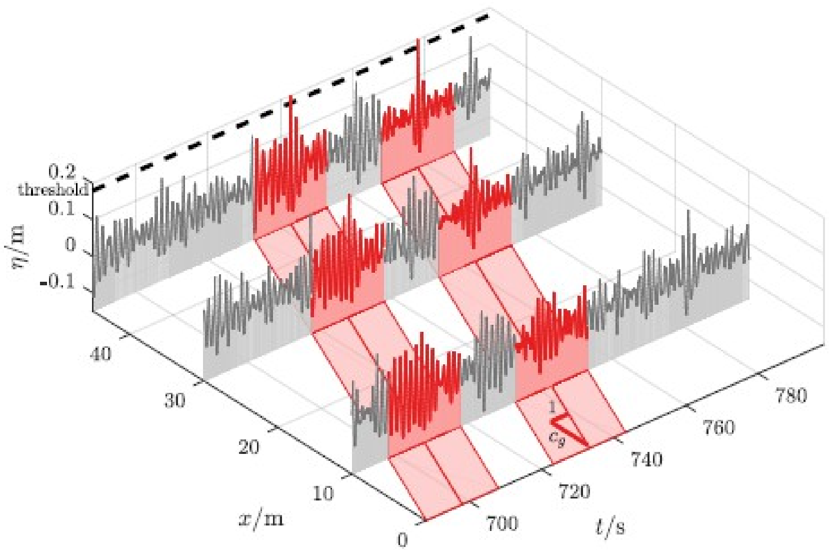


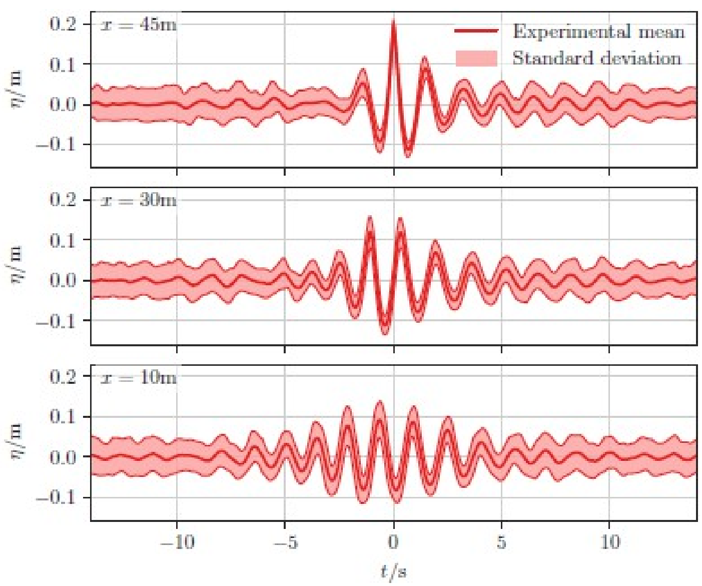
- Wave generator enforcing **<u>random initial data</u>** with Gaussian statistic and observational Jonswap spectrum

- Pick as **<u>observable</u>**: $f(\psi(L)) = \max_{t\in[0,T]}|\psi(L,t)|$

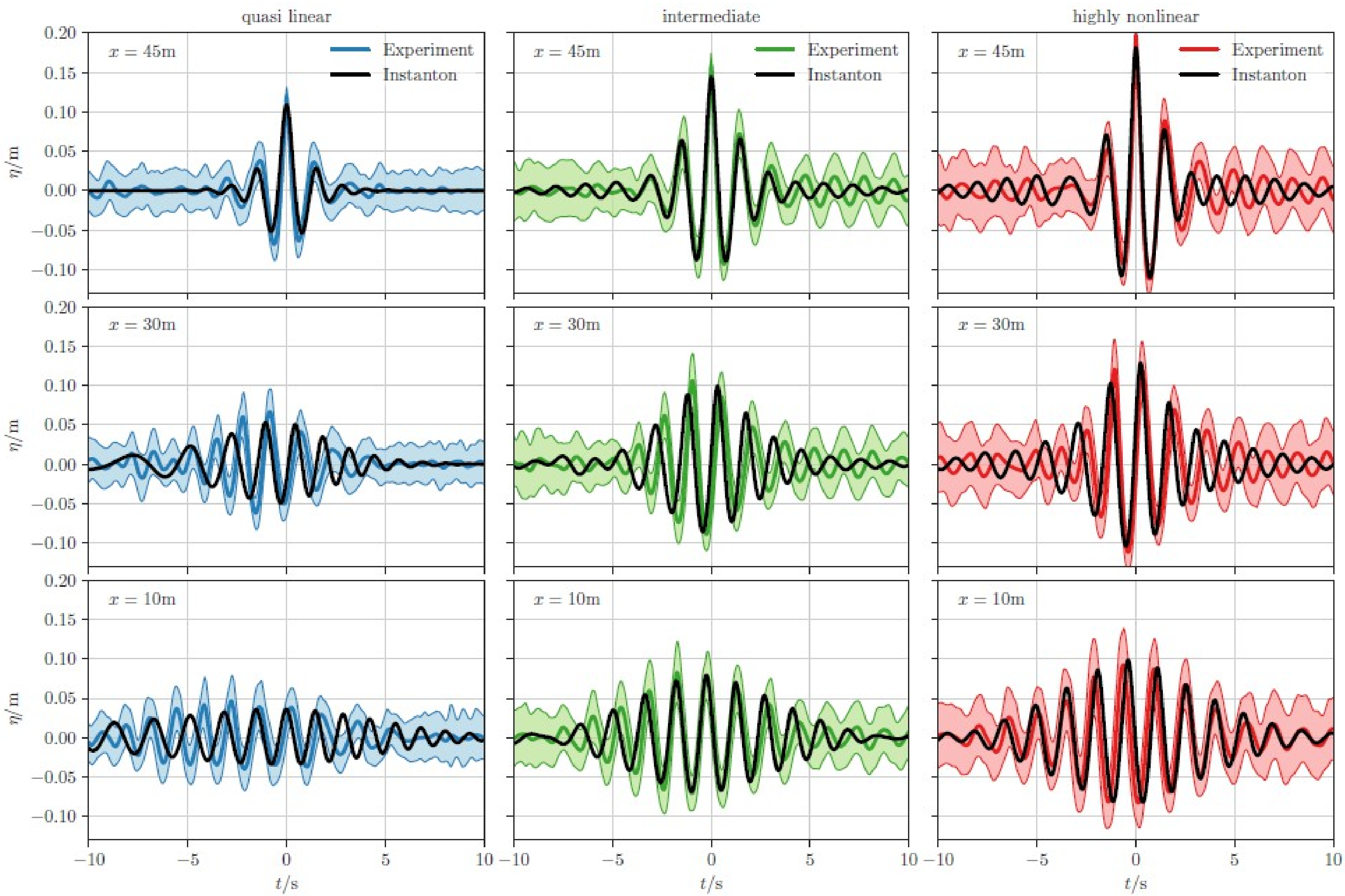
Filtering of experimental extreme events

- At fixed x along the flume, select events exceeding threshold z
- Track the wave packet backward in space with group velocity $c_g = \frac{\omega_0}{2k_0}$
- Collect extreme events (centered at $t = 0$) and their precursors
- Compute the average extreme event and the standard deviation

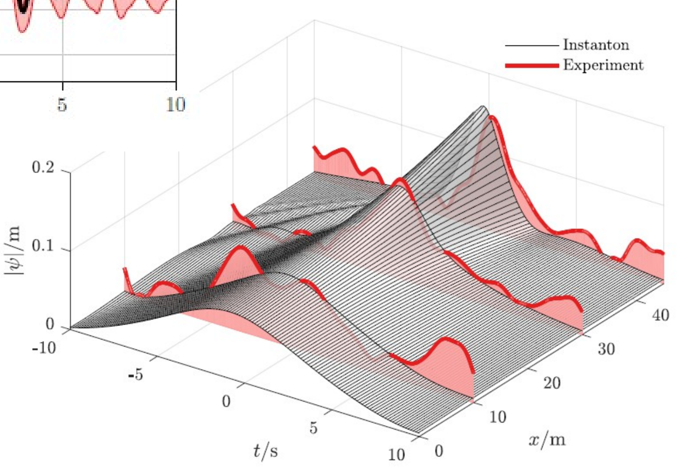




Comparison: Experiment *vs* Instanton



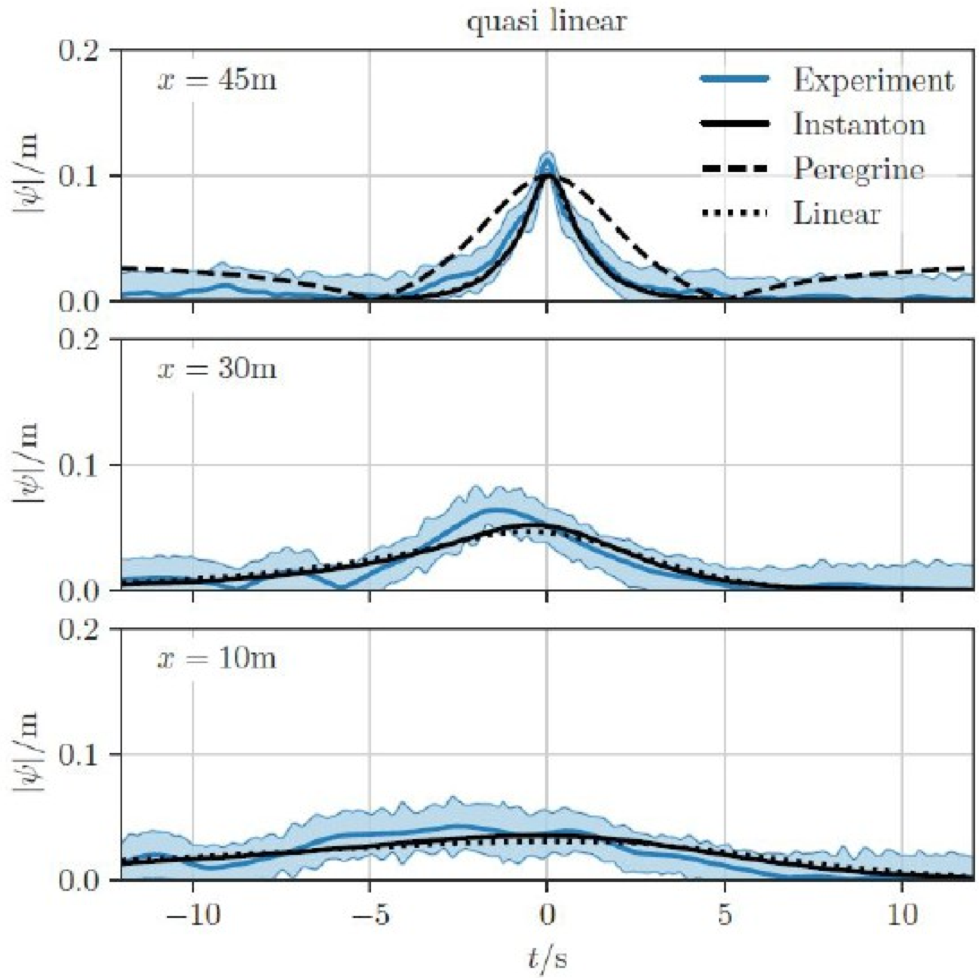
Set	γ	H_s (m)	ϵ
quasi-linear	1	0.11	0.077
intermediate	3.3	0.13	0.56
highly nonlinear	6	0.15	1.11



Experiment *vs* Instanton: quasi-linear regime

Tending to the linear regime, the surface elevation is Gaussian.

- Numerical method converges to the covariance, inverse Fourier transform of the spectrum
- Analytical solution is available: retrieving Lindgren-Boccotti result*
- Past/future history found by evolving backward/forward the (NL) Schrödinger equation

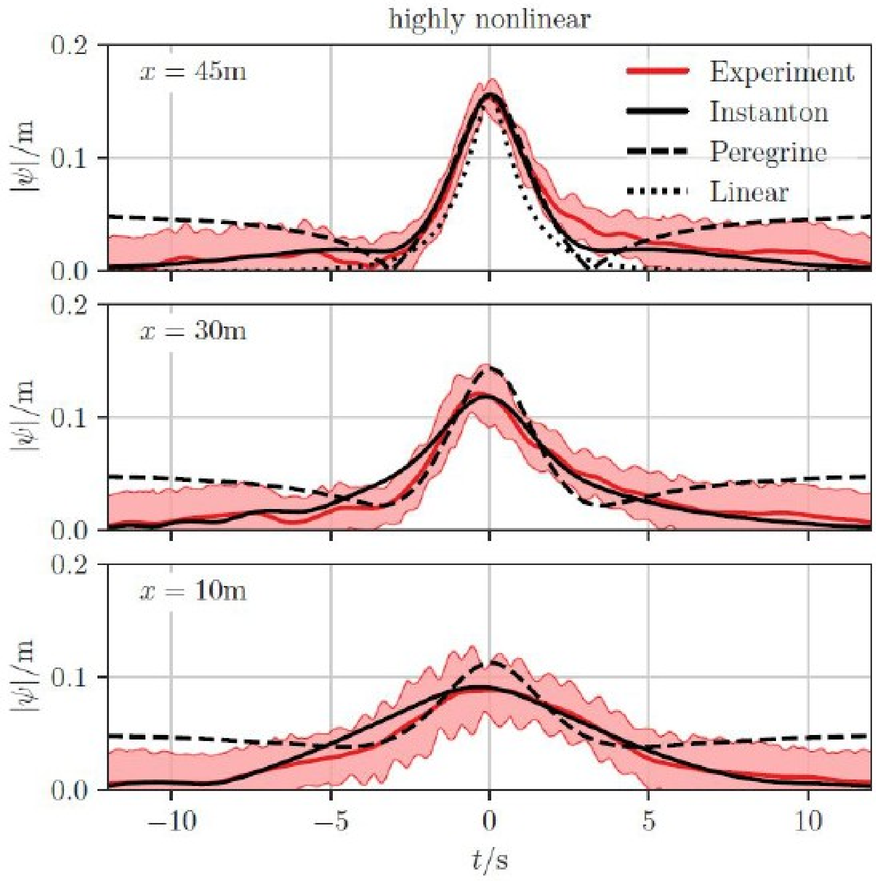


* *G Lindgren. Local maxima of Gaussian fields. In: Arkiv for matematik 10.1-2 (1972), pp. 195–218; P Boccotti. Wave mechanics for ocean engineering. Vol. 64. Elsevier (2000).*

Experiment *vs* Instanton: highly-nonlinear regime

If nonlinearity is strong, approach the semi-classical regime of NLS

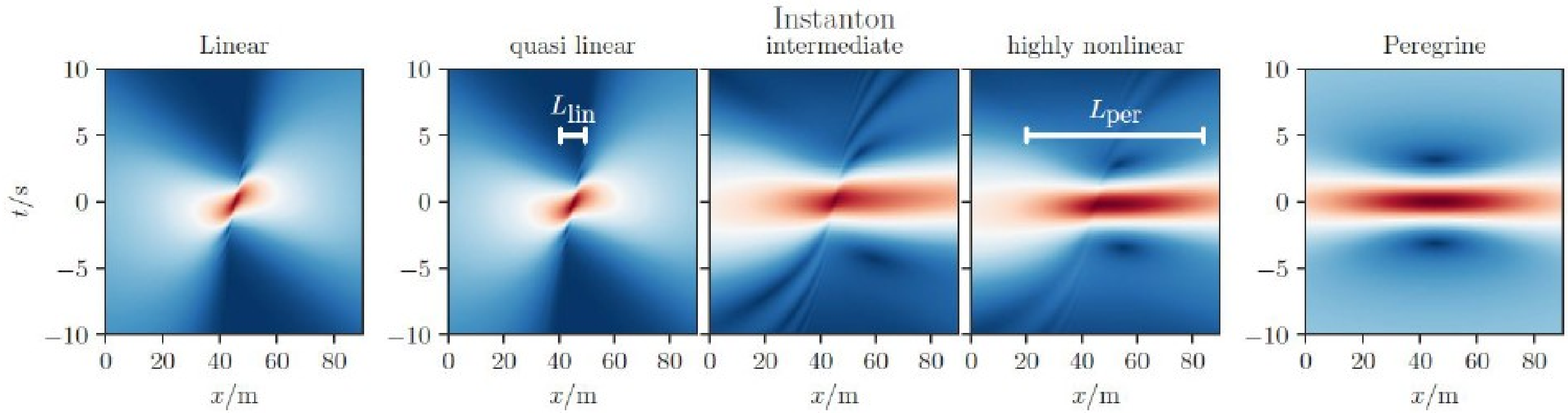
- Any single localised pulse on negligible background leads to emergence of **Peregrine soliton**†
- By scale invariance, this can be attained if peaks are large and focused enough for nonlinearity to rule over dispersion
- Envelope locally (both in space and time) converges to the Peregrine



† *M Bertola and A Tovbis. Universality for the focusing nonlinear Schrödinger equation at the gradient catastrophe point. CPAM 66.5 (2013), pp. 678–752.*

A Tikan, et al. Universality of the Peregrine soliton in the focusing dynamics of the cubic nonlinear Schrödinger equation. PRL 119, 033901 (2017).

A unifying picture of rogue waves



Quasi-linear regime

- Linear length scale: $L_{lin} = \omega_0^2/(k_0 \Delta\omega^2)$
- Wave packet $\simeq$ 9 m
- Linear superposition dominates

Highly-nonlinear regime

- Peregrine length scale: $L_{Per} = \sqrt{L_{lin} L_{nl}}$, where $L_{nl} = 8/(k_0^3 H_s^2)$ is the modulational instabiity scale
- Wave packet $\simeq$ 65 m

Instanton interpolates smoothly between the two regimes!

Intermediate regimes show mixed linear and nonlinear features which are captured by the instanton

Outline

- Introduction
- Large Deviation Theory: fundamentals
- Computation of instantons in wave systems
- Experimental instantons in a wave flume
- **Rogue waves in a real ocean?**
- Outlook: future directions and applications

Framework for the problem in deep sea

Framework shift to a **stationary deep sea** (for simplicity: still 1D, narrow-banded, with observational JONSWAP spectrum)

No well-defined initial condition like in the flume! What to do?

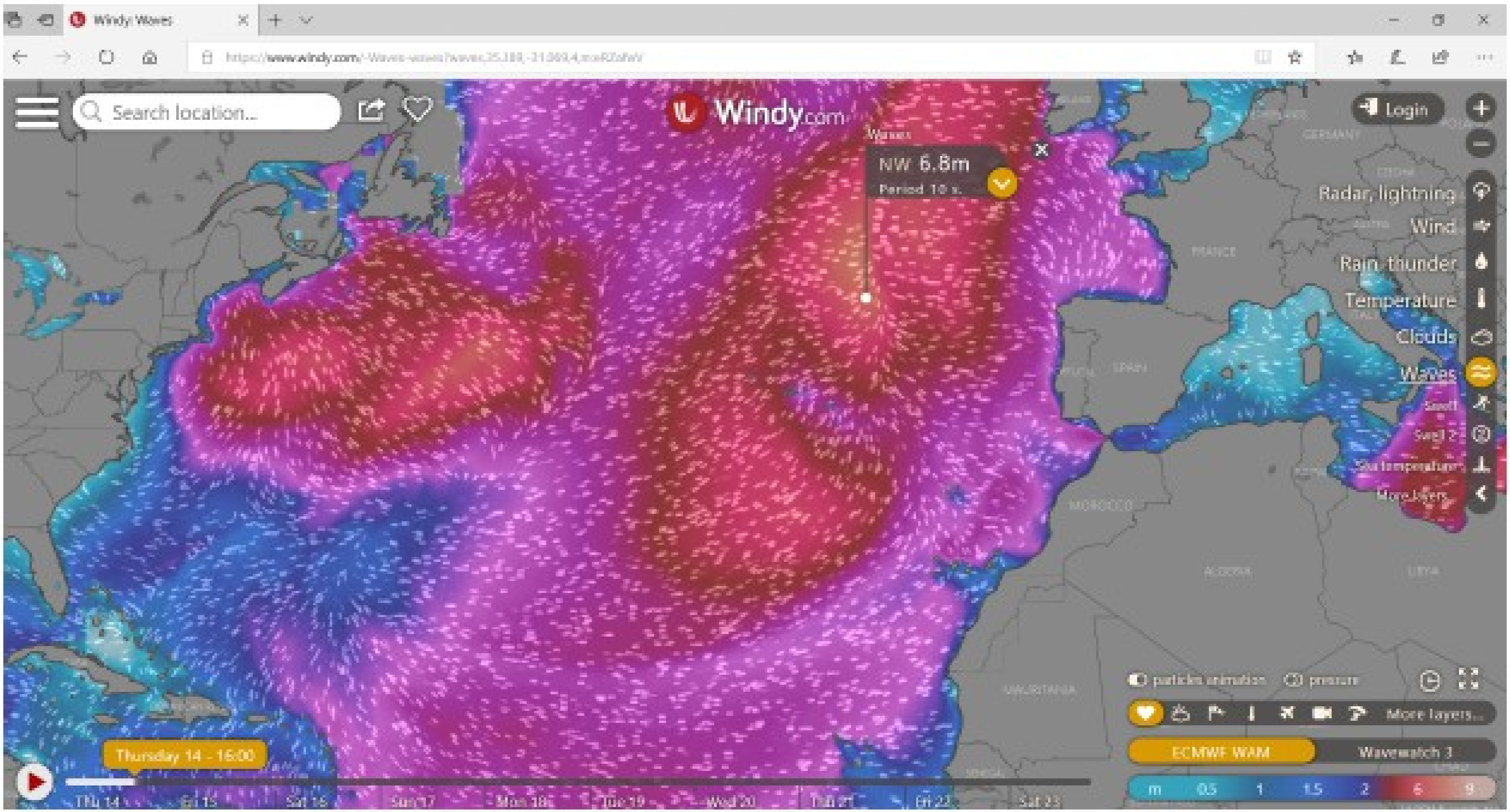
Bayesian-like framework

- **Scale-separation** assumption: spectrum is stationary on time/length scales where NLS is the governing equation
- Assume **Gaussian statistics** (and use this as **prior**) on the basis of Wave Turbulence theory + max entropy distribution
- **Gaussian prior is inaccurate in the tail** (like CLT valid only for core of distribution), supplementing it with the dynamics we approach the invariant distribution: **posterior** (for now, conjecture)
- Need for *a posteriori* consistency check

More challenging ground, but closer to the real interesting problem.
Conceptual change, but **same optimization problem**

- GD, T Grafke, & E Vanden-Eijnden. Rogue waves and large deviations in deep sea. PNAS, 115(5), 855-860 (2018)

Slow spectral dynamics on large scales



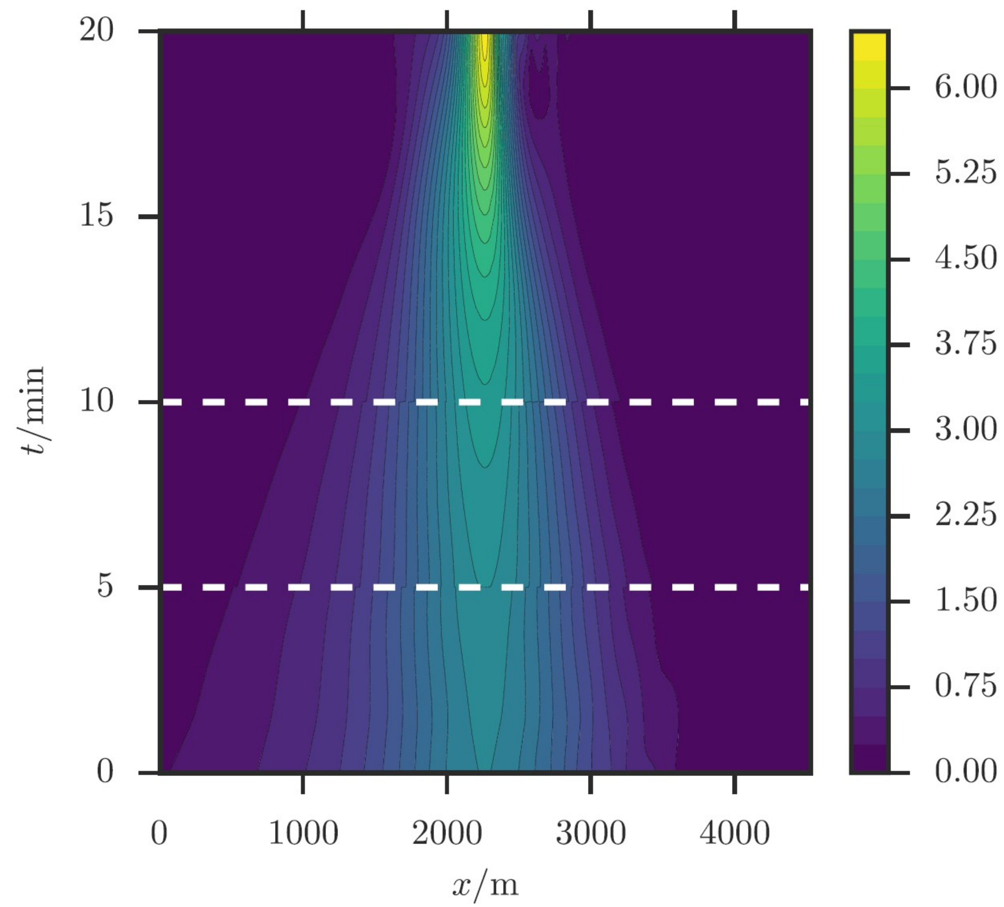
Snapshot of ECMWF prediction of a macroscopic state of the Northern Atlantic Ocean surface – from the app *Windy.com*

Modified NLS (Dysthe)

$$\partial_t u + \frac{1}{2}\partial_x u + \frac{i}{8}\partial_x^2 u - \frac{1}{16}\partial_x^3 u + \frac{i}{2}|u|^2 u$$
$$+ \frac{3}{2}|u|^2\partial_x u + \frac{1}{4}u^2\partial_x \bar{u} - \frac{i}{2}\,|\partial_x|\,|u|^2 = 0$$

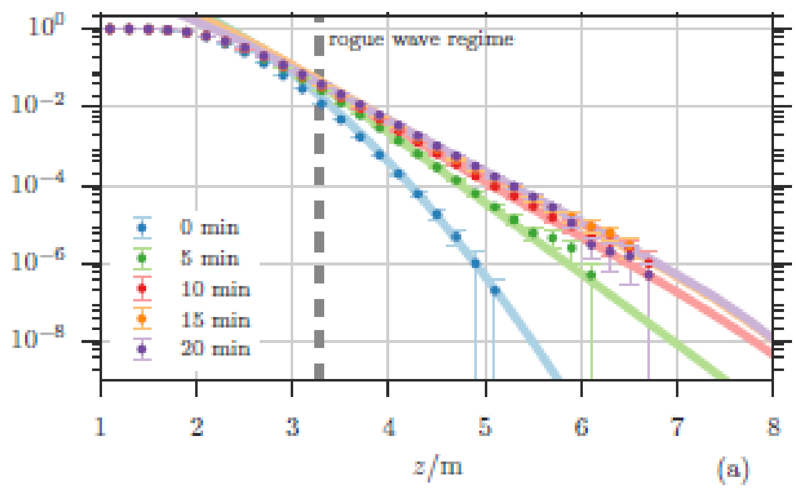
Dysthe KB (1979) Note on a modification to the nonlinear Schrödinger equation for application to deep water waves. *Proc R Soc Lond A* 369:105–114.
Stiassnie M (1984) Note on the modified nonlinear Schrödinger equation for deep water waves. *Wave motion* 6:431–433.

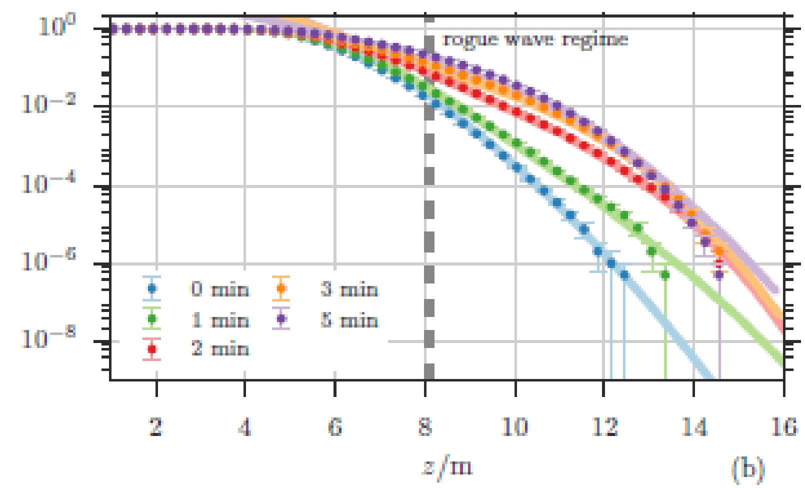
Non-integrable.
Instantons qualitatively similar to NLS but with asymmetry



LDT estimate of the tails

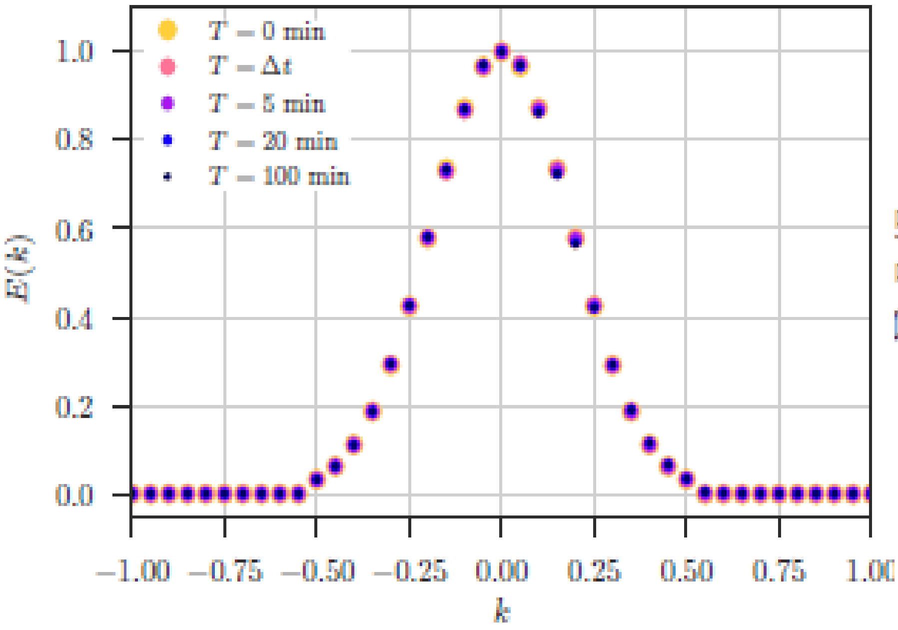
- Want to estimate $\mathbb{P}(\max_x |\psi(x,T)| \geq z)$
- Prior inaccurate in the tail, with the dynamics we approach the invariant distribution: can **observe convergence to invariant state** after a **transient in agreement with Peregrine** time scale!
- LDT estimate: great agreement with MC sampling, but gain in efficiency: **statistics of rare events dominated by single realizations**: instantons

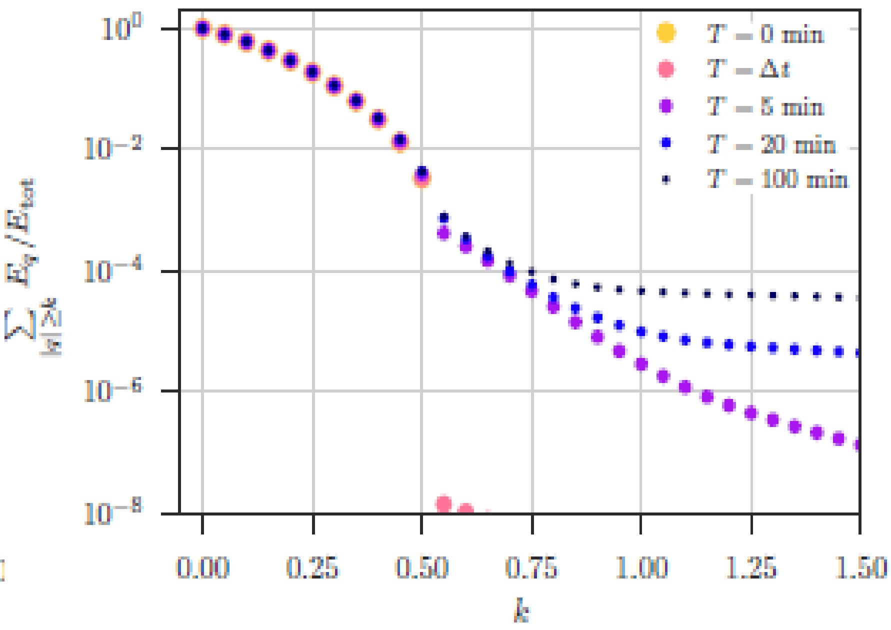




$H_s = 3.3$ m $\quad T_{Per} \simeq 18$ min $\qquad H_s = 8.2$ m $\quad T_{Per} \simeq 8$ min

Consistency check: spectral invariance during evolution

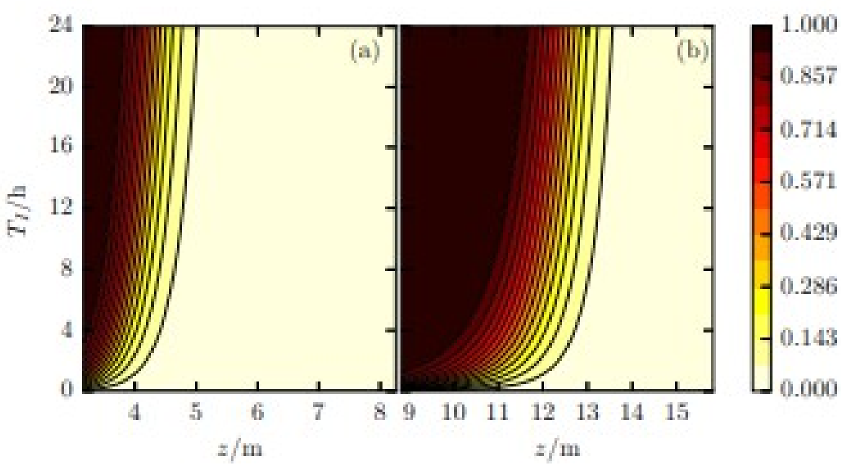




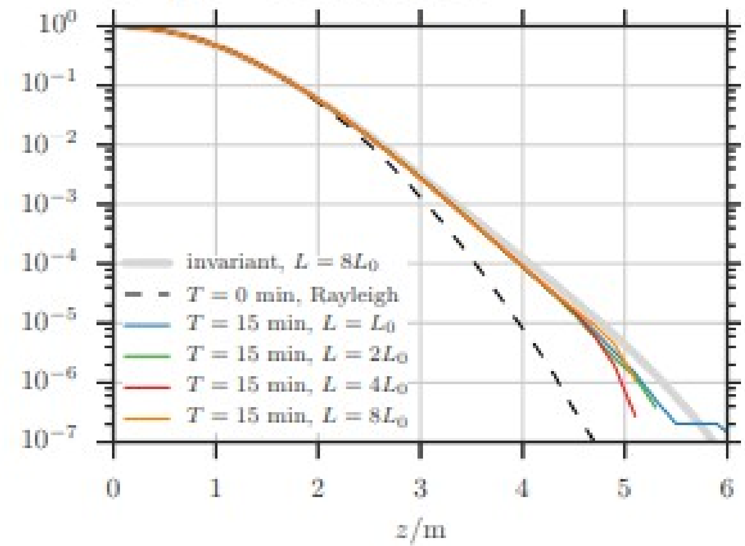
Statistics in large space-time domains

- Initial gaussian distribution tends to the invariant distribution as the observation time increases (prior $\rightarrow$ posterior).
- This can be used to extend the statistics of extremes to a wider space-time domain, through a boxing argument

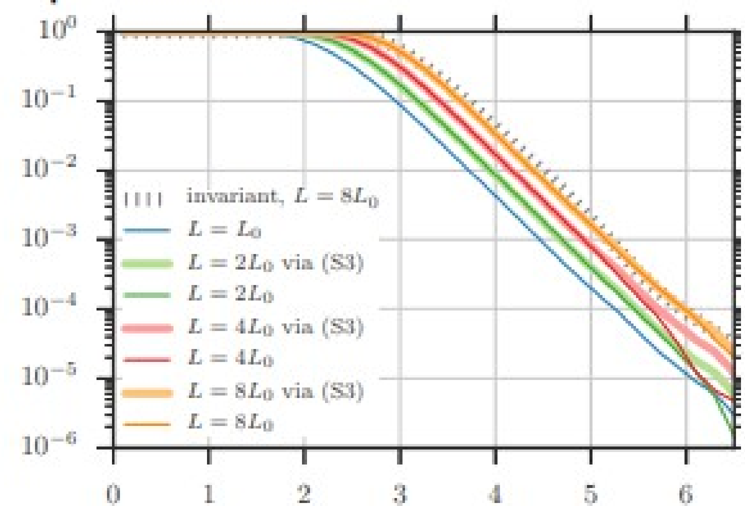
$$\mathbb{P}(\max_{(t,x)\in\mathcal{D}} |u(t,x)| > z) \sim 1 - [1 - \mathbb{P}(|u| > z)]^{N_{\mathcal{D}}}, \quad \text{dove } N_{\mathcal{D}} = |\mathcal{D}|/(\lambda_c \tau_c)$$



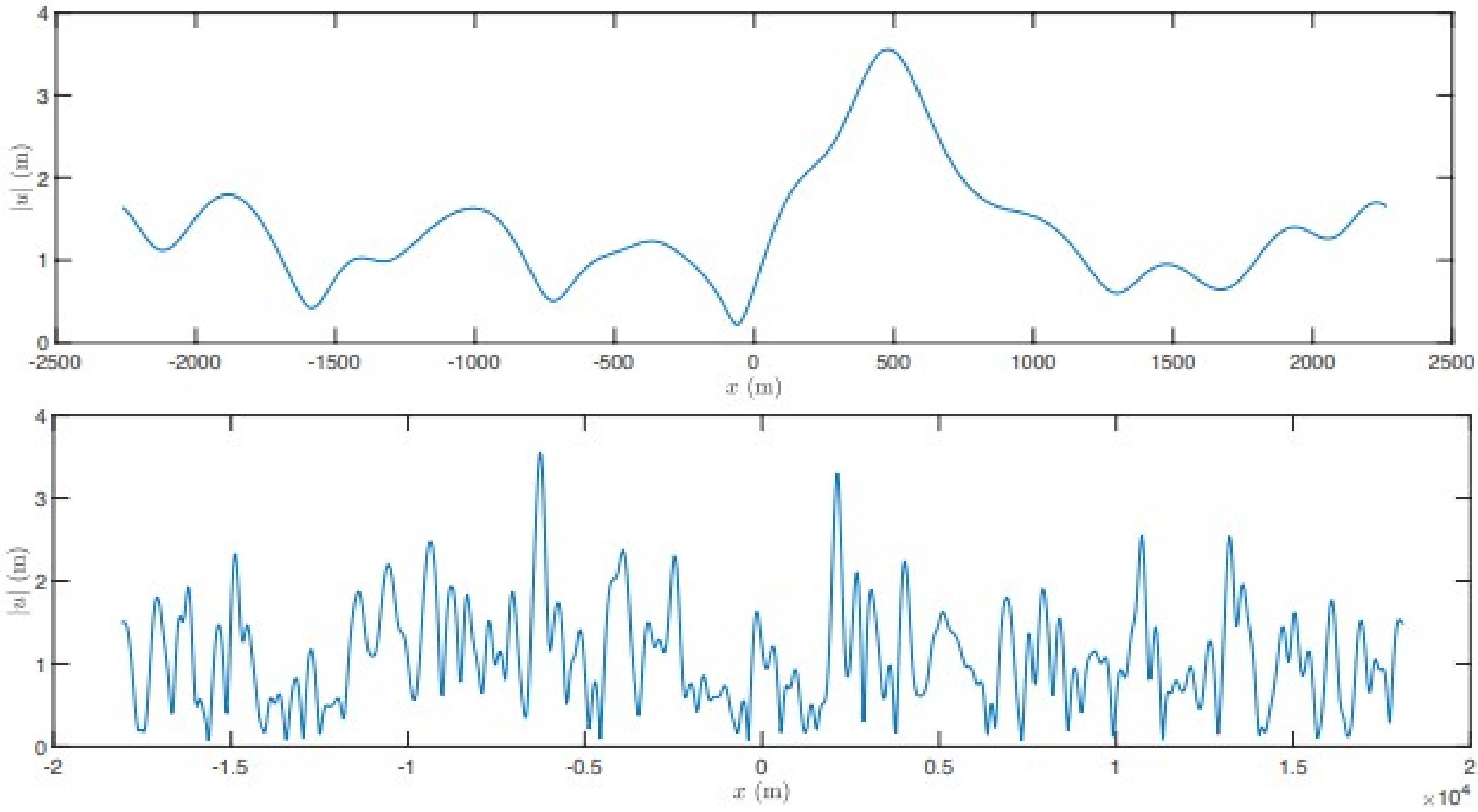
Fixed-point statistics



Spatial max statistics



Accounting for entropic effect



Nice interplay with statistics of extrema:

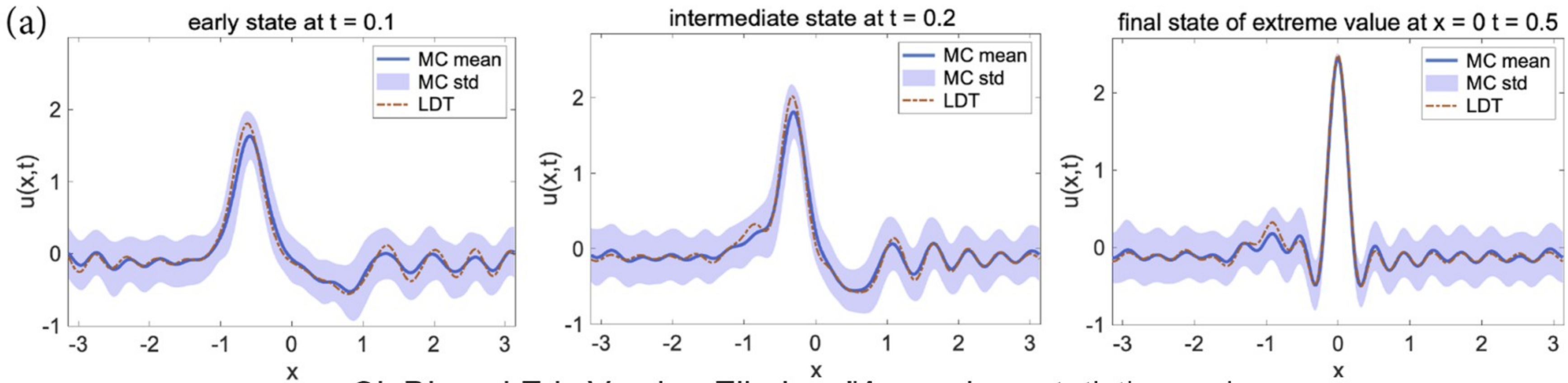
- **LDT**: provides the **effective mechanism and likelihood**
- **Statistics of extrema**: straightforward extension of results to **domains of arbitrary size**

Outline

- Introduction
- Large Deviation Theory: fundamentals
- Computation of instantons in wave systems
- Experimental instantons in a wave flume
- Rogue waves in a real ocean?
- **Outlook: future directions and applications**

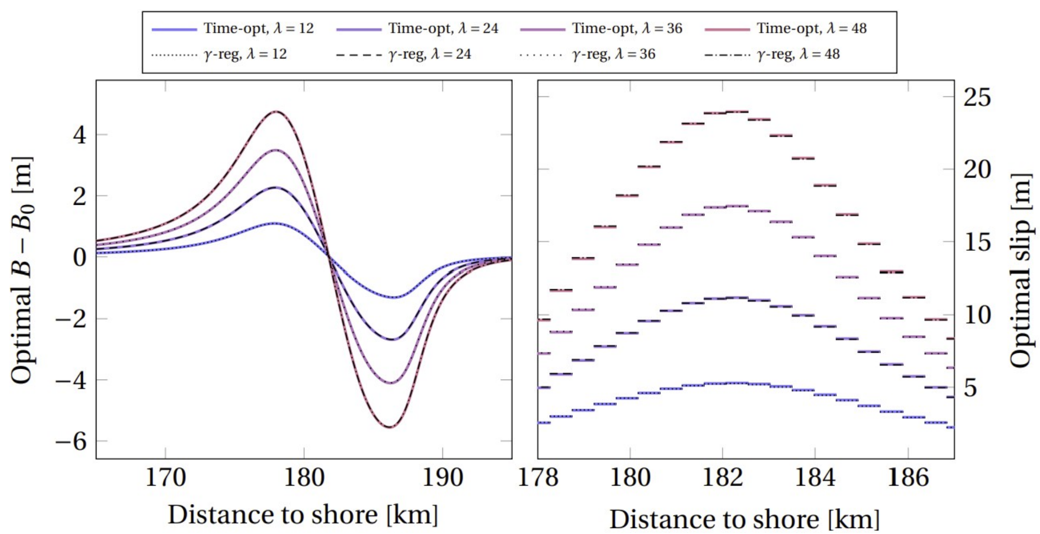
Application to other wave equations

KdV past a step



Qi, Di, and Eric Vanden-Eijnden. "Anomalous statistics and large deviations of turbulent water waves past a step." *AIP Advances* 12.2 (2022).

Shallow water equations for tsunami prediction

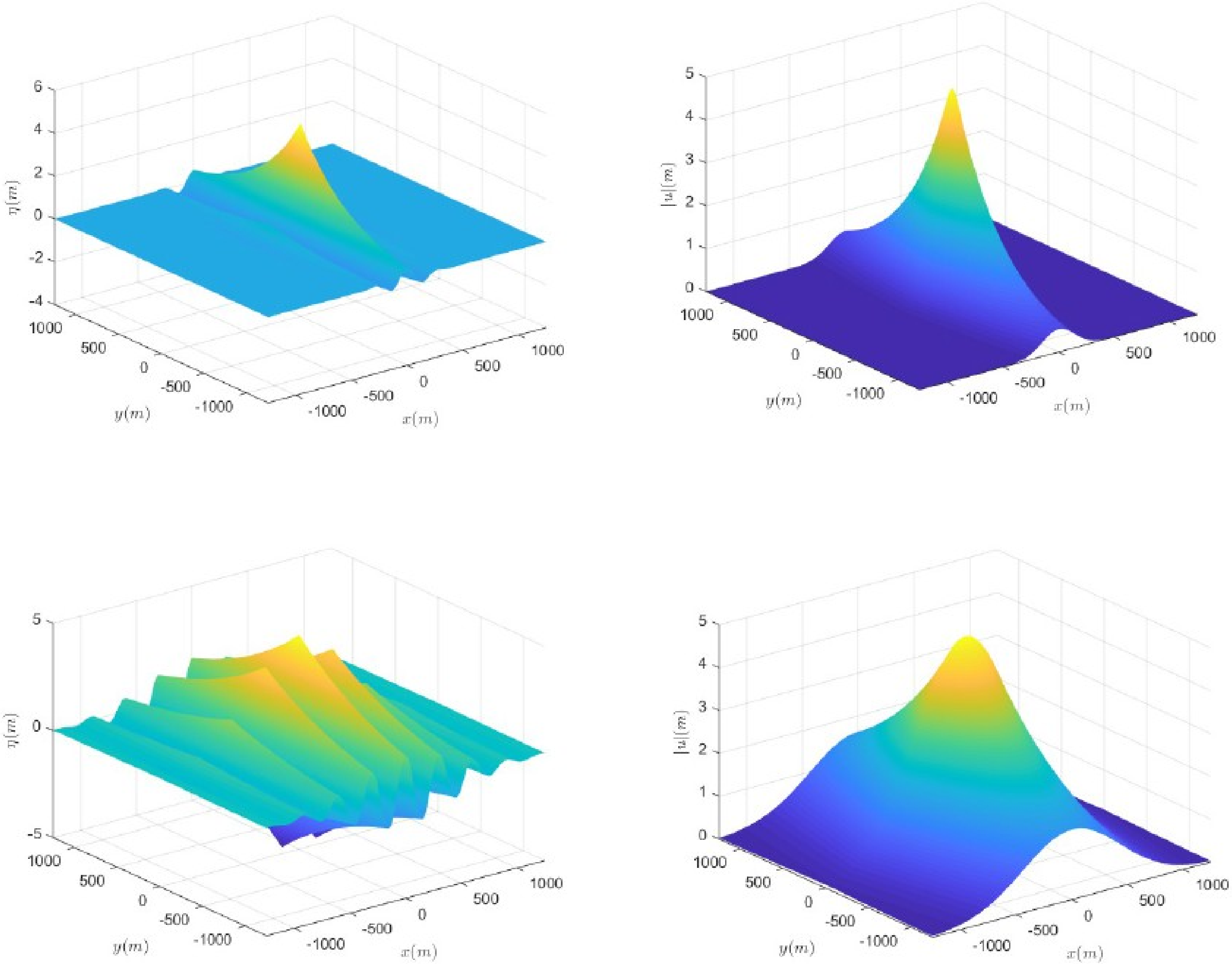


Tong, S., Vanden-Eijnden, E., & Stadler, G. (2021). Extreme event probability estimation using PDE-constrained optimization and large deviation theory, with application to tsunamis. *Communications in Applied Mathematics and Computational Science*, *16*(2), 181-225.

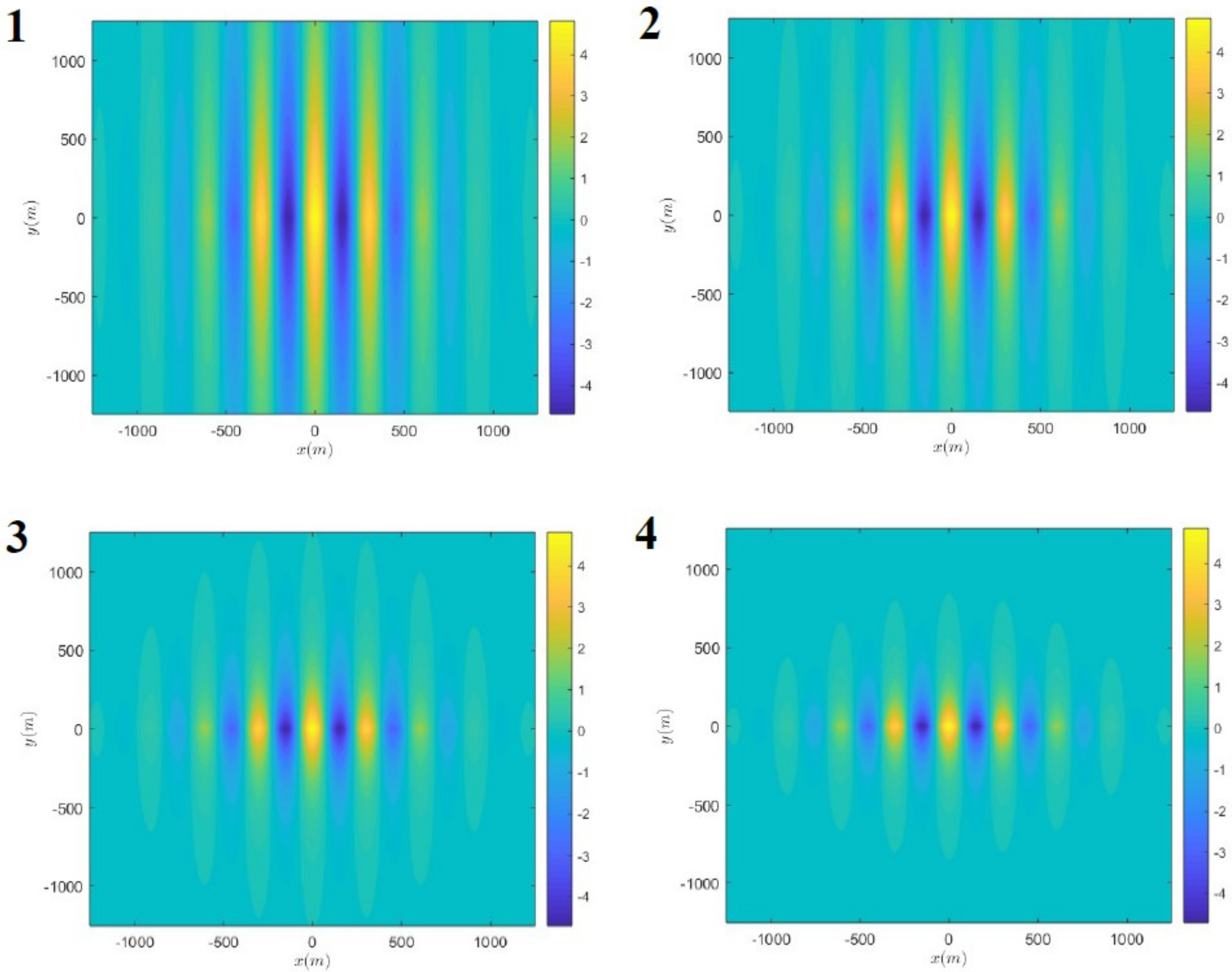
Extension of "LDP" for extreme events to include second-order prefactor correction

Extending to 2 dimensions (linear)

(from MS thesis by Alessandro Falchi)



2D linear case

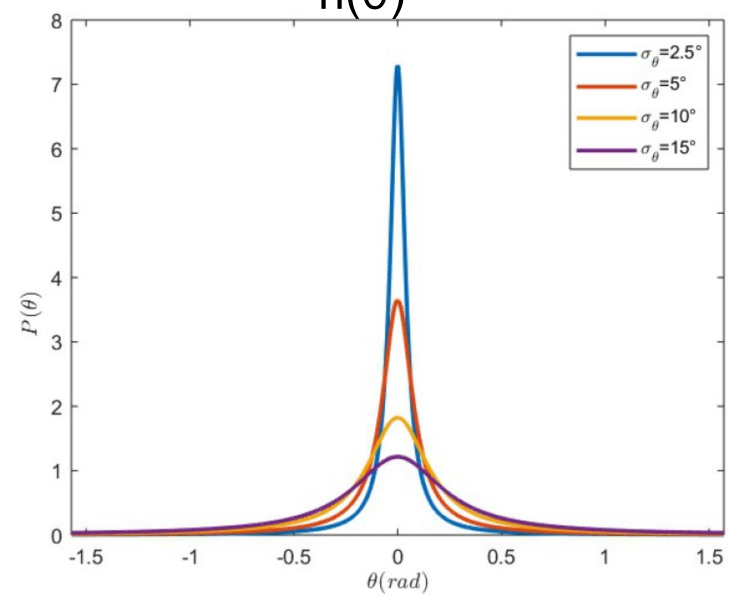


Straightforward generalization of the 1D case: again same result as Lindgren-Boccotti

$$n(x,\theta) = n(x)n(\theta)$$

$$n(\theta)=B(N)\cos(\theta)^N$$

$n(\theta)$



Instantons in a 2D nonlinear evolution? to be done (see works by Fedele extending quasi-determinism to nonlinear bound modes)

Generalize to random parameters / adjoint method

$$E(u,\theta) = I(\theta) - \lambda f(u(T)))\,,$$
$$\boxed{\partial_t u = b(u,\theta),} \qquad u(t=0) = u_0(\theta).$$

Now the dynamics itself can depend on random parameters, not only ICs

$$\nabla_\theta E(u(T,\theta),\theta) = \nabla_\theta I - \lambda\, J^\top(T,\theta)\, \partial_u f(u(T,\theta))$$
$$\partial_t J = \partial_u b\, J + \partial_\theta b, \qquad J(0) = \nabla_\theta\, u_0.$$

Direct method

- J evolution: dim(u) x dim(θ)

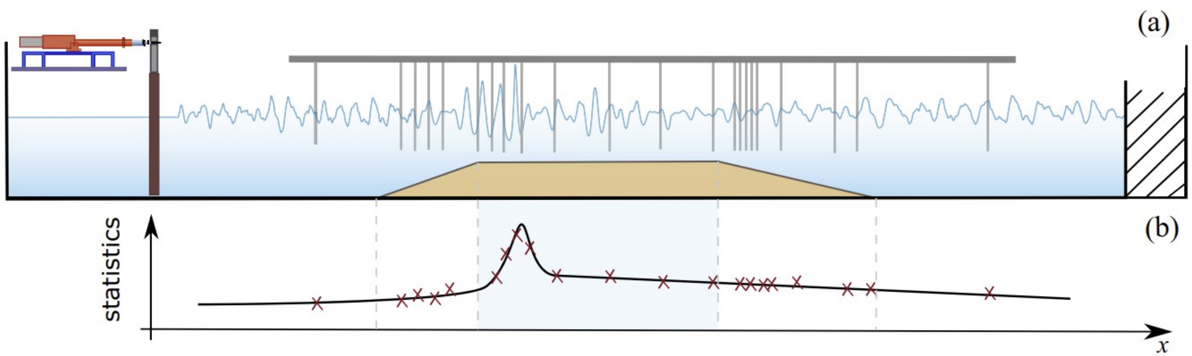
$$\nabla_\theta E = \nabla_\theta I - (\nabla_\theta u_0)^\top p(0,\theta) - \int_0^T (\partial_\theta b)^\top p\, dt$$
$$\partial_t p = -(\partial_u b)^\top p, \qquad p(T,\theta) = \lambda \partial_u f(u(T,\theta)).$$

Adjoint method

- p evolution: dim(u) !!

All technical details in: *GD et al. 2019* *and* *Tong et al. 2021*

Random parameters in the dynamics: e.g. variable-depth wave propagation in coastal waters?
Optimal bathymetry for rogue wave formation?



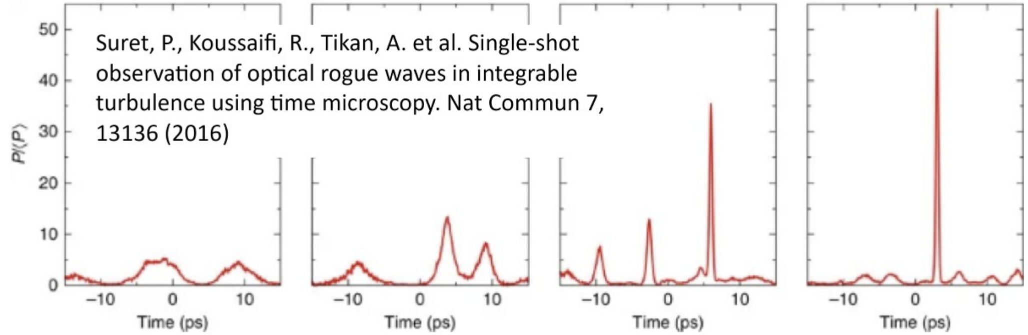
Picture from

Li Y and Chabchoub A (2023). On the formation of coastal rogue waves in water of variable depth. *Cambridge Prisms: Coastal Futures*, **1**, e33, 1–7

see also e.g.

Majda AJ, Moore MN, Qi D. Statistical dynamical model to predict extreme events and anomalous features in shallow water waves with abrupt depth change. Proceedings of the National Academy of Sciences. 2019 Mar 5;116(10):3982-7.

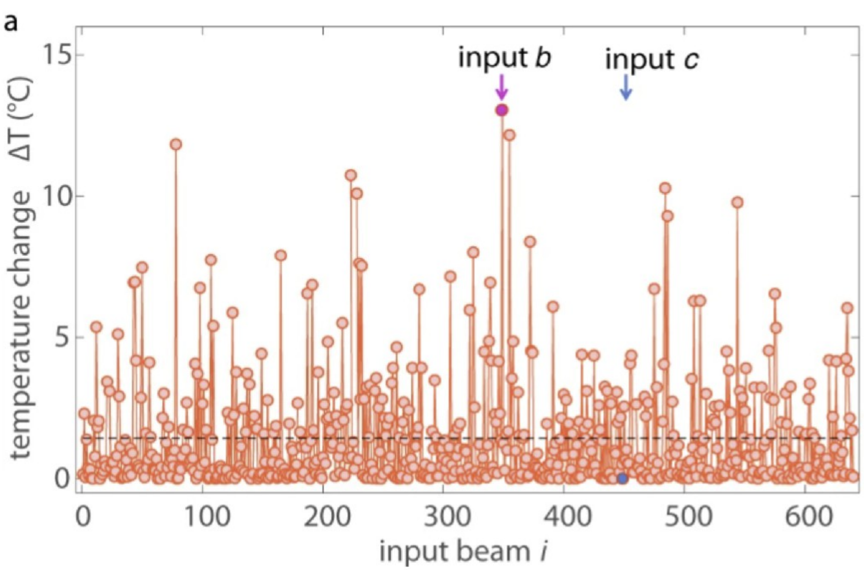
Application in nonlinear optics



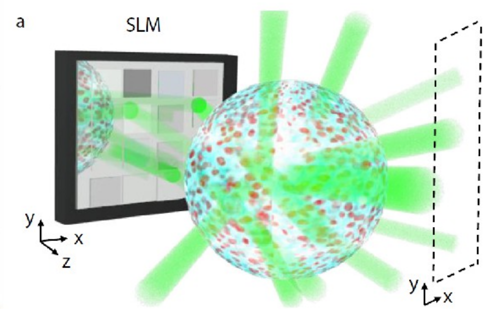
Suret, P., Koussaifi, R., Tikan, A. et al. Single-shot observation of optical rogue waves in integrable turbulence using time microscopy. Nat Commun 7, 13136 (2016)

Experimental observation of optical rogue waves in nonlinear fiber optics.
NLS governing dynamics

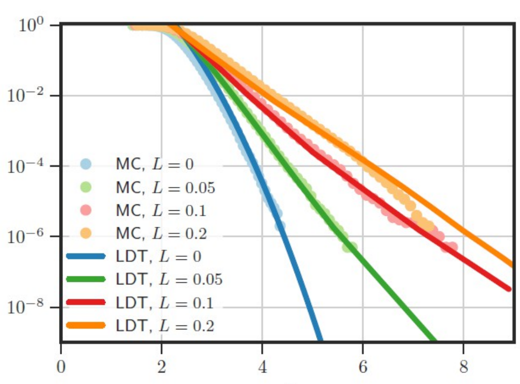
Fig. 6: Local temperature increase induced by extreme waves.



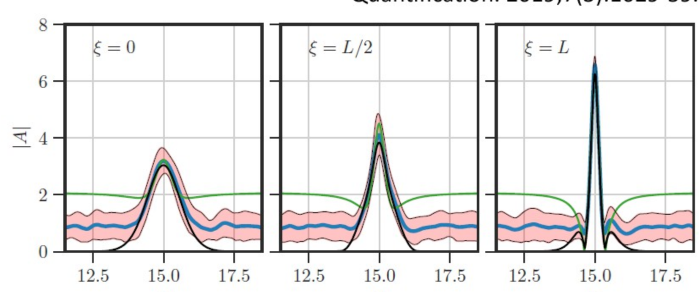
Pierangeli, D., Perini, G., Palmieri, V. et al. Extreme transport of light in spheroids of tumor cells. Nat Commun 14, 4662 (2023)



Optical rogue waves studied as potential treatment. Tumor cells illuminated by randomly modulated laser beams. Need for control of energy delivered by focused beams: Could be posed as optimization problem?



DG, Grafke T, Vanden-Eijnden E. Extreme event quantification in dynamical systems with random components. SIAM/ASA Journal on Uncertainty Quantification. 2019;7(3):1029-59.



Instantons and LDP shown to be highly relevant. Promising application not really exploited so far

Rigorous LDP

Communications on
PURE AND APPLIED MATHEMATICS

RESEARCH ARTICLE

Large deviations principle for the cubic NLS equation

Miguel Angel Garrido, Ricardo Grande, Kristin M. Kurianski, Gigliola Staffilani

First published: 21 July 2023 | https://doi.org/10.1002/cpa.22131

(small nonlinearity regime)

Theorem 1.1 (Large Deviation Principle). *Consider the NLS equation on the circle $\mathbb{T} = [0, 2\pi]$:*

$$\begin{cases} i\partial_t u + \Delta u = \varepsilon^2 |u|^2 u \\ u(0,x) = \sum_{k\in\mathbb{Z}} c_k \eta_k e^{ikx} \end{cases} \tag{1.9}$$

with initial data as in (1.8). Consider the probability of seeing a large wave of height $z(\varepsilon) := z_0\, \varepsilon^{-1/2} > 0$ at time $t > 0$ and fixed $z_0 > 0$. If $t \lesssim \varepsilon^{-1}$ we have that

$$\lim_{\varepsilon\to 0^+} \varepsilon \log \mathbb{P}\left(\sup_{x\in\mathbb{T}} |u(t,x)| > z_0\, \varepsilon^{-1/2} \right) = -\frac{z_0^2}{\sum_{k\in\mathbb{Z}} c_k^2}. \tag{1.10}$$

Rigorous LDP with "speed" given by smallness parameter ε
(proof uses Gärtner-Ellis)

Theorem 1.9. *Consider the set $\mathcal{U}(\varepsilon)$ given by (1.12)-(1.13). Then $\mathcal{U}(\varepsilon)$ satisfies the same LDP, (1.10), as the set of rogue waves (1.11). Moreover, $\mathcal{U}(\varepsilon)$ is almost entirely contained in the set $\mathcal{D}(t, z_0\varepsilon^{-1/2} - \varepsilon)$. More precisely,*

$$\log \mathbb{P}\left(\mathcal{U}(\varepsilon) - \mathcal{D}(t, z_0\varepsilon^{-1/2} - \varepsilon) \right) \lesssim -\exp(c\varepsilon^{-1/2})) \qquad \text{as } \varepsilon \to 0^+. \tag{1.14}$$

"All extreme waves larger than z look like the optimizer (instanton)"

Thank you for your attention!