Introduction to the analysis of integrable PDEs when the number of solitons goes to ∞, with an aim towards analytical and computational research projects

Bob Jenkins and Ken McLaughlin.
Collaborations with Manuela Girotti, Tamara Grava, and Alexander Minakov.
Also using some work of Bertola, Grava, and Orsatti

Nonlinear Schrödinger equation: $$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0, \qquad \psi(x,0) = \psi_0(x)$$

Zakharov and Shabat showed complete integrability, with Lax pair:

$$(\partial_x - \mathcal{L})\Phi = 0, \qquad \mathcal{L} = -iz\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & \psi(x,t) \\ -\psi(x,t)^* & 0 \end{pmatrix},$$

$$(i\partial_t - \mathcal{B})\Phi = 0, \qquad \mathcal{B} = iz\mathcal{L} + \frac{1}{2}\begin{pmatrix} -|\psi|^2 & -\psi_x \\ -\psi_x^* & |\psi|^2 \end{pmatrix},$$

$$\exists \Phi(x,t) \iff \psi(x,t) \text{ solves NLS.}$$

Scattering theory for $(\partial_x - \mathcal{L})\,\Phi = 0$ yields solution procedure.

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Jost solutions: $\displaystyle \lim_{x\to+\infty} \Phi^{(+)} \begin{pmatrix} e^{izx} & 0 \\ 0 & e^{-izx} \end{pmatrix} = I \;, \quad \lim_{x\to-\infty} \Phi^{(-)} \begin{pmatrix} e^{izx} & 0 \\ 0 & e^{-izx} \end{pmatrix} = I$

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t is viewed as a parameter.

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Some important notation

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -iz & 0 \\ 0 & iz \end{pmatrix} = -iz\sigma_3$$

$$\begin{pmatrix} e^{izx} & 0 \\ 0 & e^{-izx} \end{pmatrix} = e^{izx\sigma_3}$$

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So the special solutions $\Phi^{\pm}$ satsify

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$$i\partial_x m = -iz[\sigma_3, m] + \Psi m, \qquad \lim_{x\to\pm\infty} m^{(\pm)}(x,z) = I.$$

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Existence and uniqueness: $z \in \mathbb{R}$

$$m^{(\pm)}(z) = I + \int_{\pm\infty}^{x} e^{iz(y-x)\sigma_3} \begin{pmatrix} 0 & \psi(y) \\ -\psi(y)^* & 0 \end{pmatrix} m^{(\pm)}(y) e^{-iz(y-x)\sigma_3} dy$$

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$$m^{(\pm)} = \left(m_1^{(\pm)}, m_2^{(\pm)}\right) = \begin{pmatrix} m_{11}^{(\pm)} & m_{12}^{(\pm)} \\ m_{21}^{(\pm)} & m_{22}^{(\pm)} \end{pmatrix}.$$

$$m_1^{(-)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{x} \begin{pmatrix} 0 & \psi(y) \\ -\psi(y)^* e^{2iz(x-y)} & 0 \end{pmatrix} m_1^{(-)}(y) dy$$

$m_1^{(-)}$: Analytic for $z \in \mathbb{C}_+$

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$$\Phi^{(+)}(x,z) = m^{(+)}(x,z)e^{-izx\sigma_3}$$

Properties!

$$\Phi^{(-)}(x,z) = m^{(-)}(x,z)e^{-izx\sigma_3}$$

$$\det\left(\Phi^{(\pm)}(x,z)\right) = \det\left(m^{(\pm)}\right) \equiv 1$$

$\Phi^{(+)}(x,z)$ and $\Phi^{(-)}(x,z)$: fundamental solutions for each $z \in \mathbb{R}$

$$\Phi^{(-)}(x;z) = \Phi^{(+)}(x;z)S(z), \qquad z \in \mathbb{R},$$

$$S(z) = \begin{pmatrix} a(z) & -b(z)^* \\ b(z) & a(z)^* \end{pmatrix}, \qquad \det S(z) = |a(z)|^2 + |b(z)|^2 = 1$$

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reflection coefficient: $r(z) = \dfrac{b(z)}{a(z)}$

transmission coefficient: $\tau(z) = \dfrac{1}{a(z)}$

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$$1 + |r(z)|^2 = |\tau(z)|^2$$

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$a(z) \to 1$ as $z \to \infty$, $z \in \mathbb{C}_+$.

Properties!

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$\mathbb{R}$

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similar properties for $m_2^{(-)}(x,z), m_1^{(+)}(x,z)$ in $\mathbb{C}_-$

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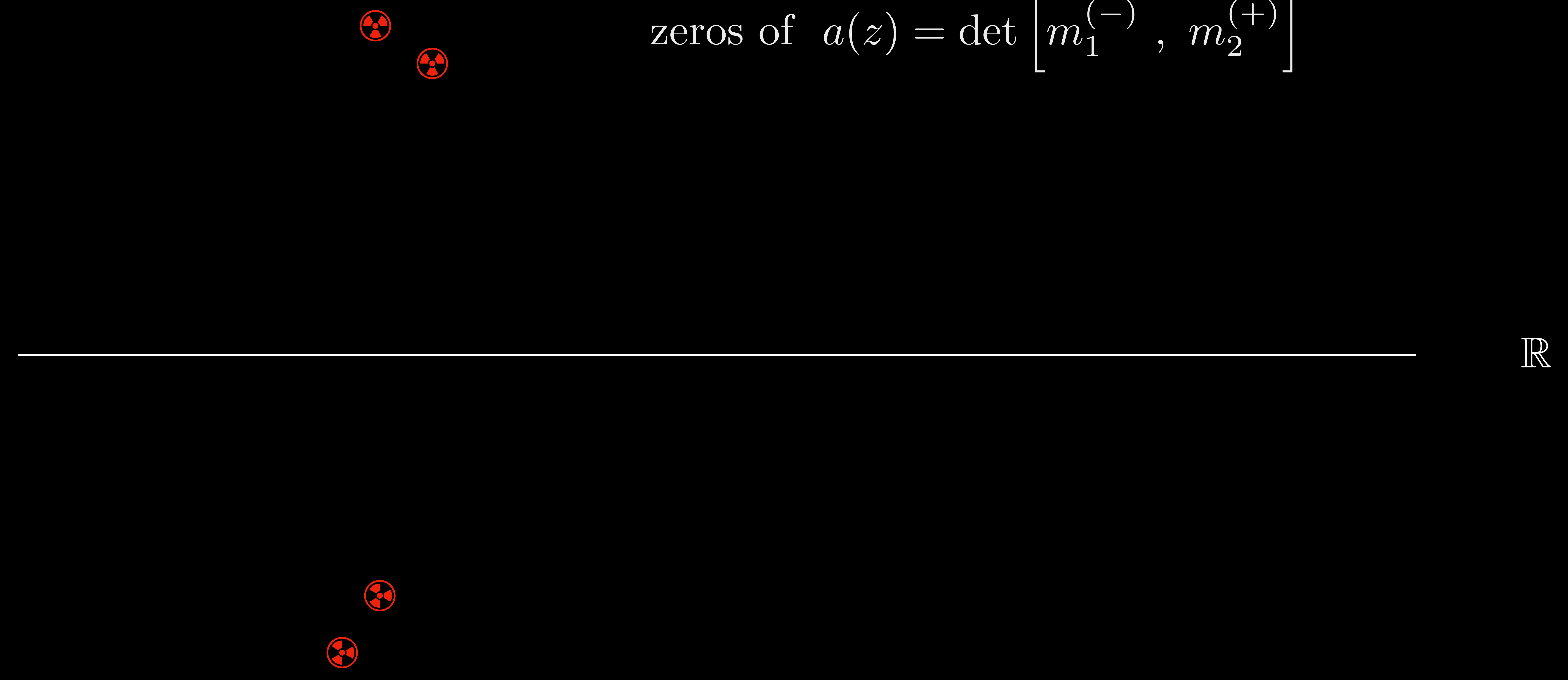
zeros of $a(z) = \det\left[m_1^{(-)}\,,\ m_2^{(+)}\right]$



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$\Phi_1^{(-)}(x,z_0)$ decaying as $x \to -\infty$

$\Phi_2^{(+)}(x,z_0)$ decaying as $x \to +\infty$

Linearly dependent: $\Phi_1^{(-)}(x,z_0) = \tilde{c}_0 \Phi_2^{(+)}(x,z_0)$

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$z = z_0$ is an eigenvalue!

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$$\psi(x,0) \quad \text{yields} \quad \mathcal{D} = \{r(z), \{z_k, c_k\}_{k=1}^{N}\}$$

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0, \qquad \psi(x,0) = \psi_0(x).$$

$$\psi(x,t) \text{ yields } \mathcal{D}(t) = \{r(z,t), \{z_k,(t)c_k(t)\}_{k=1}^{N}\}$$

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$$\psi_0(x) \in H^{j,k}(\mathbb{R}) = \left\{f \in L^2(\mathbb{R}) \; : \partial_x^j f, \; |x|^k f \in L^2(\mathbb{R})\right\} + \{ \text{ no real zeros of } a(z)\}$$
$$\implies r(z) \in H^{k,j}$$

The inverse problem

$$\Phi^{(+)}(x,z) = m^{(+)}(x,z)e^{-izx\sigma_3}$$

The inverse problem!

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$\psi(x,t)$ is actually determined by $\mathcal{D}(t) = \left\{ r(z)e^{2itz^2}, \{z_k, c_k e^{2itz_k^2}\}_{k=1}^N \right\}$

Going from $\mathcal{D}(t)$ to $\psi(x,t)$: The inverse problem

$\Phi^{(+)}(x,z) = m^{(+)}(x,z)e^{-izx\sigma_3}$ **The inverse problem!** $\Phi^{(-)}(x,z) = m^{(-)}(x,z)e^{-izx\sigma_3}$

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Going from $\mathcal{D}(t)$ to $\psi(x,t)$: The inverse problem

If we could construct

$$M(z) = M(z;x,t) := \begin{cases} \left[\dfrac{m_1^{(-)}(x,t;z)}{a(z)}, m_2^{(+)}(x,t;z) \right] & : \quad z \in \mathbb{C}^+ \\ \sigma_2 M(z^*;x,t)^* \sigma_2 & : \quad z \in \mathbb{C}^-. \end{cases}$$

We can extract $\psi(x,t)$:

$$M = I + \frac{1}{2iz}\begin{pmatrix} -\int_x^\infty |\psi(s,t)|^2 ds & \psi(x,t) \\ \psi(x,t)^* & \int_x^\infty |\psi(s,t)|^2 ds \end{pmatrix} + o\left(z^{-1}\right),$$

Riemann-Hilbert problem

Find a meromorphic function $M : \mathbb{C}\backslash(\mathbb{R}\cup\{z_j\}\cup\{z_j^*\}) \to SL_2(\mathbb{C})$ *satisfying*

1. $M(z) = I + o\left(z^{-1}\right)$ as $z \to \infty$.

2. *For each* $z \in \mathbb{R}$,

$$M_+(z) = M_-(z)\begin{pmatrix} 1+|r(z)|^2 & r^*(z)e^{-2it\theta(z)} \\ r(z)e^{2it\theta(z)} & 1 \end{pmatrix}, \tag{1}$$

where

$$\theta = \theta(z;x,t) = z^2 - 2\xi z = (z-\xi)^2 - \xi^2, \quad \xi = -x/(2t).$$

3. $M(z)$ *has simple poles at each* z_k *and* z_k^* *and*

$$\begin{aligned} \operatorname*{Res}_{z_k} M &= \lim_{z\to z_k} M \begin{pmatrix} 0 & 0 \\ c_k e^{2it\theta} & 0 \end{pmatrix}, \\ \operatorname*{Res}_{z_k^*} M &= \lim_{z\to z_k^*} M \begin{pmatrix} 0 & -c_k^* e^{-2it\theta} \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{2}$$

Find a meromorphic function $M : \mathbb{C}\setminus(\mathbb{R}\cup\{z_j\}\cup\{z_j^*\}) \to SL_2(\mathbb{C})$ *satisfying*

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1. $M(z) = I + o\left(z^{-1}\right)$ as $z \to \infty$.

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3. $M(z)$ *has simple poles at each* z_k *and* z_k^* *and*

$$\begin{aligned} \operatorname*{Res}_{z_k} M &= \lim_{z \to z_k} M \begin{pmatrix} 0 & 0 \\ c_k e^{2it\theta} & 0 \end{pmatrix}, \\ \operatorname*{Res}_{z_k^*} M &= \lim_{z \to z_k^*} M \begin{pmatrix} 0 & -c_k^* e^{-2it\theta} \\ 0 & 0 \end{pmatrix}. \end{aligned} \qquad (2)$$

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If we could construct $M(z) = M(z; x, t) := \begin{cases} \left[\dfrac{m_1^{(-)}(x,t;z)}{a(z)}, m_2^{(+)}(x,t;z) \right] & : \quad z \in \mathbb{C}^+ \\ \sigma_2 M(z^*; x, t)^* \sigma_2 & : \quad z \in \mathbb{C}^-. \end{cases}$

We can extract $\psi(x,t)$: $M = I + \dfrac{1}{2iz} \begin{pmatrix} -\int_x^\infty |\psi(s,t)|^2 ds & \psi(x,t) \\ \psi(x,t)^* & \int_x^\infty |\psi(s,t)|^2 ds \end{pmatrix} + o\left(z^{-1}\right),$

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Poles due to $a(z)$

Residue conditions: at z_k, $m_1^{(-)} = c_k m_2^{(+)}$

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Simplest case: $r \equiv 0$ and 1 pole in $\mathbb{C}_+$

$$\mathcal{D}(t) = \left\{ \{z_0, c_0 e^{2itz_0^2}\} \right\} = \left\{ \{\mu + i\eta, c_0 e^{2it(\xi+i\eta)^2}\} \right\}$$

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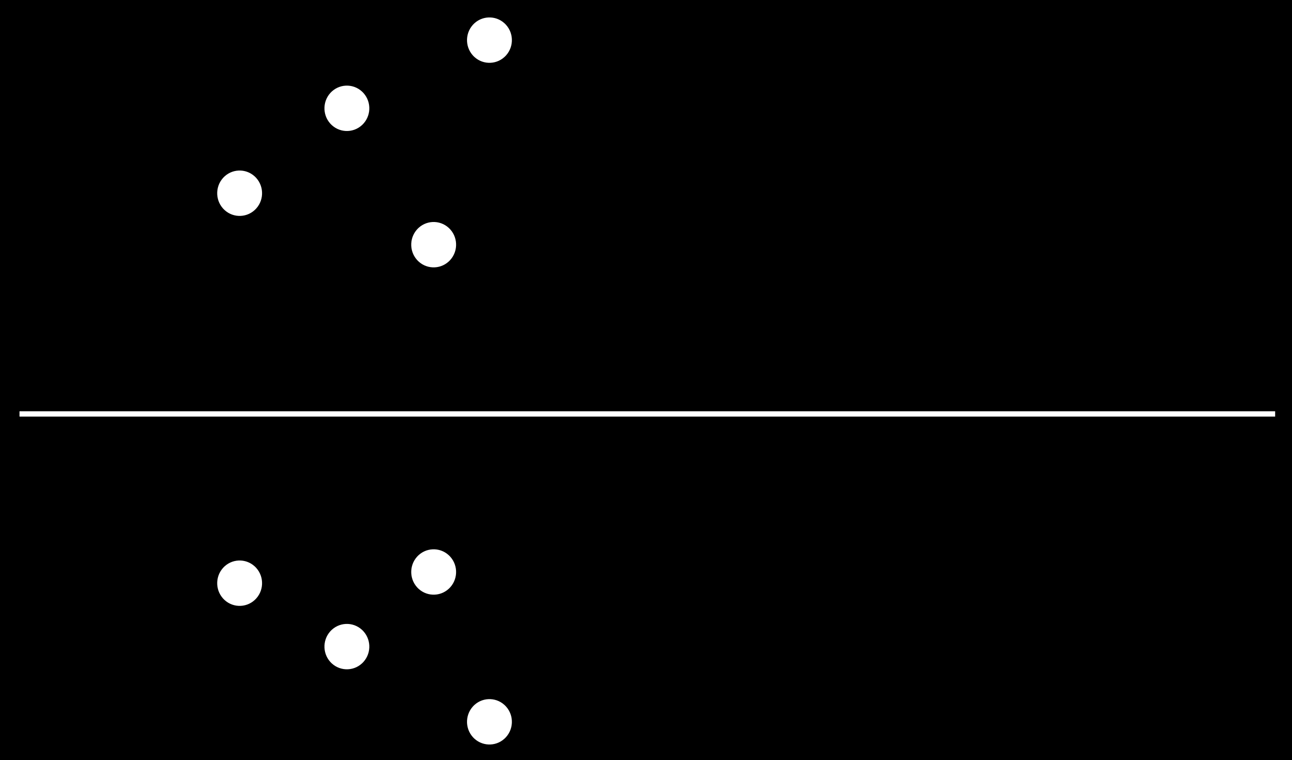
$$\psi(x, t; \sigma_d) = 2\eta \operatorname{sech}\left(2\eta(x + 2\mu t - x_0)\right) e^{-2i(\mu x + (\mu^2 - \eta^2)t)} e^{-i\phi_0},$$

$$x_0 = \frac{1}{2\eta} \log \left| \frac{c_0}{2\eta} \right|, \qquad \phi_0 = \frac{\pi}{2} + \arg(c_0),$$

The N-soliton case: $r \equiv 0$, poles at $\{z_k, z_k^*\}_{k=1}^N$



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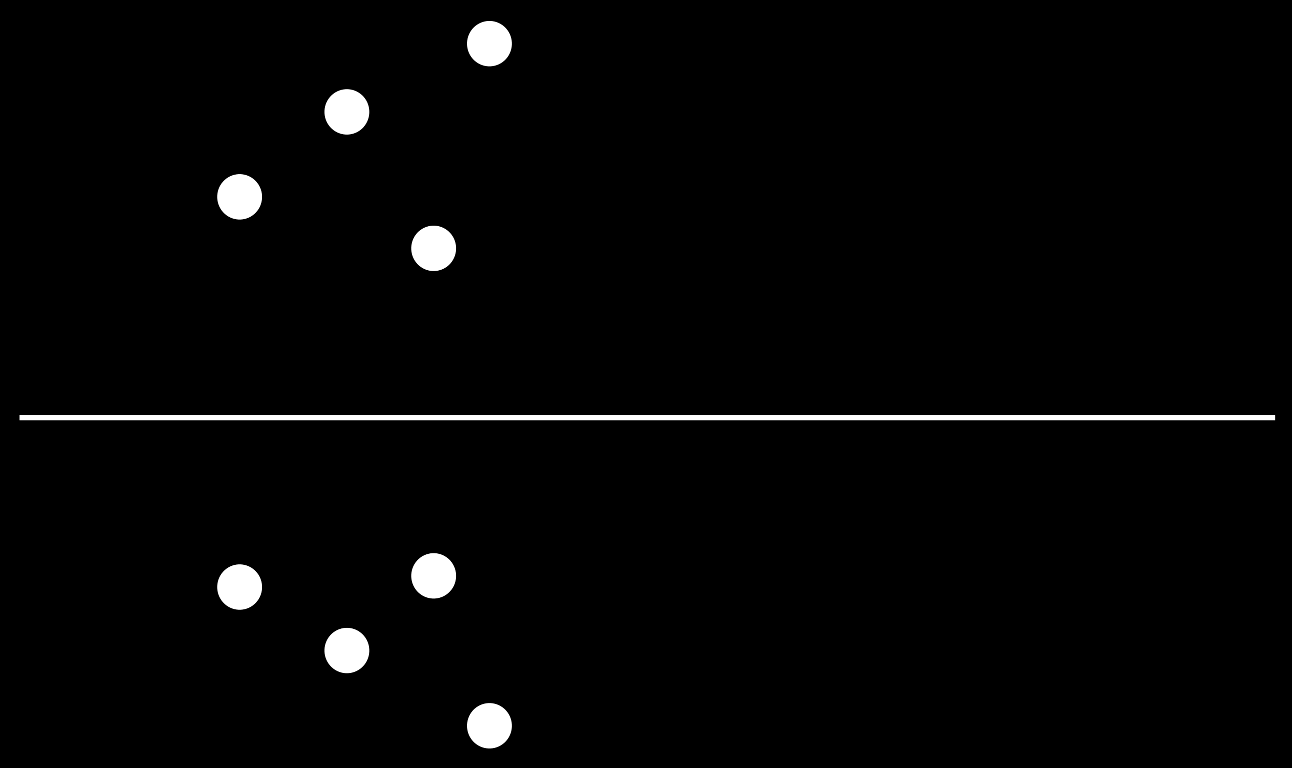

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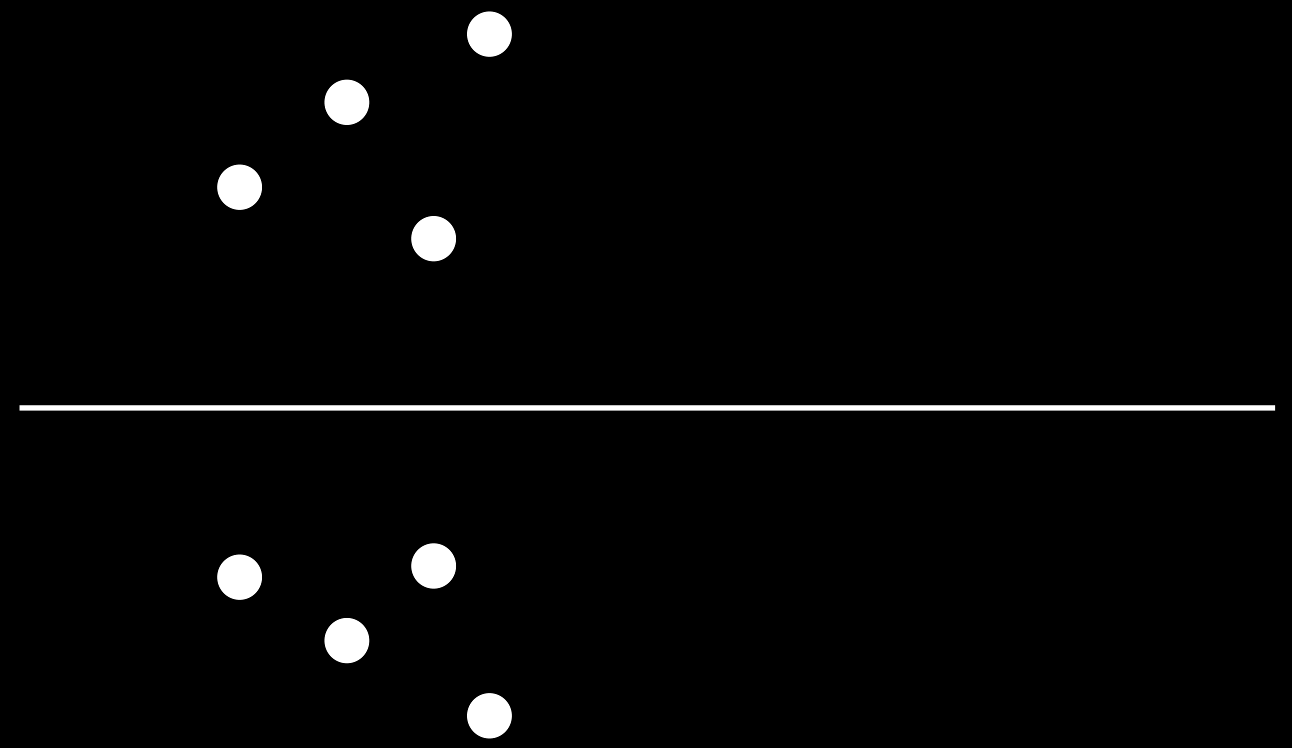

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Task: Determine $\{\alpha_k(x,t), \beta_k(x,t)\}_{k=1}^N$ and then $\psi(x,t) = -2i \sum_{k=1}^N \beta^*(x,t)$

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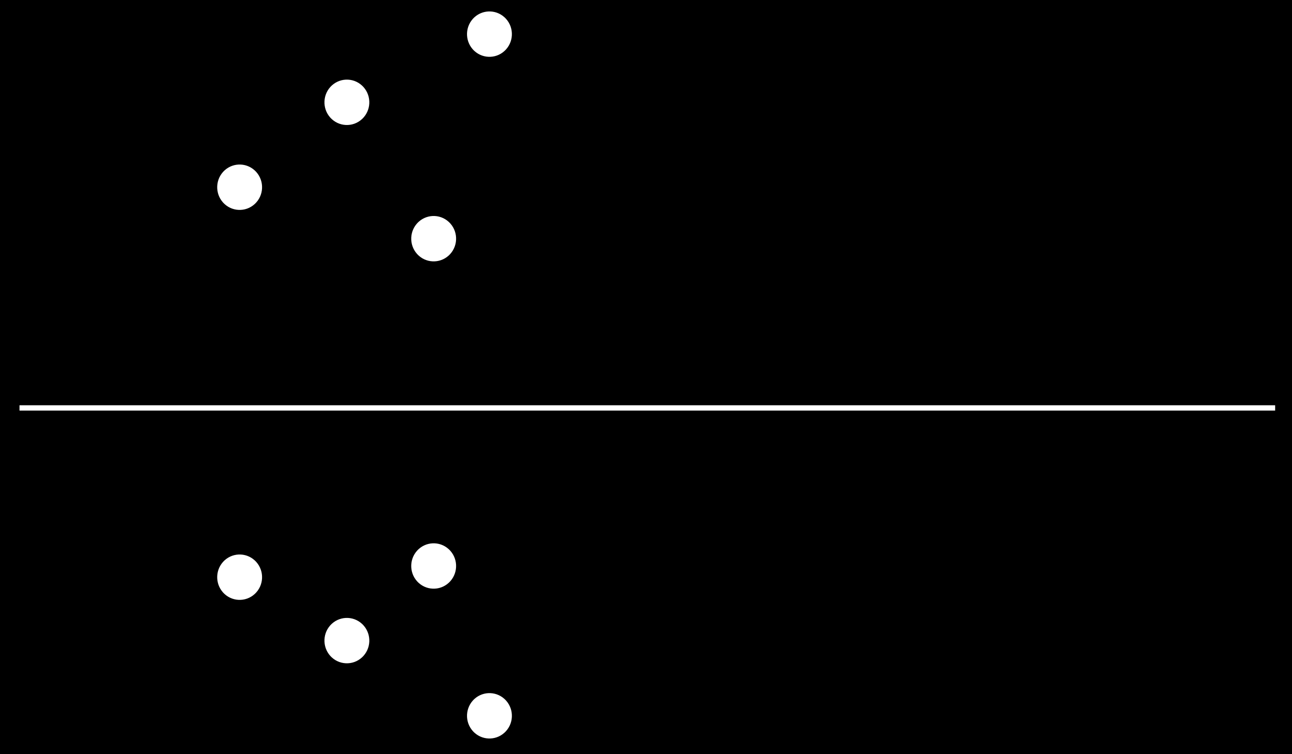

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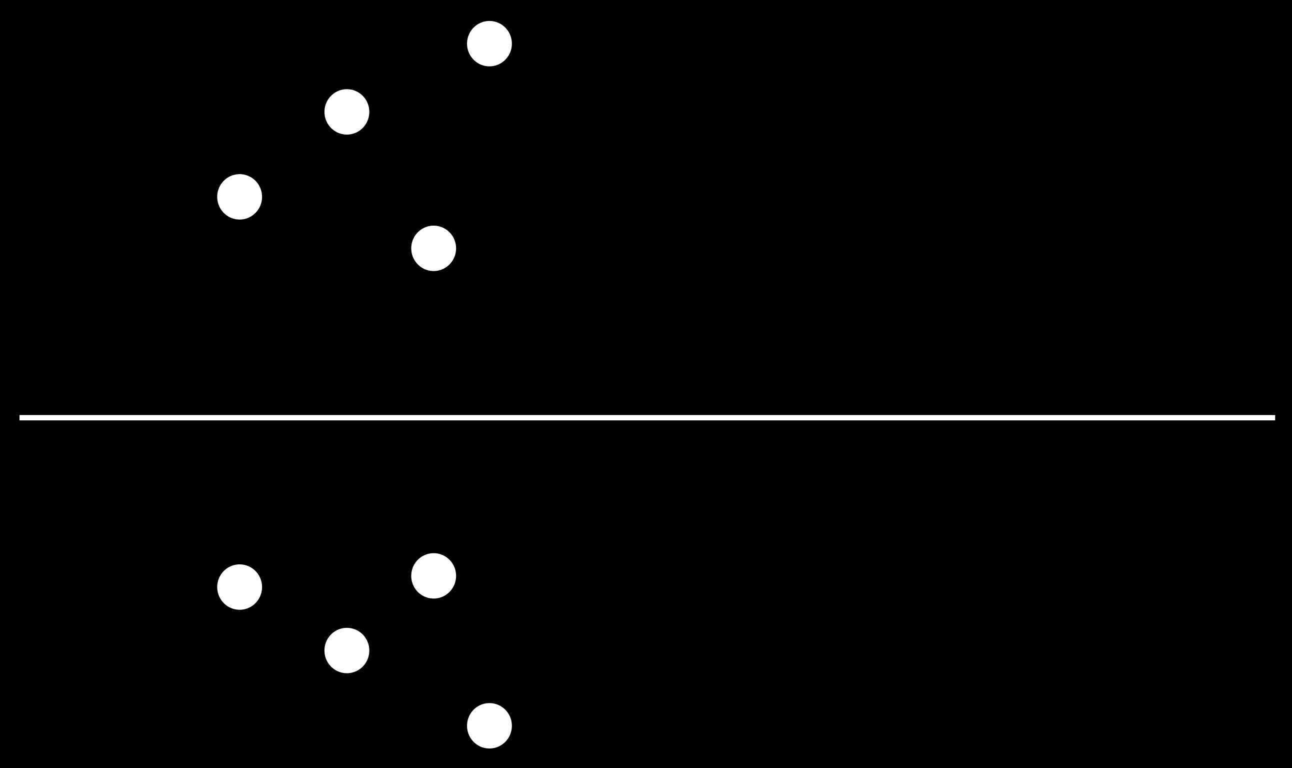

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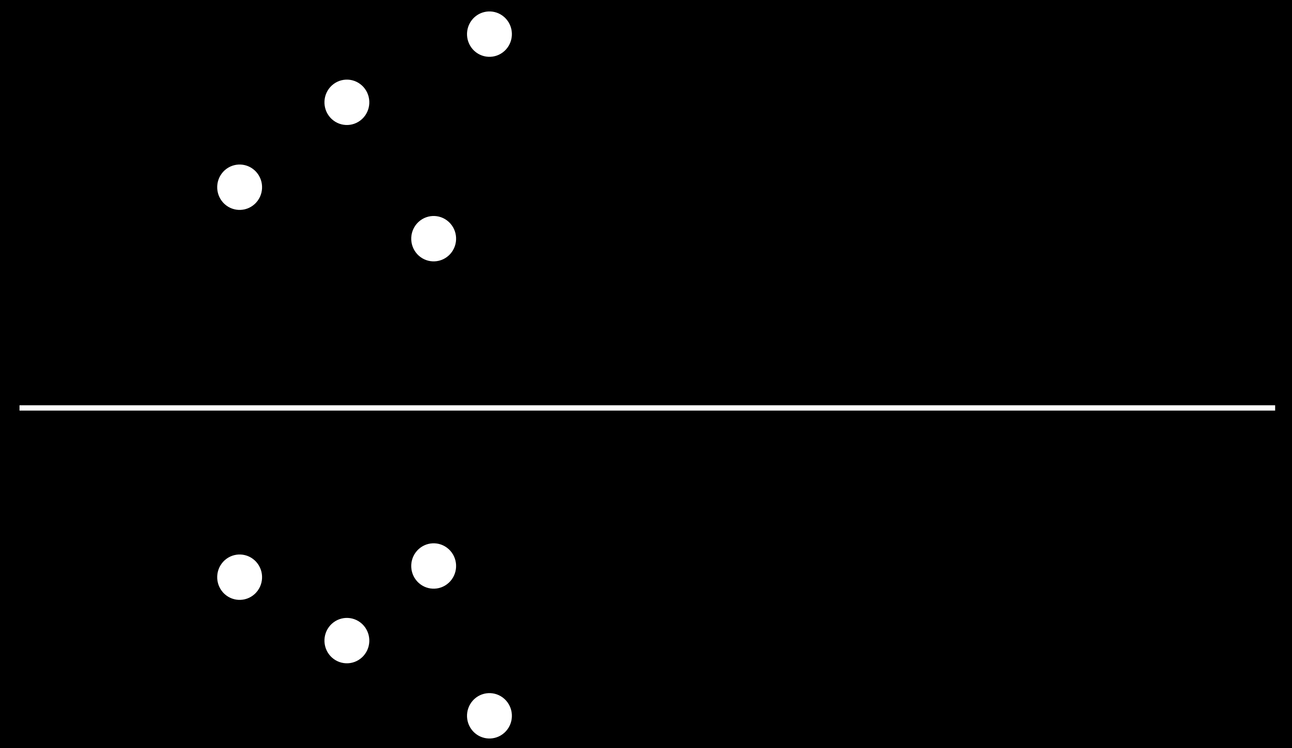

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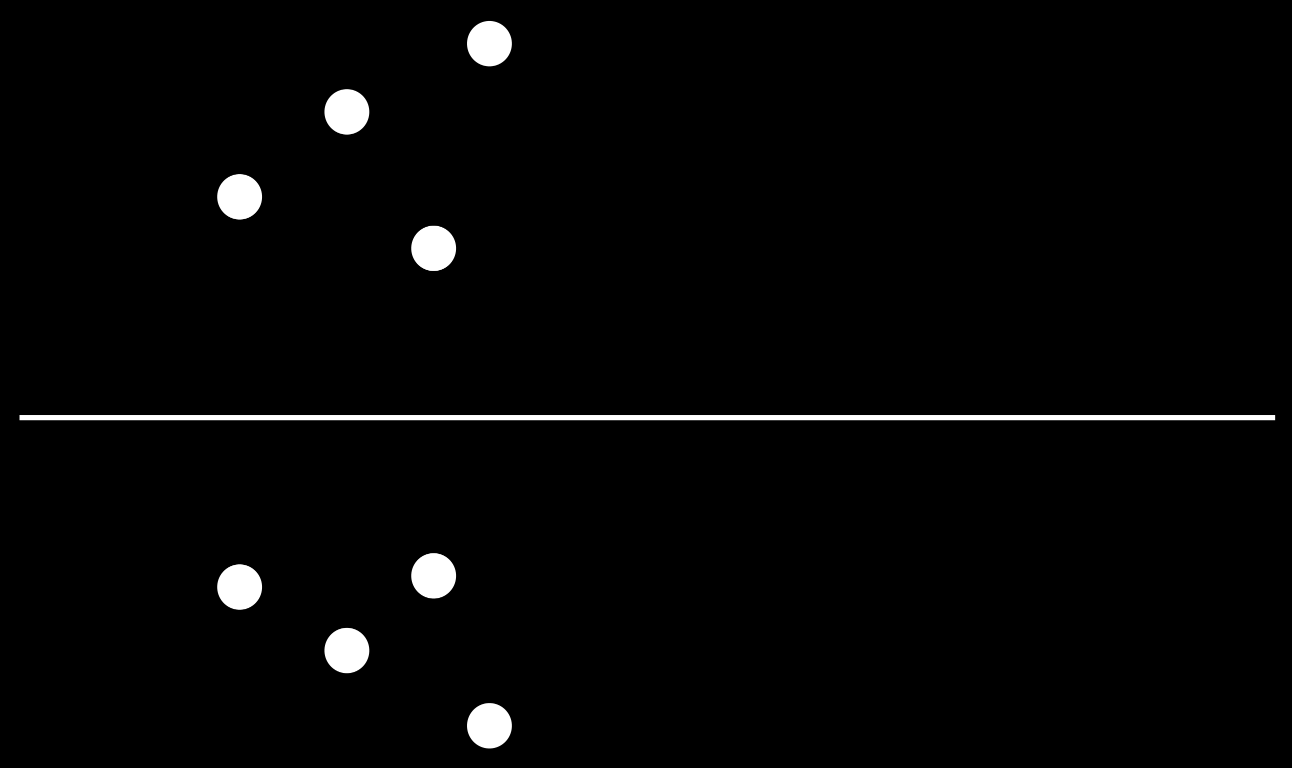
$$\begin{pmatrix} -\beta_k^* \\ \alpha_k^* \end{pmatrix} = \begin{pmatrix} -c_k e^{-2it\overline{\theta}} \\ 0 \end{pmatrix} - \left(\sum_{j=1}^{N} \frac{1}{z_k^* - z_j} \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} \right) c_k^* e^{-2it\overline{\theta}}$$

$$\psi(x,t) = -2i \sum_{k=1}^{N} \beta^*(x,t)$$

The N-soliton case: $r \equiv 0$, poles at $\{z_k, z_k^*\}_{k=1}^N$



$$M(z) = I + \sum_{j=1}^{N} \frac{1}{z - z_j} \begin{pmatrix} \alpha_j(x,t) & 0 \\ \beta_j(x,t) & 0 \end{pmatrix} + \frac{1}{z - z_j^*} \begin{pmatrix} 0 & -\beta_j(x,t)^* \\ 0 & \alpha_j(x,t)^* \end{pmatrix}$$

$$\operatorname*{Res}_{z_k} M = \lim_{z \to z_k} M \begin{pmatrix} 0 & 0 \\ c_k e^{2it\theta} & 0 \end{pmatrix}$$

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Research / Homework:

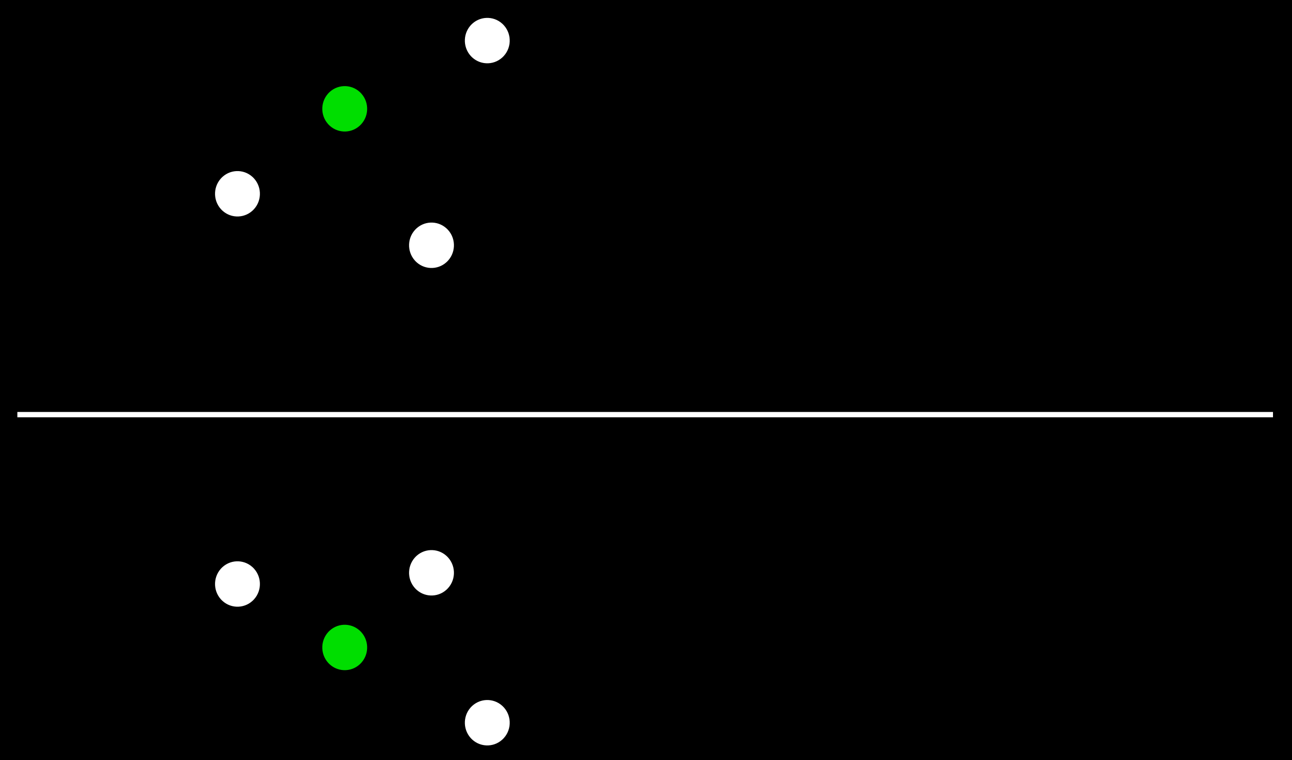
Develop numerical code for $N = 2$, then general N:

Inputs: N, $\{z_k = \mu_k + i\eta_k\}_{k=1}^N$, $\{c_k\}_{k=1}^N$, grid of space-time values

Output: solution ψ obtained by iteration, on the space-time grid

The N-soliton case: $r \equiv 0$. Can you just add a pole?

"Darboux transform"



M_{N-1}: solution with $N-1$ poles

M_N: solution with N poles

Form $E = M_N \left(M_{N-1}\right)^{-1}$ $\quad$ E only has poles at z_N and z_N^*

$$\operatorname*{Res}_{z_N} E = \operatorname*{Res}_{z_N} M_N \left(M_{N-1}\right)^{-1}$$

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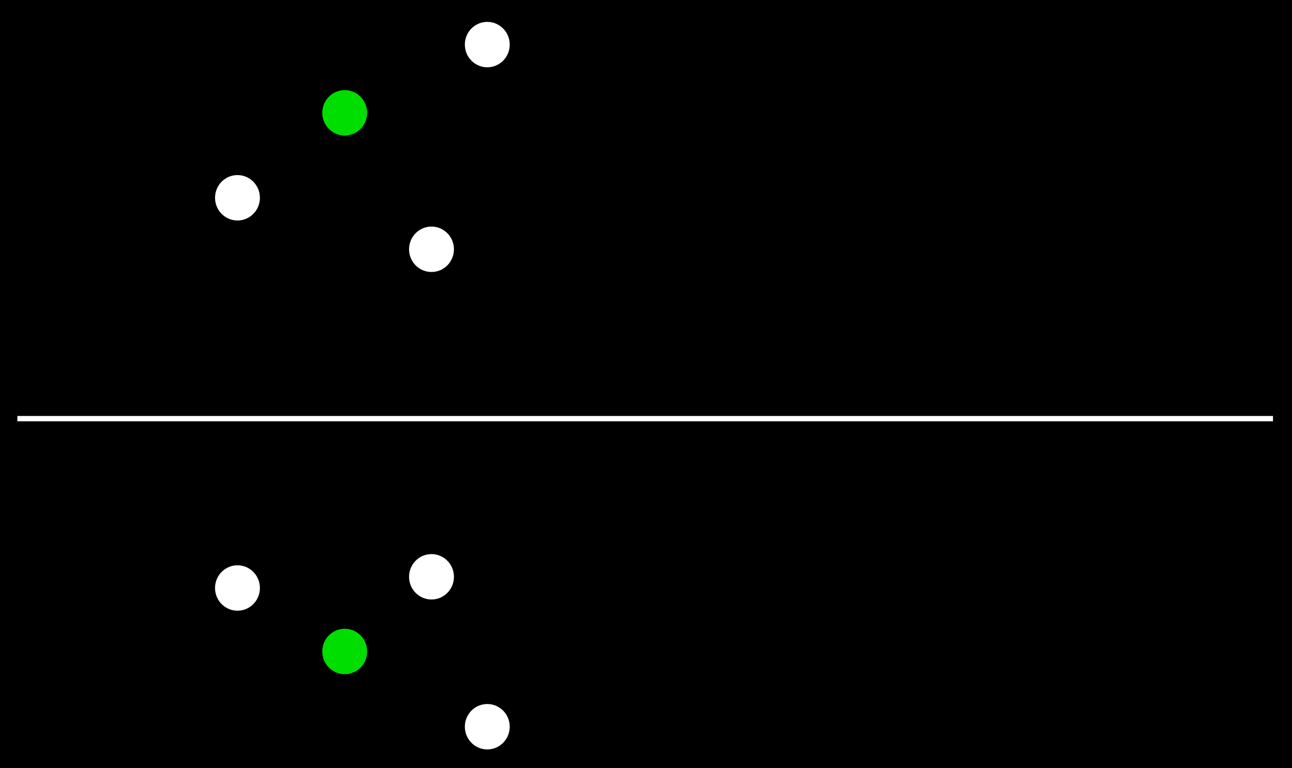

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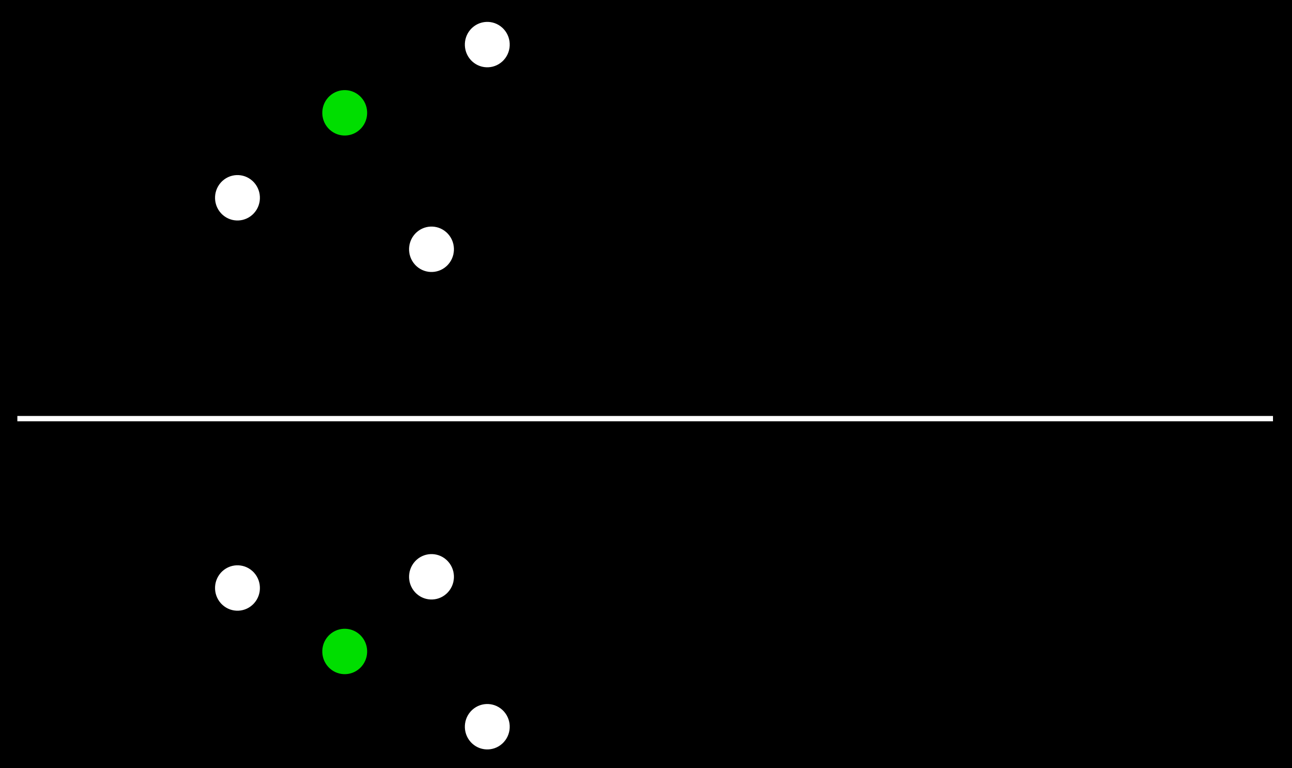

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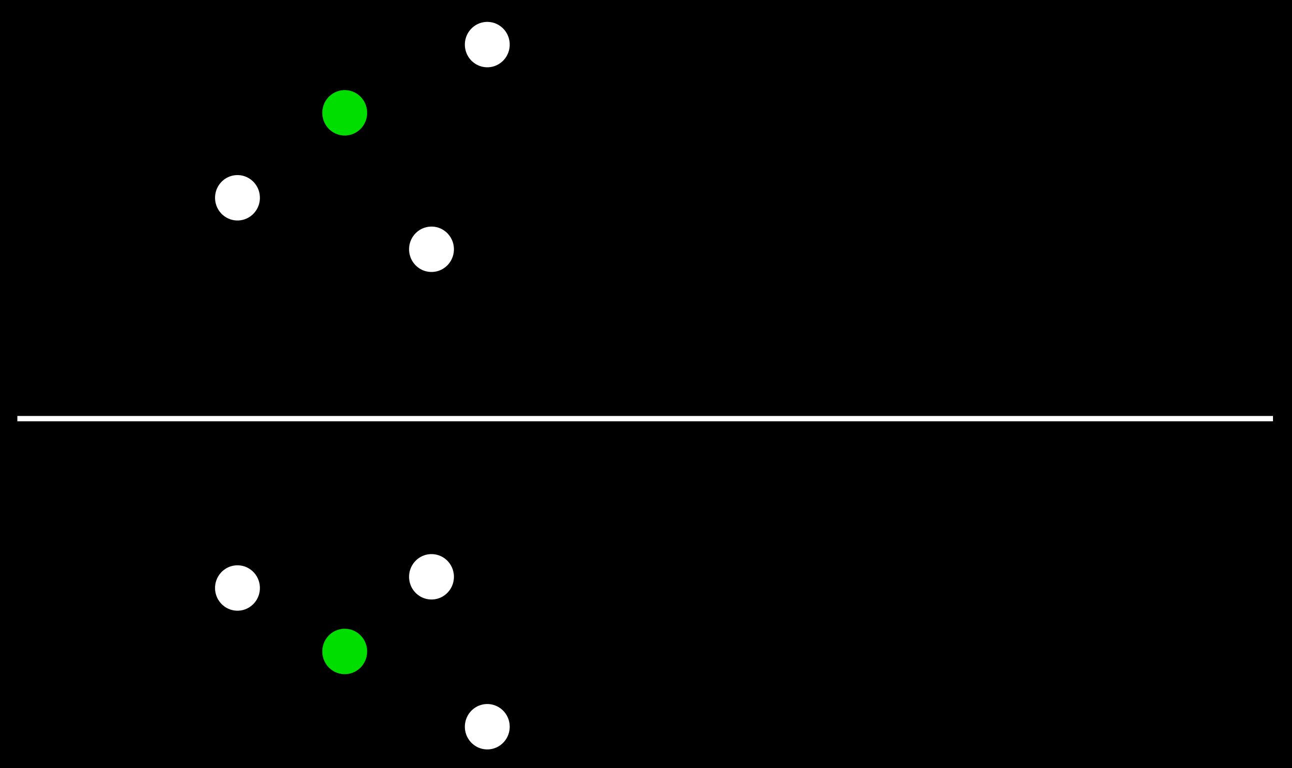

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$$= \lim_{z\to z_N} E M_{N-1} \begin{pmatrix} 0 & 0 \\ c_N e^{2it\theta} & 0 \end{pmatrix} \left(M_{N-1}\right)^{-1}$$

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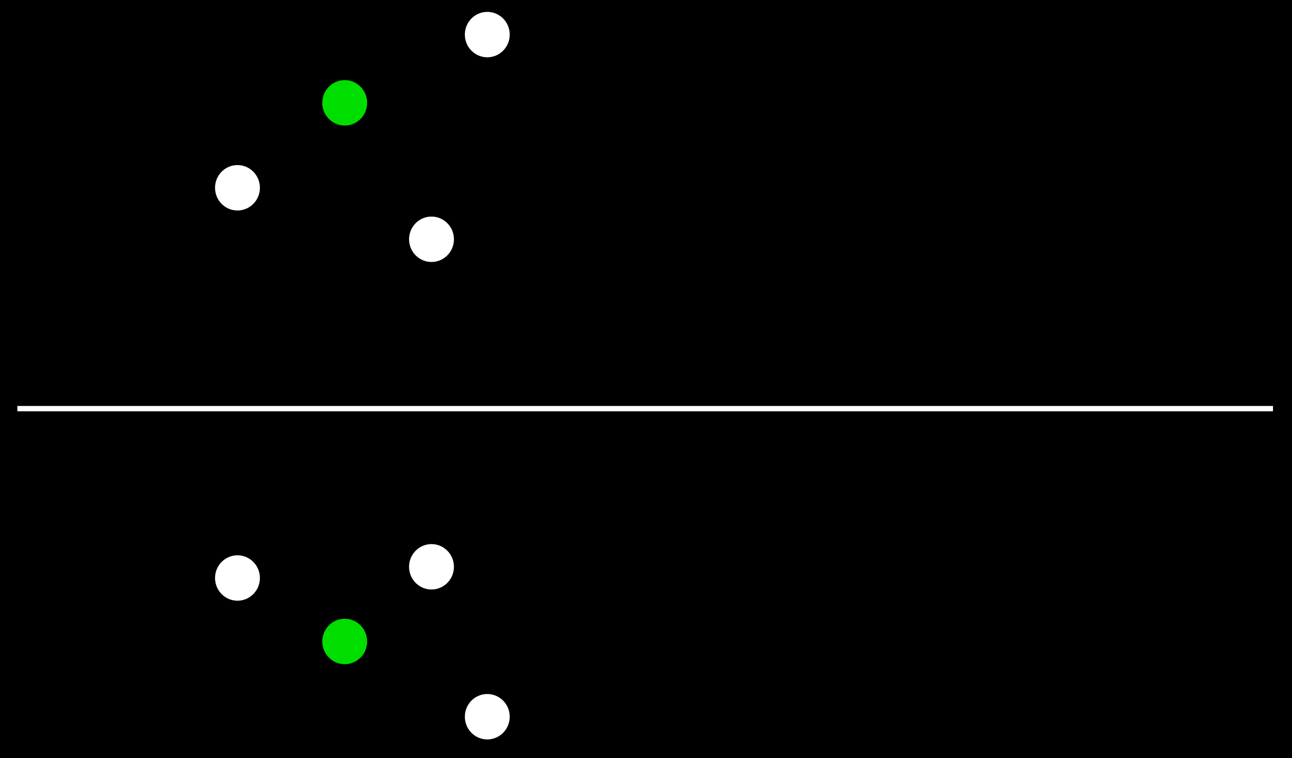

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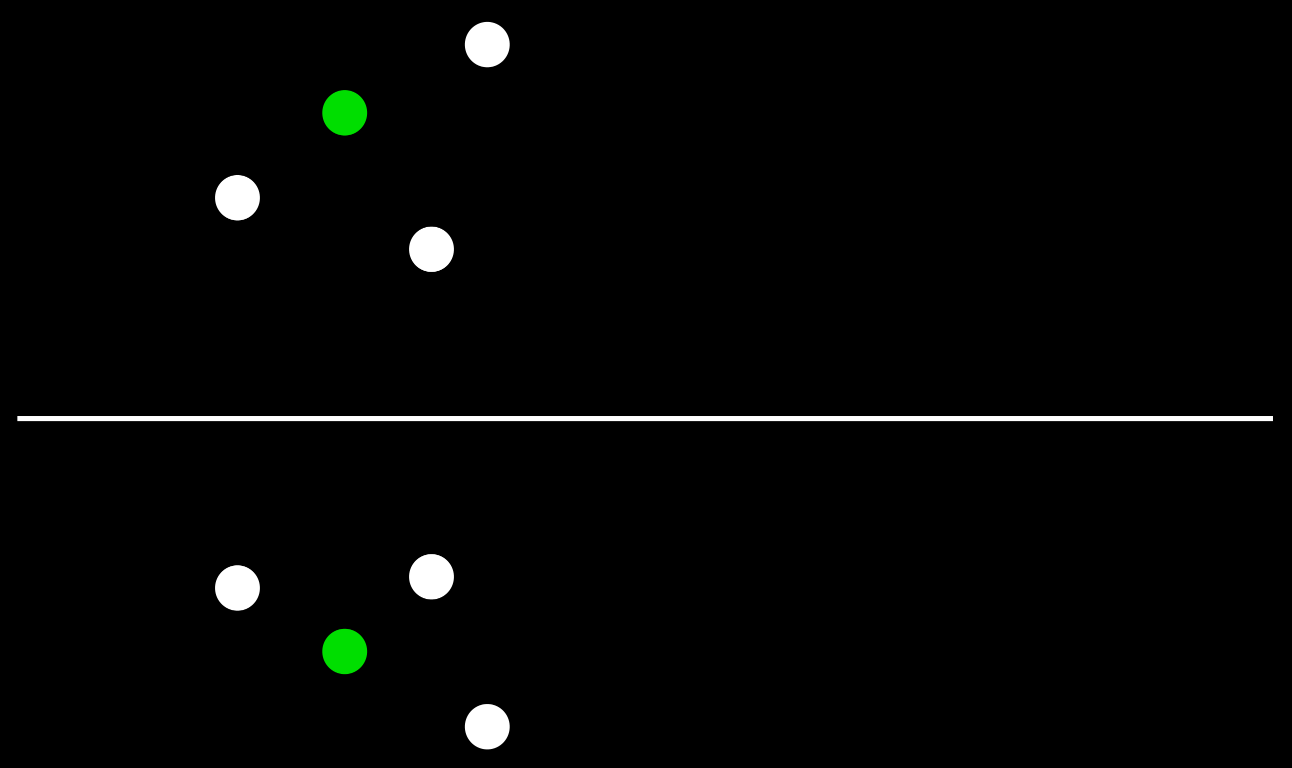

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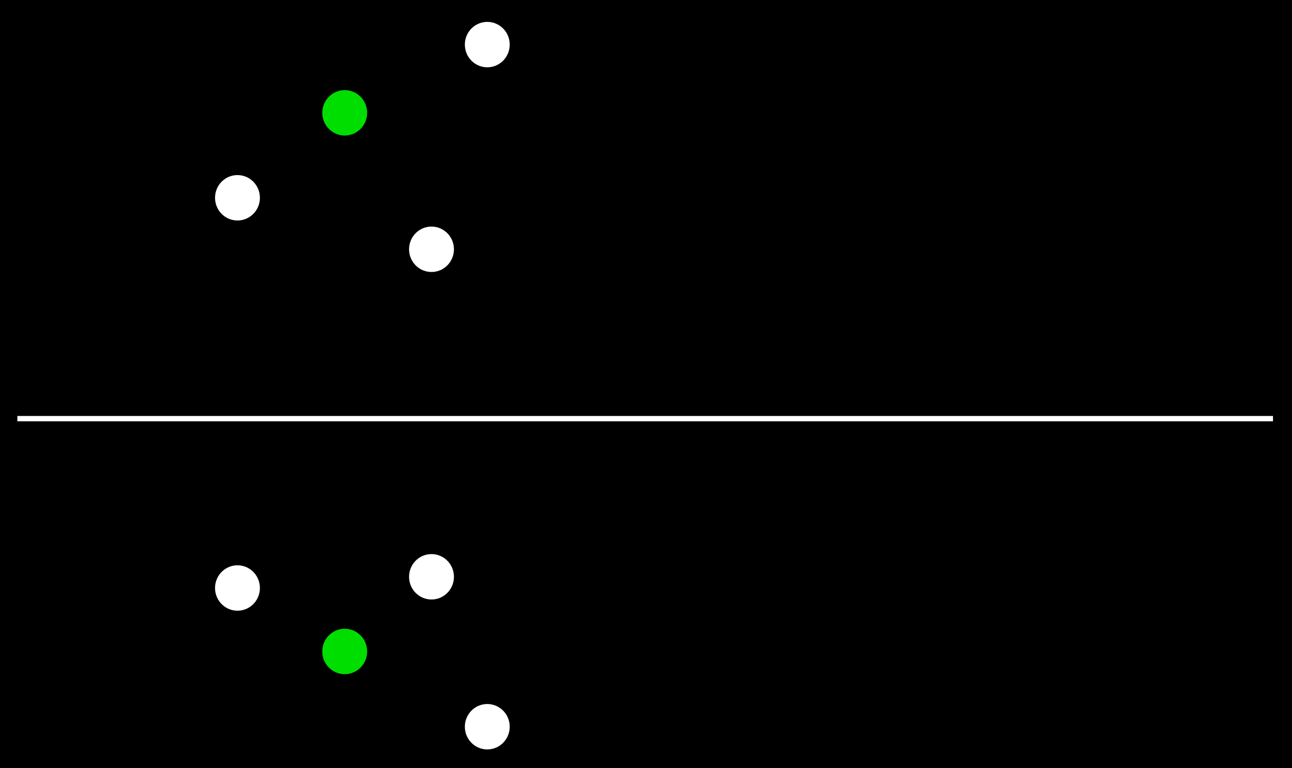

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$$\operatorname*{Res}_{z_N^*} E = \lim_{z \to z_N^*} E \left(M_{N-1} \begin{pmatrix} 0 & -c_N^* e^{-2it\theta} \\ 0 & 0 \end{pmatrix} (M_{N-1})^{-1} \right)$$

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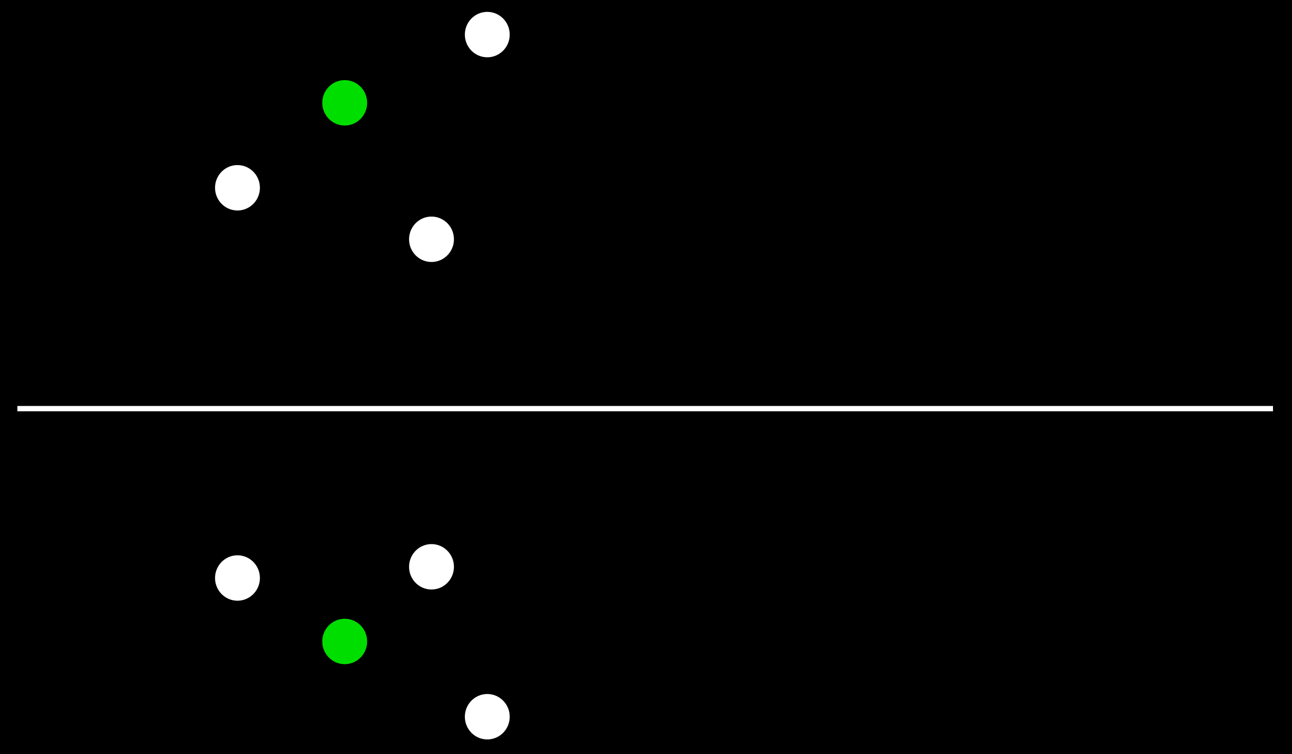

$$\operatorname*{Res}_{z_N} E = \lim_{z \to z_N} E \left(M_{N-1} \begin{pmatrix} 0 & 0 \\ c_N e^{2it\theta} & 0 \end{pmatrix} (M_{N-1})^{-1} \right)$$

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$$E(z) = I + \frac{P(x,t)}{z - z_N} + \frac{Q(x,t)}{z - z_N^*}$$

must have

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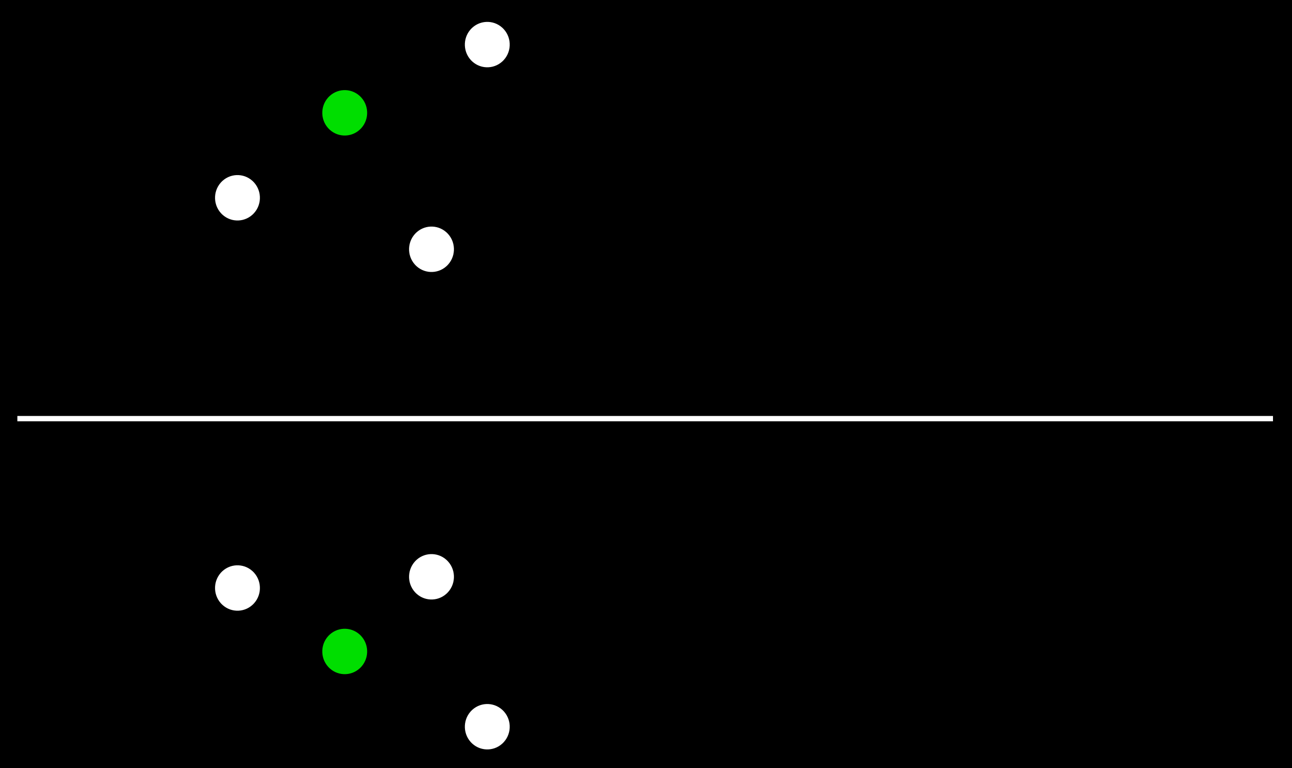

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$$E(z) = I + \frac{P(x,t)}{z - z_N} + \frac{Q(x,t)}{z - z_N^*}$$

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$$P(x,t) \cdot \left(M_{N-1} \begin{pmatrix} 0 & 0 \\ c_N e^{2it\theta} & 0 \end{pmatrix} \right) \Bigg|_{z=z_N} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$Q(x,t) \cdot \left(M_{N-1} \begin{pmatrix} 0 & -c_N^* e^{-2it\theta} \\ 0 & 0 \end{pmatrix} \right) \Bigg|_{z=z_N^*} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P(x,t) = \left(I + \frac{Q(x,t)}{z_N - z_N^*} \right) \left[\left(M_{N-1} \begin{pmatrix} 0 & 0 \\ c_N e^{2it\theta} & 0 \end{pmatrix} (M_{N-1})^{-1} \right) \Bigg|_{z=z_N} \right]$$

$$Q(x,t) = \left(I + \frac{P(x,t)}{z_N^* - z_N} \right) \left[\left(M_{N-1} \begin{pmatrix} 0 & -c_N^* e^{2-it\theta} \\ 0 & 0 \end{pmatrix} (M_{N-1})^{-1} \right) \Bigg|_{z=z_N^*} \right]$$

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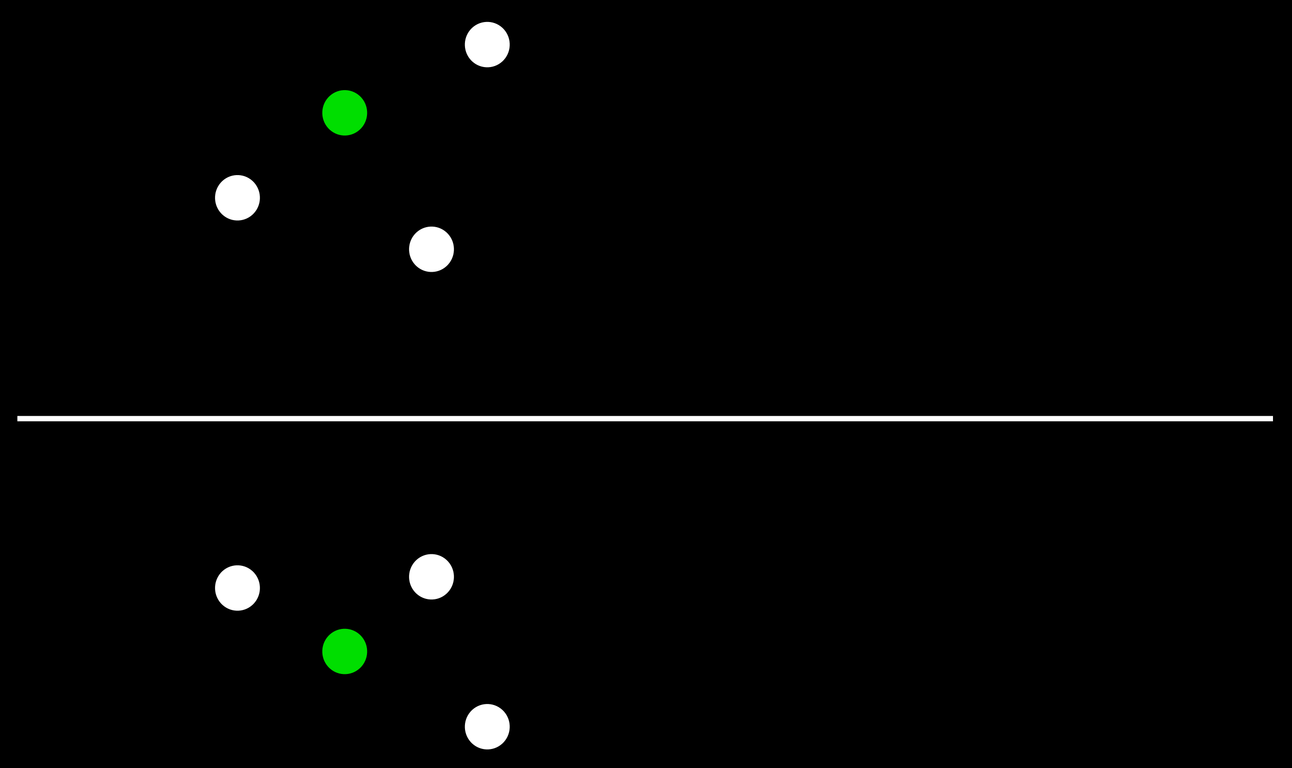

$$\operatorname*{Res}_{z_N} E = \lim_{z \to z_N} E \left(M_{N-1} \begin{pmatrix} 0 & 0 \\ c_N e^{2it\theta} & 0 \end{pmatrix} (M_{N-1})^{-1} \right)$$

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Research / Homework: Develop numerical code for $N = 2$, then general N:

Inputs: N, $\{z_k = \mu_k + i\eta_k\}_{k=1}^N$, $\{c_k\}_{k=1}^N$, grid of space-time values, carefully ordered,

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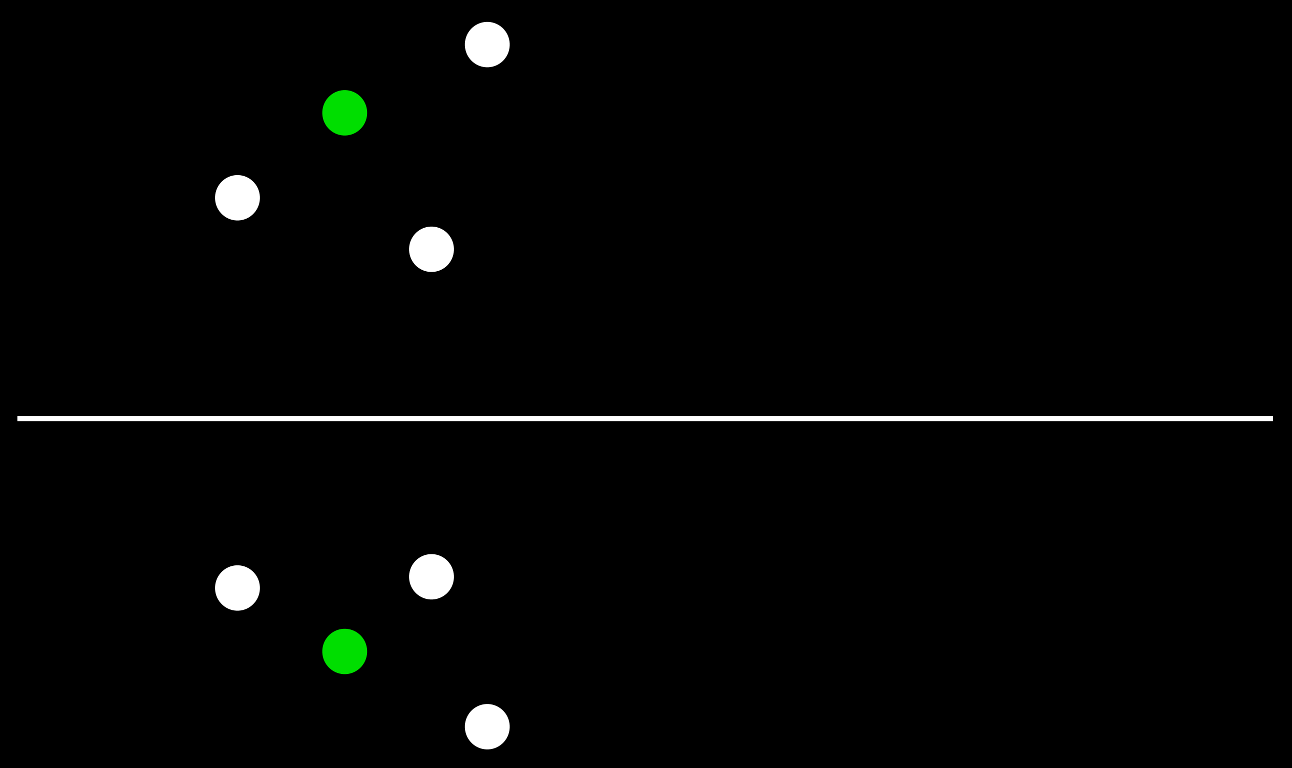

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A challenge: understand analytically how to make stable algorithms