Quasi-periodic steady invariant structures in incompressible fluids

(joint work with N.Masmoudi and R. Montalto)

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Outline

Euler equation in the channel

- Domain: two-dimensional finite channel $\mathbb{R} \times [-1, 1]$;
- Stationary Euler equation in vorticity-stream function formulation

$$\{\psi, \Delta\psi\} := \psi_x (\Delta\psi)_y - \psi_y (\Delta\psi)_x = 0\,; \qquad (1)$$

- Impermeability condition at the boundary: $\psi_x = 0$ on $\{y = \pm 1\}$;
- Velocity field: $(u(x, y), v(x, y)) := \nabla^{\perp}\psi(x, y) = (\partial_y \psi(x, y), -\partial_x \psi(x, y))$.

Informal Theorem

Let $\kappa_0 \in \mathbb{N}$. There exist $\varepsilon_0 > 0$ small enough and a family of stationary solutions $(\psi_\varepsilon(x, y) = \breve{\psi}_\varepsilon(\mathbf{x}, y)|_{\mathbf{x}=\widetilde{\omega}x})_{\varepsilon \in [0, \varepsilon_0]}$ of the Euler equation (1) in the finite channel $(x, y) \in \mathbb{R} \times [-1, 1]$ that are quasi-periodic in the horizontal direction $x \in \mathbb{R}$ for some frequency vector $\widetilde{\omega} \in \mathbb{R}^{\kappa_0}$, with $\mathbf{x} = \widetilde{\omega}x \in \mathbb{T}^{\kappa_0}$. Such family bifurcates from a shear equilibrium $\psi_{\mathfrak{m}}(y)$ and can be chosen to be arbitrarily close to the Couette flow $\psi_{\rm cou}(y) := \frac{1}{2}y^2$ in $H^s_{\mathbf{x}} H^{7/2-}_y(\mathbb{T}^{\kappa_0} \times [-1, 1])$, with $s > 0$ sufficiently large.

The hydrodynamics (in)stability problem

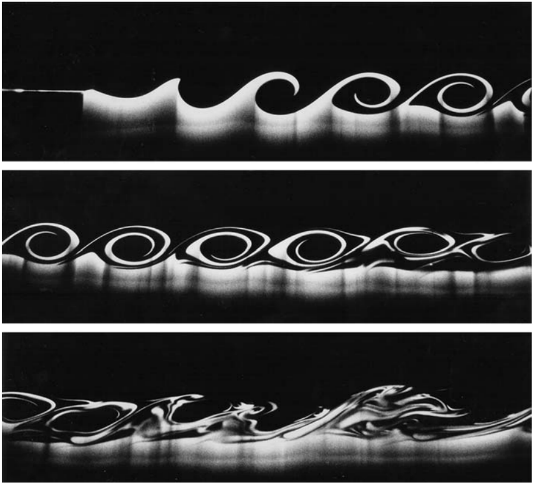


Figure: Kelvin-Helmholtz instability for two layered shear flows (credits: Lawrence et at. (1991) - Cushman-Roisin (2005))

Literature overview: inviscid dynamics around shear flows

- **Linear and nonlinear damping for Vlasov-Poisson** [Mouhot & Villani (2011)]
- **Linear inviscid damping for Euler close to Couette** [Kelvin (1887), Orr (1907), **Lin & Zeng** (2011)];
- **Nonlinear inviscid damping** [Bedrossian & Masmoudi (2015), Deng & Masmoudi (2018), Ionescu & Jia (2020)];
- **Stratified fluids with flows close to Couette** [Yang & Lin (2018), Bianchini, Coti-Zelati & Dolce (2020)];
- **Compressible fluids** [Antonelli, Dolce & Marcati (2021)];
- **Linear inviscid damping for other shear flows** [Zillinger (2017)];
- **Oscillatory stationary flows** [Li & Lin (2011), **Lin & Zeng** (2011), Coti-Zelati, Elgindi & Widmayer (2020)].

Literature overview: quasi-periodic flows

- **Time quasi-periodic solution in fluid dynamics**
  - **2D water waves equation** [Berti & Montalto (2017), Baldi, Berti, Haus & Montalto (2018), Berti, F. & Maspero (2020,2021), Feola & Giuliani (2020)];
  - **Vortex patches in active scalar equations** [Berti, Hassainia & Masmoudi (2022), Hmidi & Roulley (2021), Hassainia, Hmidi & Masmoudi (2021), Hassainia & Roulley (2022), Hassainia, Hmidi & Roulley (2023), Roulley (2022), Garcia, Hassainia & Roulley (2023), Gómez-Serrano, Ionescu & Park (2023)];
  - **Forced Euler and Navier-Stokes** [Baldi & Montalto (2021), Montalto (2021), F. & Montalto (2022)];
  - **Non-resonant Euler flows** [Crouseilles & Faou (2013), Enciso, Peralta-Salas & Torres de Lizaur (2022)];
- **Space quasi-periodic and "spatial dynamics" in PDE**
  - **Space bi-periodic analysis** [Scheurle (1983), Iooss & Los (1990), Iooss & Mielke (1991), Bridges & Rowland (1994), Bridges & Dias (1996)];
  - **Quasi-periodic for semilinear elliptic PDE** [Valls (2006), Poláčic & Valdebenito (2017)].
  - **Remark!** In these results and in ours, **space quasi-periodic** means quasi-periodic in **ONE selected** space direction!

Lin & Zeng $\frac{3}{2}$-threshold for periodic flows

Theorem (Lin & Zeng (ARMA, '11))

Let $T > 0$ be a fixed period. The following hold:

• $s \in [0, \frac{3}{2})$*: For any* $\varepsilon > 0$*, there exists* $\psi_\varepsilon(x, y)$ *solution of* (1) *such that* ψ_ε *has minimal* x*-period* T*,*

$$\|\Delta\psi_\varepsilon - 1\|_{H^s((0,T)\times(-1,1))} < \varepsilon \,,$$

and $-\partial_x\psi_\varepsilon(x, y)$ *is non-trivial in* $\mathbb{R} \times [-1, 1]$*;*

• $s > \frac{3}{2}$*: There exists* $\varepsilon_0 > 0$ *such that, for any traveling solution* $\psi(x - ct, y)$*,* $c \in \mathbb{R}$*, of the Euler equation* $\partial_t\Delta\psi + \{\psi, \Delta\psi\} = 0$ *on* $\mathbb{R} \times [-1, 1]$ *with* x*-period* T *and satisfying*

$$\|\Delta\psi - 1\|_{H^s((0,T)\times(-1,1))} < \varepsilon_0 \,,$$

we must have $\partial_x\psi(x - ct, y) \equiv 0$ *in the whole channel* $\mathbb{R} \times [-1, 1]$ *for all times* $t \in \mathbb{R}$*.*

Important consequences:

• When $s > \frac{3}{2}$, the non-existence of non-trivial invariant structures is a hint for the nonlinear damping for inviscid flows close to Couette in H^s;

• When $s \in [0, \frac{3}{2})$, there exist inviscid flows close to Couette in H^s that cannot damp down to a shear.

Lin-Zeng construction of periodic solutions

- **Starting point:** a particular (monotone) shear flow ($A > 0$, $0 < \gamma \ll 1$)

$$U_{\gamma,A}(y) = y + A\gamma^2 \mathrm{erf}\left(\tfrac{y}{\gamma}\right) := y + \tfrac{2}{\sqrt{\pi}} A\gamma^2 \int_0^{\frac{y}{\gamma}} e^{-s^2}\, \mathrm{d}s\,, \quad y \in [-1,1]\,,$$

with associated stream function $\psi_{\gamma,A}(y)$ (i.e. $\psi'_{\gamma,A} = U_{\gamma,A}$) satisfying

$$\psi''_{\gamma,A} = F(\psi_{\gamma,A})\,, \quad \text{for some } \; F : \mathbb{R} \to \mathbb{R} \; \text{ inherited from } \; \psi_{\gamma,A}\,;$$

- **Goal:** Construct the steady stream function $\psi(x,y) = \psi_{\gamma,A}(y) + \phi(x,y)$, periodic in x with period $2\pi/\alpha$, that solves

$$\Delta\psi(x,y) = F(\psi(x,y)) \quad \text{with the same nonlinearity } \; F : \mathbb{R} \to \mathbb{R}\,; \tag{2}$$

if (2) is fulfilled, then $\psi(x,y)$ is a solution of the Euler equation (1)!

- When we linearize (2) around $\psi_{\gamma,A}$, we get

$$\partial_x^2 \phi = -\partial_y^2 \phi + F'(\psi_{\gamma,A})\phi + o(|\phi|^2) \quad \text{and} \quad F'(\psi_{\gamma,A}) = \frac{\psi'''_{\gamma,A}}{\psi'_{\gamma,A}} = \frac{U''_{\gamma,A}}{U_{\gamma,A}} \ldots$$

What leads to periodic oscillations

Lemma

Let $\mathcal{L}_{\gamma,A} : H^2(-1,1) \to L^2(-1,1)$ be the Schrödinger operator

$$\mathcal{L}_{\gamma,A} = -\partial_y^2 + Q_{\gamma,A}(y)\,, \quad Q_{\gamma,A}(y) := F'(\psi_{\gamma,A}(y)) := \frac{U''_{\gamma,A}(y)}{U_{\gamma,A}(y)}\,,$$

with zero Dirichlet boundary conditions at $\{y = \pm 1\}$. For any fixed $A > \frac{1}{2}$, for $\gamma > 0$ small enough, the operator $\mathcal{L}_{\gamma,A}$ has a (unique) **negative** *eigenvalue $-\beta^2_{\gamma,A}$. The remaining part of the spectrum consists of positive eigenvalues.*

- *Key property:* in the limit $\gamma \to 0$, the potential

$$Q_{\gamma,A}(y) = -4A\frac{1}{\gamma\sqrt{\pi}}e^{-\left(\frac{y}{\gamma}\right)^2}\left(1 + A\gamma\,\frac{\gamma}{y}\mathrm{erf}\left(\frac{y}{\gamma}\right)\right)^{-1} \overset{\gamma\to 0}{\to} -4A\delta_0(y)$$

in the sense of distributions, and the delta potential carries one negative eigenvalue for the Schrödinger operator

$$\mathcal{L}_{0,A} := -\partial_y^2 - 4A\delta_0(y)\,.$$

The bifurcation argument in Lin & Zeng

Proposition

Assume $U \in \mathcal{C}^5[-1,1]$, **monotone** *in* $[-1,1]$ *and with* $U'(0) > 0$, $U''(0) = 0$. *Define*

$$\mathcal{L} := -\partial_y^2 + Q(y) : H^2(-1,1) \to L^2(-1,1)\,, \quad Q(y) = \frac{U''(y)}{U(y) - U(0)}\,,$$

acting with zero Dirichlet conditions at $\{y = \pm 1\}$*. If* $\mathcal{L}$ *has a negative eigenvalue* $-k_0^2$ *with positive eigenfunction* $\phi_0(y)$*, then there exists* $\varepsilon_0 > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0)$*, there exists a steady solution* $(u_\varepsilon(x,y), v_\varepsilon(x,y))$ *to the Euler equation, periodic in* x *with minimal period* $T_\varepsilon \to \frac{2\pi}{k_0}$ *as* $\varepsilon \to 0$*, such that*

$$\Delta\psi_\varepsilon = F(\psi_\varepsilon)\,, \quad \|\Delta\psi_\varepsilon - U'(y)\|_{H^2} = \varepsilon$$

and the vector field near $\{y = 0\}$ *has leading order in* $\varepsilon \to 0$ *given by*

$$\begin{cases} u_\varepsilon(x,y) \sim U(y) + \varepsilon\, \phi_0'(y) \cos(\frac{2\pi}{T_\varepsilon} x) \\ v_\varepsilon(x,y) \sim \qquad\quad - \varepsilon\, \frac{2\pi}{T_\varepsilon} \phi_0(y) \sin(\frac{2\pi}{T_\varepsilon} x) \end{cases}$$

[Kelvin's cat-eyes flow].

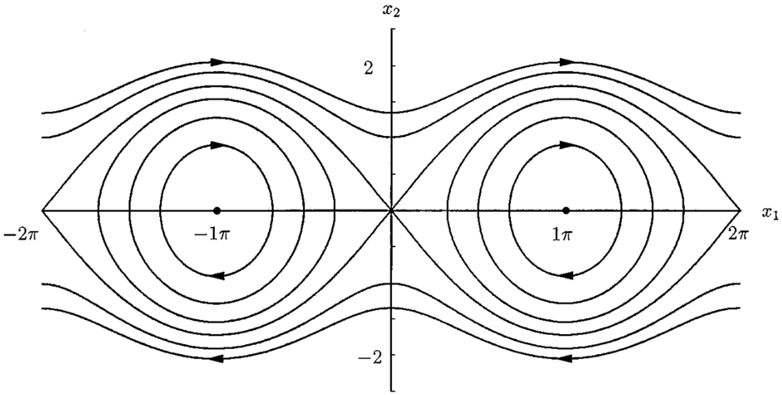


FIGURE: Streamlines of the Kelvin–Stuart cat's-eye flow (source: Majda-Bertozzi book)

Main steps of the proof:

- Construction of a nonlinearity $F \in \mathcal{C}_0^2(\mathbb{R})$, $[\max \psi_0, \min \psi_0] \subseteq \mathrm{spt}(f)$, where $\phi_0' = U$, by solving the Cauchy problem (assuming $U(y)$ odd symmetric!)

$$\begin{cases} F'(z) = Q(\psi_0^{-1}(z)) \\ F(\psi_0(0)) = \psi_0''(0) \end{cases} \rightsquigarrow \quad F(\psi_0(y)) = \psi_0''(y)\,, \quad F'(\psi_0(y)) = Q(y)\,;$$

- Look for a stream function of the form $\psi(x,y) = \psi_0(y) + \phi(\xi, y)|_{\xi=\alpha x}$ with $\phi(\xi, y)$ 2π-periodic in x, such that

$$\begin{cases} \Delta\psi = F(\psi), \\ \psi(x, \pm 1) = \psi_0(\pm 1) \end{cases} \rightsquigarrow \begin{cases} \alpha^2 \partial_\xi^2 \phi + \partial_y^2 \phi - \big(F(\phi + \psi_0) - F(\psi_0)\big) = 0, \\ \phi(\xi, \pm 1) = 0\,. \end{cases}$$

- Apply Crandall-Rabinowitz Theorem with the nonlinear functional

$$\mathcal{F}(\phi, \alpha^2) := \alpha^2 \partial_\xi^2 \phi + \partial_y^2 \phi - \big(F(\phi + \psi_0) - F(\psi_0)\big) = 0,$$

bifurcating from the kernel $\mathrm{Ker}(\mathcal{G}) := \{\cos(\xi)\phi_0(y)\}$ of the linearized operator

$$\mathcal{G} := \mathrm{d}_\phi \mathcal{F}(0, k_0^2) := k_0^2 \partial_\xi^2 + \partial_y^2 - F'(\psi_0) = k_0^2 \partial_\xi^2 - \mathcal{L}.$$

What leads to space quasi-periodic flows

- **Key object:** a **well** prescribed analytic potential $Q_{\mathfrak{m}}(y)$, even in y, depending on a parameter $\mathfrak{m} \gg 1$ such that, in the limit $\mathfrak{m} \to \infty$, it uniformly approaches the classical potential well

$$Q_{\mathfrak{m}}(y) = Q_{\mathfrak{m}}(\mathtt{E}, \mathtt{r}; y) \overset{\mathfrak{m}\to\infty}{\to} Q_{\infty}(\mathtt{E}, \mathtt{r}; y) := \begin{cases} 0 & |y| > \mathtt{r}\,, \\ -\mathtt{E}^2 & |y| < \mathtt{r}\,; \end{cases}$$

- **Constrain:** the depth $\mathtt{E} > 1$ and the width $\mathtt{r} \in (0,1)$ are related by

$$\mathtt{E}\mathtt{r} = \kappa_0(\pi + \tfrac{1}{4}),$$

for a given $\kappa_0 \in \mathbb{N}$, **fixed from the very beginning**, which counts the exact number of negative eigenvalues $-\lambda_{1,\mathfrak{m}}^2(\mathtt{E}), \ldots, -\lambda_{\kappa_0,\mathfrak{m}}^2(\mathtt{E}) < 0$ for the operator

$$\mathcal{L}_{\mathfrak{m}} := -\partial_y^2 + Q_{\mathfrak{m}}(y)\,, \quad \text{with eigenfunctions} \quad \mathcal{L}_{\mathfrak{m}}\phi_{j,\mathfrak{m}} = -\lambda_{j,\mathfrak{m}}^2\phi_{j,\mathfrak{m}}\,,$$

with Dirichlet boundary conditions on $[-1,1]$. The rest of the spectrum $(\lambda_{j,\mathfrak{m}}^2(\mathtt{E}))_{j\geqslant\kappa_0+1}$ is strictly positive.

The shear equilibrium

- We define the stream function $\psi_{\mathfrak{m}}(y)$ as the solution of the linear ODE

$$\psi_{\mathfrak{m}}'''(y) = Q_{\mathfrak{m}}(y)\psi_{\mathfrak{m}}'(y)\,, \quad y \in [-1,1]\,. \tag{3}$$

- When $|y| > \mathtt{r}$, $\psi_{\mathfrak{m}}(y)$ behaves as the Couette flow, with

$$\psi_{\mathfrak{m}}'(y) \overset{\mathfrak{m}\to\infty}{\to} y - A_{\rm out}\,\mathrm{sgn}(y)\,, \quad |y| > \mathtt{r}\,;$$

- When $|y| < \mathtt{r}$, $\psi_{\mathfrak{m}}(y)$ ceases to be monotone and exhibits oscillations, with

$$\psi_{\mathfrak{m}}'(y) \overset{\mathfrak{m}\to\infty}{\to} A_{\rm in}\sin(\mathtt{E}y)\,, \quad |y| < \mathtt{r}\,.$$

In particular, $\psi_{\mathfrak{m}}(y)$ has exactly $2\kappa_0 + 1$ critical points

$$0 =: \mathtt{y}_{0,\mathfrak{m}} < |\mathtt{y}_{1,\mathfrak{m}}| < \ldots < |\mathtt{y}_{\kappa_0,\mathfrak{m}}| < \mathtt{r}\,.$$

Proposition (Proximity to the Couette flow)

There exists an even stream function $\psi_{\mathfrak{m}}(y)$, solution to (3), such that

$$\|\psi_{\mathfrak{m}} - \psi_{\rm cou}\|_{H^3[-1,1]} \lesssim \sqrt{\mathtt{r}} \quad \textit{and} \quad \|\psi_{\mathfrak{m}} - \psi_{\rm cou}\|_{H^4[-1,1]} \gtrsim \frac{1}{\sqrt{\mathtt{r}}}\,.$$

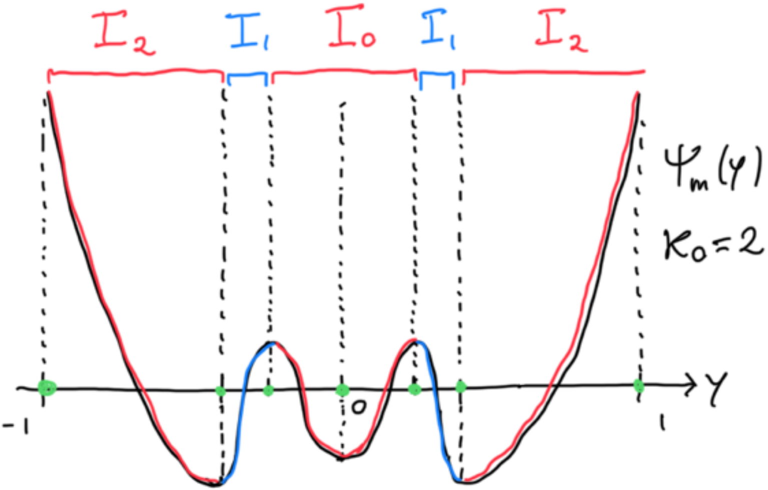


FIGURE: A (non-scaled) picture of the stream function $\psi_{\mathfrak{m}}(y)$, with $\kappa_0 = 2$

The (unperturbed) nonlinearity

- We write $[-1,1] = \bigcup_{p=0,1,\ldots,\kappa_0} \mathtt{I}_p$, where

$$\mathtt{I}_p := \{y \in \mathbb{R} : \mathtt{y}_{p,\mathfrak{m}} \leqslant |y| \leqslant \mathtt{y}_{p+1,\mathfrak{m}}\}, \quad p = 1, \ldots, \kappa_0 \,, \quad \mathtt{y}_{\kappa_0+1,\mathfrak{m}} := 1 \,.$$

The stream function $\psi_{\mathfrak{m}}(y)$ solves **locally** on each set $\mathtt{I}_p$ a second-order nonlinear ODE.

Theorem (Local nonlinearities)

Let $S \in \mathbb{N}$ and let $\mathfrak{m} \geqslant \overline{\mathfrak{m}}(\mathtt{r}) \gg 1$. For any $p = 0, 1, \ldots, \kappa_0$, there exists a nonlinear function $F_{p,\mathfrak{m}} \in \mathcal{C}_0^{S+1}(\mathbb{R})$, $\psi \to F_{p,\mathfrak{m}}(\psi)$, such that

$$(\partial_\psi F_{p,\mathfrak{m}})(\psi_{\mathfrak{m}}(y)) = Q_{\mathfrak{m}}(y) \,, \ \ y \in [-1,1] \Rightarrow \ \psi''_{\mathfrak{m}}(y) = F_{p,\mathfrak{m}}(\psi_{\mathfrak{m}}(y)) \,, \ \ y \in \mathtt{I}_p \,.$$

We have $\mathcal{C}^{S+1}$-continuity at $\psi = \psi_{\mathfrak{m}}(y)$ at the critical points $y = \pm \mathtt{y}_{p,\mathfrak{m}}$, $p = 1, \ldots, \kappa_0$, meaning that, for any $n = 0, 1, \ldots, S+1$,

$$\lim_{|y| \to \mathtt{y}_{p,\mathfrak{m}}^-} \partial_y^n (F_{p-1,\mathfrak{m}}(\psi_{\mathfrak{m}}(y))) = \lim_{|y| \to \mathtt{y}_{p,\mathfrak{m}}^+} \partial_y^n (F_{p,\mathfrak{m}}(\psi_{\mathfrak{m}}(y))) = \psi_{\mathfrak{m}}^{(n+2)}(\mathtt{y}_{p,\mathfrak{m}}) \,.$$

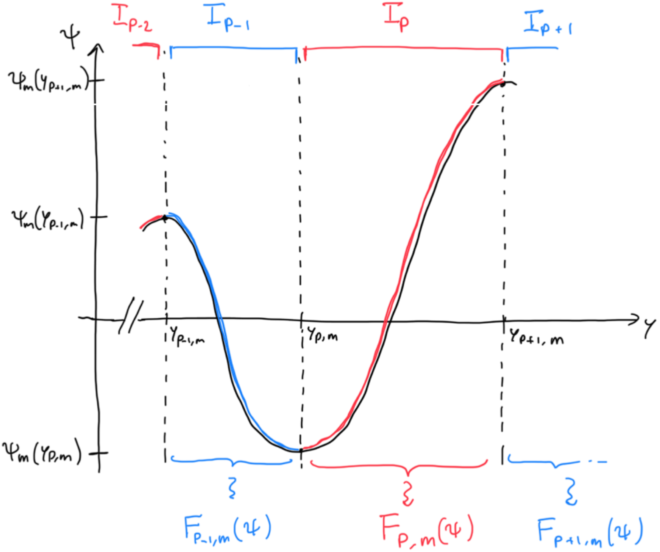


FIGURE: Picture of the stream function $\psi_{\mathfrak{m}}(y)$ close to the critical point $y_{p,\mathfrak{m}}$.

Around the shear equilibrium

- We search for nontrivial stream functions close to $\psi_{\mathfrak{m}}(y)$ of the form

$$\psi(x,y) = \psi_{\mathfrak{m}}(y) + \varphi(x,y)\,, \quad \text{with } \varphi(x,y) \textbf{ space quasi-periodic in } x \in \mathbb{R}$$

Definition

Let $\kappa_0 \in \mathbb{N}$. A function $\mathbb{R} \ni x \mapsto u(x)$ is **quasi-periodic** if there exist a function $\mathbb{T}^{\kappa_0} \ni \mathbf{x} \mapsto \breve{u}(\mathbf{x})$ and a frequency vector $\omega \in \mathbb{R}^{\kappa_0} \backslash \{0\}$ such that

$$u(x) = \breve{u}(\mathbf{x})|_{\mathbf{x}=\omega x}\,, \quad \text{with } \omega \cdot \ell \neq 0 \quad \forall\, \ell \in \mathbb{Z}^{\kappa_0} \backslash \{0\}\,.$$

- *Domain:* $\mathcal{D} := \mathbb{T}^{\kappa_0} \times [-1,1] \hookrightarrow \mathbb{R} \times [-1,1]$, $\mathbb{T}^{\kappa_0} := (\mathbb{R}/2\pi\mathbb{Z})^{\kappa_0}$;
- *Equation for the perturbation:* having $\varphi(x,\pm 1) = 0$,

$$\{\psi, \Delta\psi\} = 0 \overset{\psi=\psi_{\mathfrak{m}}(y)+\varphi}{\rightsquigarrow} \{\psi_{\mathfrak{m}}, \Delta_\omega \breve{\varphi}\} + \{\breve{\varphi}, \psi_{\mathfrak{m}}''\} + \{\breve{\varphi}, \Delta_\omega \breve{\varphi}\} = 0$$

where

$$\Delta := \partial_x^2 + \partial_y^2 \rightsquigarrow \Delta_\omega := (\omega \cdot \partial_{\mathbf{x}})^2 + \partial_y\,.$$

- If we search for $\psi(x,t)$ such that, for any $p = 0, 1, ..., \kappa_0$

$$\psi''_{\mathfrak{m}}(y) = F_{p,\mathfrak{m}}(\psi_{\mathfrak{m}}(y)) \rightsquigarrow \Delta\psi(x,y) = F_{p,\mathfrak{m}}(\psi(x,y)), \quad (x,y) \in \mathbb{R} \times \mathtt{I}_p$$

then we immediately lose continuity at $\{|y| = \mathtt{y}_{p,\mathfrak{m}}\}$: in general

$$\lim_{|y| \to \mathtt{y}_{p,\mathfrak{m}}^-} F_{p-1,\mathfrak{m}}(\psi_{\mathfrak{m}}(y) + \varphi(x,y)) \neq \lim_{|y| \to \mathtt{y}_{p,\mathfrak{m}}^+} F_{p,\mathfrak{m}}(\psi_{\mathfrak{m}}(y) + \varphi(x,y))$$

unless $\varphi(x, \pm\mathtt{y}_{p,\mathfrak{m}}) = 0$ for any $x \in \mathbb{R}$ (too strict!)

- **Key idea**: if you look for pertubations $\varphi(x,y) = O(\varepsilon)$, perturb the nonlinearities as well!

$$F_{p,m}(\psi) \rightsquigarrow F_{p,\varepsilon}(\psi), \quad \varepsilon \in (0,1);$$

Properties that we want:

  - It is perturbative: $F_{p,\varepsilon}(\psi) \to F_{p,\mathfrak{m}}(\psi)$ uniformly as $\varepsilon \to 0$;
  - Away from critical values, $F_{p,\varepsilon}$ essentially equals to $F_{p,\mathfrak{m}}$;
  - There is "room enough" to "accomodate" the perturbation $\varphi(x,y)$ and reobtain continuity everywhere.

The equation that we solve

- *Equation for the perturbation:* the starting point is Euler:

$$\{\breve\psi, \Delta_\omega \breve\psi\} = 0 \overset{\psi=\psi_{\mathfrak{m}}(y)+\varphi}{\rightsquigarrow} \{\psi_{\mathfrak{m}}, \Delta_\omega \breve\varphi\} + \{\breve\varphi, \psi''_{\mathfrak{m}}\} + \{\breve\varphi, \Delta_\omega \breve\varphi\} = 0.$$

The corresponding "elliptic" equation is (recalling $\psi''_{\mathfrak{m}}(y) = F_{p,\mathfrak{m}}(\psi_{\mathfrak{m}}(y))$)

$$\Delta_\omega \breve\psi = F_{p,\varepsilon}(\breve\psi) \rightsquigarrow \Delta_\omega \breve\varphi(\mathbf{x}, y) = F_{p,\varepsilon}(\psi_{\mathfrak{m}}(y) + \breve\varphi(\mathbf{x}, y)) - F_{p,\mathfrak{m}}(\psi_{\mathfrak{m}}(y))$$

where $(\mathbf{x}, y) \in \mathbb{T}^{\kappa_0} \times \mathtt{I}_p$, $p = 0, 1, ..., \kappa_0$;

- *Boundary conditions*: $\breve\varphi(\mathbf{x}, -1) = \breve\varphi(\mathbf{x}, 1) = 0$;
- *Symmetries*: we search for *(space) reversible* $\breve\varphi(\mathbf{x}, y)$, namely

$$\breve\varphi(\mathbf{x}, y) \in \mathrm{even}(\mathbf{x})\mathrm{even}(y)\,;$$

- *Functional spaces*: Sobolev spaces $H^{s,\rho} := H^s(\mathbb{T}^{\kappa_0}, H_0^\rho([-1,1]))$, with

$$H^{s,\rho} := \Big\{ u(\mathbf{x}, y) = \sum_{\ell \in \mathbb{Z}^{\kappa_0}} u_\ell(y) e^{\mathrm{i}\ell\cdot\mathbf{x}} \ :\ \|u\|_{s,\rho}^2 := \sum_{\ell \in \mathbb{Z}^{\kappa_0}} \langle \ell \rangle^{2s} \|u_\ell\|_{H_0^\rho([-1,1])}^2 < \infty \Big\}.$$

The linear problem at the equilibrium

- Linearizing the Euler equation at $\varphi = 0$, we get, for $(\mathbf{x}, y) \in \mathbb{T}^{\kappa_0} \times [-1, 1]$

$$\{\psi_{\mathfrak{m}}, \Delta_\omega \breve{\varphi}\} + \{\breve{\varphi}, \psi''_{\mathfrak{m}}\} = 0 \;\rightsquigarrow\; (\omega \cdot \partial_{\mathbf{x}})^2 \breve{\varphi}(\mathbf{x}, y) = \mathcal{L}_{\mathfrak{m}} \breve{\varphi}(\mathbf{x}, y)$$

  (recall that $\mathcal{L}_{\mathfrak{m}} = -\partial_y^2 + Q_{\mathfrak{m}}$ and that $\psi'''_{\mathfrak{m}} = Q_{\mathfrak{m}} \psi'_{\mathfrak{m}}$);
- Family of space quasi-periodic solutions

$$\varphi(x, y) = \sum_{j=1}^{\kappa_0} A_j \cos(\lambda_{j,\mathfrak{m}}(\mathtt{E}) x) \phi_{j,\mathfrak{m}}(y), \quad A_j \in \mathbb{R} \setminus \{0\} \tag{4}$$

  with frequency vector $\omega \equiv \vec{\omega}_{\mathfrak{m}}(\mathtt{E}) := (\lambda_{1,\mathfrak{m}}(\mathtt{E}), ..., \lambda_{\kappa_0,\mathfrak{m}}(\mathtt{E})) \in \mathbb{R}^{\kappa_0} \setminus \{0\}$ (recall that $-\lambda_{1,\mathfrak{m}}^2(\mathtt{E}), ..., -\lambda_{\kappa_0,\mathfrak{m}}^2(\mathtt{E}) < 0$ are the negative eigenvalues of $\mathcal{L}_{\mathfrak{m}}$).

The role of the parameter E

- Recall the constrain $\mathtt{Er} = \kappa_0\left(\pi + \frac{1}{4}\right)$ and $\vec{\omega}_{\mathtt{m}}(\mathtt{E}) := (\lambda_{1,\mathtt{m}}(\mathtt{E}), ..., \lambda_{\kappa_0,\mathtt{m}}(\mathtt{E}))$;

Proposition ($\vec{\omega}_{\mathtt{m}}(\mathtt{E})$ is Diophantine)

Let $\mathtt{E}_2 > \mathtt{E}_1 > (\kappa_0 + \frac{1}{4})\pi$*. Given* $\overline{\upsilon} \in (0,1)$ *and* $\overline{\tau} \gg 1$*, there exists a Borel set*

$$\overline{\mathcal{K}} = \overline{\mathcal{K}}(\overline{\upsilon}, \overline{\tau}) := \left\{ \mathtt{E} \in [\mathtt{E}_1, \mathtt{E}_2] \, : \, |\vec{\omega}_{\mathtt{m}}(\mathtt{E}) \cdot \ell| \geqslant \overline{\upsilon} \langle \ell \rangle^{-\overline{\tau}} \, , \, \forall \, \ell \in \mathbb{Z}^{\kappa_0} \backslash \{0\} \right\},$$

such that $\mathtt{E}_2 - \mathtt{E}_2 - |\overline{\mathcal{K}}| = o(\overline{\upsilon})$.

- **<u>Goal</u>**: existence of a small amplitude, reversible **space quasi-periodic** function $\breve{\varphi}(\mathbf{x}, y)$, solution of the equation

$$\Delta_\omega \breve{\varphi}(\mathbf{x}, y) = F_{p,\varepsilon}(\psi_{\mathtt{m}}(y) + \breve{\varphi}(\mathbf{x}, y)) - F_{p,\mathtt{m}}(\psi_{\mathtt{m}}(y)) \tag{5}$$

with frequency vector $\omega \in \mathbb{R}^{\kappa_0}$ close to $\vec{\omega}_{\mathtt{m}}(\mathtt{E})$, resembling at leading order a linear solution (4) at the equilibrium, for a **fixed valued of the depth** $\mathtt{E} \in \overline{\mathcal{K}}$ and **for most values of an auxiliary parameter**

$$\mathtt{A} \in \mathcal{J}_\varepsilon(\mathtt{E}) := [\mathtt{E} - \sqrt{\varepsilon}, \mathtt{E} + \sqrt{\varepsilon}] \, . \tag{6}$$

Theorem (F.-Masmoudi-Montalto (2023))

Fix $\kappa_0 \in \mathbb{N}$ *and* $\mathtt{m} \gg 1$. *Fix also* $\mathtt{E} \in \overline{\mathcal{K}}$ *and* $\xi = (\xi_1, \ldots, \xi_{\kappa_0}) \in \mathbb{R}^{\kappa_0}_{>0}$. *Then there exist* $\overline{s} > 0$, $\varepsilon_0 > 0$ *such that the following hold.*

1) *For any* $\varepsilon \in (0, \varepsilon_0)$ *there exists a Borel set* $\mathcal{G}_\varepsilon = \mathcal{G}_\varepsilon(\mathtt{E}) \subset \mathcal{J}_\varepsilon(\mathtt{E})$, *with*$\mathcal{J}_\varepsilon(\mathtt{E})$ *as in* (6) *and with density 1 at* $\mathtt{E}$ *when* $\varepsilon \to 0$, *namely* $\lim_{\varepsilon \to 0}(2\sqrt{\varepsilon})^{-1}|\mathcal{G}_\varepsilon(\mathtt{E})| = 1$;

2) *There exists* $h_\varepsilon = h_\varepsilon(\mathtt{E}) \in H^3_0([-1,1])$, $\|h_\varepsilon\|_{H^3} \lesssim \varepsilon$, $h_\varepsilon = \mathrm{even}(y)$, *such that, for any* $\mathtt{A} \in \mathcal{G}_\varepsilon$, *the equation* (5) *has a reversible,* **space quasi-periodic** *solution of the form*

$$\breve{\varphi}_\varepsilon(\mathbf{x}, y)|_{\mathbf{x}=\tilde{\omega}(\mathtt{A})x} = h_\varepsilon(\mathtt{E}; y) + \varepsilon \sum_{j=1}^{\kappa_0} \sqrt{\xi_j} \cos(\tilde{\omega}_j(\mathtt{A})x)\phi_{j,\mathtt{m}}(\mathtt{E}; y) + \breve{r}_\varepsilon(\mathbf{x}, y)|_{\mathbf{x}=\tilde{\omega}(\mathtt{A})x}\,, \tag{7}$$

where $\breve{r}_\varepsilon = \breve{r}_\varepsilon(\mathtt{E}, \mathtt{A}; \mathbf{x}, y) \in H^{\overline{s},3}$, *with* $\lim_{\varepsilon \to 0} \frac{\|\breve{r}_\varepsilon\|_{\overline{s},3}}{\varepsilon} = 0$, *and* $\tilde{\omega} = (\tilde{\omega}_j)_{j=1,\ldots,\kappa_0} \in \mathbb{R}^{\kappa_0}$, *depending on* $\mathtt{A}$ *and* ε, *with* $|\tilde{\omega}(\mathtt{A}) - \vec{\omega}_{\mathtt{m}}(\mathtt{E})| \leqslant C\sqrt{\varepsilon}$, *with* $C > 0$ *independent of* $\mathtt{E}$ *and* $\mathtt{A}$. *Moreover for any* $\varepsilon \in [0, \varepsilon_0]$, *the stream function*

$$\psi_\varepsilon(x, y) = \breve{\psi}_\varepsilon(\mathbf{x}, y)|_{\mathbf{x}=\tilde{\omega}(\mathtt{A})x} = \psi_{\mathtt{m}}(y) + \breve{\varphi}_\varepsilon(\mathbf{x}, y)|_{\mathbf{x}=\tilde{\omega}(\mathtt{A})x}\,, \tag{8}$$

with $\varphi_\varepsilon(\mathbf{x}, y)$ *as in* (7), *defines a space quasi-periodic solution of the steady 2D Euler equation that is* **close to the Couette flow** *with estimates*

$$\|\breve{\psi}_\varepsilon - \psi_{\mathrm{cou}}\|_{\overline{s},3} \lesssim_{\overline{s}} \frac{1}{\sqrt{\mathtt{E}}} + \varepsilon\,, \quad \psi_{\mathrm{cou}}(y) := \tfrac{1}{2}y^2\,.$$

Comments on the main results

- The shear perturbation $h_\varepsilon(y)$ comes from the forced modification in (5) of the nonlinearities $F_{p,\mathfrak{m}}(\psi)$ into $F_{p,\eta}(\psi)$;
- The second term of $\varphi_\varepsilon(\mathbf{x}, y)$ in (8) retains the space quasi-periodicity of the linearized solution and is constructed with a suitable Nash-Moser iterative scheme (the eigenfunctions $(\phi_{j,\mathfrak{m}}(\mathtt{E}; y))_{j\in\mathbb{N}}$ depend on the parameter $\mathtt{E}$!);
- Such solutions exist for fixed values of the depth $\mathtt{E} \in \overline{\mathcal{K}}$ so that $\vec{\omega}_{\mathfrak{m}}(\mathtt{E})$ is Diophantine and for most values of the auxiliary parameter $\mathtt{A} \in \mathcal{J}_\varepsilon(\mathtt{E})$ so that $\widetilde{\omega} = \widetilde{\omega}(\mathtt{A}, \varepsilon)$ is non-resonant as well;
- <u>*Traveling quasi-periodic flows*</u>: the stream function

$$\begin{aligned}\psi_{\rm tr}(t, x, y) &:= cy + \psi_\varepsilon(x - ct, y)\\ &= cy + \psi_{\mathfrak{m}}(y) + \breve{\varphi}_\varepsilon(\phi, y)|_{\phi=\mathbf{x}-\vartheta=\widetilde{\omega}(x-ct)}\,, \quad \mathbf{x}, \vartheta \in \mathbb{T}^{\kappa_0}\,,\end{aligned}$$

solve the Euler equations in vorticity formulation

$$(\Omega_{\rm tr})_t + (\psi_{\rm tr})_y(\Omega_{\rm tr})_x - (\psi_{\rm tr})_x(\Omega_{\rm tr})_y = 0\,, \quad \Omega_{\rm tr} := \Delta\psi_{\rm tr}\,.$$

- *Generalized Kelvin's cat-eyes flow*: The flow generated by the stream function $\breve{\psi}_\varepsilon(\mathbf{x}, y)$ is a deformation of the near-Couette shear flow $(\psi'_{\mathfrak{m}}(y), 0)$:

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} \psi'_{\mathfrak{m}}(y) + h'_\varepsilon(y) \\ 0 \end{pmatrix} + \varepsilon \sum_{j=1}^{\kappa_0} \sqrt{\xi_j} \begin{pmatrix} \cos(\widetilde{\omega}_j x) \phi'_{j,\mathfrak{m}}(y) \\ \widetilde{\omega}_j \sin(\widetilde{\omega}_j x) \phi_{j,\mathfrak{m}}(y) \end{pmatrix} + o(\varepsilon)\,.$$

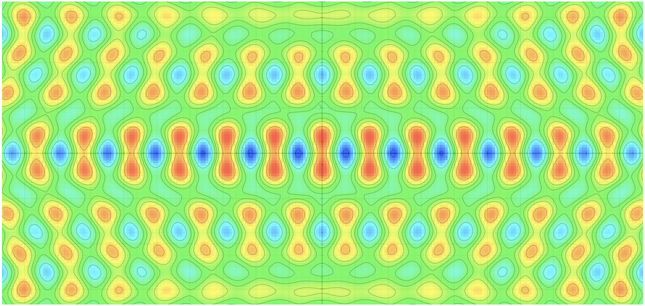


FIGURE: Streamlines of (the leading order of) a bi-periodic flow

Summary of main difficulties and novelties

- The nonlinearity of the semilinear "elliptic" problem that we solve is actually an "unknown" of the problem and it has to be constructed in such a way that one has a near Couette, space quasi-periodic solution to the Euler equation;
- Each space quasi-periodic function $\varphi_\varepsilon(\mathbf{x}, y)$ solve a nonlinear PDE with nonlinearities explicitly depending of the size ε of the solution;
- The nonlinearities have finite smoothness and their derivatives lose in size;
- The unperturbed frequencies of oscillations are only implicitly defined and their non-degeneracy property relies on an asymptotic expansion for large values of the parameter. It implies that the required non-resonance conditions are not trivial to verify;
- The basis of eigenfunctions $(\phi_{j,\mathfrak{m}}(y))_{j\in\mathbb{N}}$ of the operator $\mathcal{L}_\mathfrak{m} = -\partial_y^2 + Q_\mathfrak{m}(y)$ is not the standard exponential basis and depends explicitly on the depth parameter $\mathtt{E}$;
- The potential $Q_\mathfrak{m}(y)$ is the ruler ot the scheme!

Strategy of the proof

1. The shear equilibrium $\psi_{\mathfrak{m}}(y)$ close to Couette and its nonlinear ODE
$$\psi_{\mathfrak{m}}'''(y) = Q_{\mathfrak{m}}(y)\psi_{\mathfrak{m}}'(y) \rightsquigarrow \psi_{\mathfrak{m}}''(y) = F_{p,\mathfrak{m}}(\psi_{\mathfrak{m}}(y)), \ y \in \mathtt{I}_p;$$
2. A forced elliptic PDE for the perturbation of the shear equilibrium
$$\Delta_\omega \breve{\varphi}(\mathbf{x}, y) = F_{p,\varepsilon}(\psi_{\mathfrak{m}}(y) + \breve{\varphi}(\mathbf{x}, y)) - F_{p,\mathfrak{m}}(\psi_{\mathfrak{m}}(y));$$
3. A Nash-Moser scheme of hypothetical conjugation with the auxiliary parameter
$$\begin{aligned} \mathcal{F}(i,\alpha) = \mathcal{F}(\omega, \mathtt{A}, \mathtt{E}, \varepsilon; i, \alpha) &:= \omega \cdot \partial_{\mathbf{x}} i(\mathbf{x}) - X_{\mathcal{H}_{\varepsilon,\alpha}}(i(\mathbf{x})) = 0\,, \\ i = i(\mathbf{x}) = (\theta(\mathbf{x}), I(\mathbf{x}), z(\mathbf{x}))\,, \\ \mathcal{H}_{\varepsilon,\alpha} := \alpha \cdot I + \tfrac{1}{2}\Big(z, \begin{pmatrix} -\mathcal{L}_{\mathfrak{m}} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}\Big)_{L^2} \\ + \sqrt{\varepsilon}\Big(P_\varepsilon(A(\theta, I, z) + \tfrac{1}{\sqrt{\varepsilon}}\big(\vec{\omega}_{\mathfrak{m}}(\mathtt{E}) - \vec{\omega}_{\mathfrak{m}}(\mathtt{A})\big) \cdot I\Big)\,. \end{aligned}$$

The potential $Q_{\mathfrak{m}}(y)$ and its properties

- For $\mathfrak{m} \gg 1$ large enough, we define the even function

$$Q_{\mathfrak{m}}(y) = Q_{\mathfrak{m}}(\mathtt{E}, \mathtt{r}; y) \simeq -\mathtt{E}^2 \Big(\Big(\frac{\cosh(\frac{y}{\mathtt{r}})}{\cosh(1)} \Big)^{\mathfrak{m}} + 1 \Big)^{-1},$$

such that, in the limit $\mathfrak{m} \to \infty$, it uniformly approaches the classical potential well (on compact intervals excluding $\{|y| = \mathtt{r}\}$)

$$Q_{\mathfrak{m}}(y) = Q_{\mathfrak{m}}(\mathtt{E}, \mathtt{r}; y) \overset{\mathfrak{m}\to\infty}{\longrightarrow} Q_{\infty}(\mathtt{E}, \mathtt{r}; y) := \begin{cases} 0 & |y| > \mathtt{r}, \\ -\mathtt{E}^2 & |y| < \mathtt{r}; \end{cases}$$

Lemma (Estimates for $Q_{\mathfrak{m}}(y)$)

We have

$$\sup_{\mathfrak{m}\gg 1} \|Q_{\mathfrak{m}}\|_{L^\infty([-1,1])} \lesssim \|Q_\infty\|_{L^\infty([-1,1])} \lesssim \mathtt{E}^2 .$$

Moreover, for any fixed $\gamma > 0$ sufficiently small, we have, for any $n \in \mathbb{N}_0$,

$$|\partial_y^n (Q_{\mathfrak{m}}(y) - Q_\infty(y))| \to 0 \quad \text{uniformly in} \quad y \in [-1,1] \backslash (B_\gamma(\mathtt{r}) \cup B_\gamma(-\mathtt{r}))$$

and $\|Q_{\mathfrak{m}} - Q_\infty\|_{L^p([-1,1])} \to 0$ for any $p \in [1, \infty)$.

How to construct the nonlinearity $F_{0,\mathfrak{m}}(\psi)$

- We determine the even shear $\psi_{\mathfrak{m}}(y)$ by solving $\psi_{\mathfrak{m}}'''(y) = Q_{\mathfrak{m}}(y)\psi_{\mathfrak{m}}'(y)$;
- Since $Q_{\mathfrak{m}}(y)$ and $\psi_{\mathfrak{m}}(y)$ are even, then (recall that $\psi_{\mathfrak{m}}'(\mathtt{y}_{1,\mathfrak{m}}) = 0$)

$$Q_{\mathfrak{m}}(y) = K_{0,\mathfrak{m}}(y^2)\,, \quad \psi_{\mathfrak{m}}(y) = G_{0,\mathfrak{m}}(y^2)\,, \quad 0 \leqslant |y| < \mathtt{y}_{1,\mathfrak{m}}\,,$$

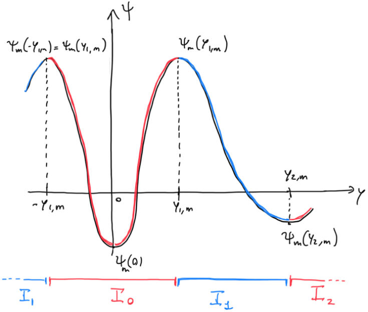
  where $K_{0,\mathfrak{m}}, G_{0,\mathfrak{m}} \in \mathcal{C}^\infty$;
- $\psi_{\mathfrak{m}}(y)$ is invertible as a function of y^2 until reaches $|y| = \mathtt{y}_{1,\mathfrak{m}}$:

$$\begin{cases} \psi_{\mathfrak{m}}'(y) \neq 0, & 0 < |y| < \mathtt{y}_{1,\mathfrak{m}} \\ \psi_{\mathfrak{m}}''(0) \neq 0\,, \end{cases} \Rightarrow G_{0,\mathfrak{m}}(z) \text{ is invertible for } 0 \leqslant z < \sqrt{\mathtt{y}_{1,\mathfrak{m}}}\,;$$

- We define $F_{0,\mathfrak{m}}(\psi)$ as solution of the Cauchy problem

$$\begin{cases} F_{0,\mathfrak{m}}'(\psi) = K_{0,\mathfrak{m}}\big(G_{0,\mathfrak{m}}^{-1}(\psi)\big), & \psi \in \psi_{\mathfrak{m}}([0, \mathtt{y}_{1,\mathfrak{m}} - \gamma_0])\,, \\ F_{0,\mathfrak{m}}(\psi_{\mathfrak{m}}(0)) = \psi_{\mathfrak{m}}''(0)\,. \end{cases}$$

How to construct the nonlinearity $F_{1,\mathfrak{m}}(\psi)$



- $Q_{\mathfrak{m}}(y)$ and $\psi_{\mathfrak{m}}(y)$ are not even-symmetric with respect to $y = \mathtt{y}_{1,\mathfrak{m}}$, BUT they are close to be: we can write, for $||y| - \mathtt{y}_{1,\mathfrak{m}}| < \mathtt{r}_{1,\pm}$, with $\mathtt{r}_{1,-} := \mathtt{y}_{1,\mathfrak{m}} - \mathtt{y}_{0,\mathfrak{m}}$ and $\mathtt{r}_{1,+} := \mathtt{y}_{2,\mathfrak{m}} - \mathtt{y}_{1,\mathfrak{m}}$.

$$Q_{\mathfrak{m}}(y) = K_{1,\mathfrak{m},\pm}\big((|y| - \mathtt{y}_{1,\mathfrak{m}})^2\big)\,, \quad K_{1,\mathfrak{m},\pm} \in \mathcal{C}^S(B_{\sqrt{\mathtt{r}_{1,\pm}}}(0))\,,$$
$$\psi_{\mathfrak{m}}(y) = G_{1,\mathfrak{m},\pm}\big((|y| - \mathtt{y}_{1,\mathfrak{m}})^2\big)\,, \quad G_{1,\mathfrak{m},\pm} \in \mathcal{C}^{S+1}(B_{\sqrt{\mathtt{r}_{1,\pm}}}(0))\,;$$

- In their regions (where $\psi'_{\mathfrak{m}}(y)$ does not change sign), both $G_{1,\mathfrak{m},-}(z)$ and $G_{1,\mathfrak{m},+}(z)$ are invertible (also here, $\psi''_{\mathfrak{m}}(\mathtt{y}_{1,\mathfrak{m}}) \neq 0$).

- In the region $0 < |y| \leqslant \mathtt{y}_{1,\mathtt{m}}$, we consider

$$\begin{cases} F'_{0,\mathtt{m}}(\psi) = K_{1,\mathtt{m},-}\big(G^{-1}_{1,\mathtt{m},-}(\psi)\big), & \psi \in \psi_{\mathtt{m}}([\gamma_{1,-}, \mathtt{y}_{1,\mathtt{m}}])\,, \\ F_{0,\mathtt{m}}(\psi_{\mathtt{m}}(\mathtt{y}_{1,\mathtt{m}})) = \psi''_{\mathtt{m}}(\mathtt{y}_{1,\mathtt{m}})\,, \end{cases}$$

  which has to coincide with the $F_{0,\mathtt{m}}(\psi)$ constructed before, because

$$F'_{0,\mathtt{m}}(\psi_{\mathtt{m}}(y)) = Q_{\mathtt{m}}(y) \quad \forall\, |y| \in [0, \mathtt{y}_{1,\mathtt{m}}]$$

- In the region $\mathtt{y}_{1,\mathtt{m}} \leqslant |y| < \mathtt{y}_{2,\mathtt{m}}$, we define $F_{1,\mathtt{m}}$ as the solution of

$$\begin{cases} F'_{1,\mathtt{m}}(\psi) = K_{1,\mathtt{m},+}\big(G^{-1}_{1,\mathtt{m},+}(\psi)\big), & \psi \in \psi_{\mathtt{m}}([\mathtt{y}_{1,\mathtt{m}}, \mathtt{y}_{2,\mathtt{m}} - \gamma_{1,+}])\,, \\ F_{1,\mathtt{m}}(\psi_{\mathtt{m}}(\mathtt{y}_{1,\mathtt{m}})) = \psi''_{\mathtt{m}}(\mathtt{y}_{1,\mathtt{m}})\,, \end{cases}$$

- Thanks to the "approximate local evenness" we have $\mathcal{C}^{S+1}$-continuity when $\psi = \psi_{\mathtt{m}}(y)$ at $y = \pm\mathtt{y}_{1,\mathtt{m}}$: for any $n = 0, 1, ..., S+1$,

$$\lim_{|y| \to \mathtt{y}_{1,\mathtt{m}}^{-}} \partial_y^n(F_{0,\mathtt{m}}(\psi_{\mathtt{m}}(y))) = \lim_{|y| \to \mathtt{y}_{1,\mathtt{m}}^{+}} \partial_y^n(F_{1,\mathtt{m}}(\psi_{\mathtt{m}}(y))) = \psi_{\mathtt{m}}^{(n+2)}(\mathtt{y}_{1,\mathtt{m}})\,.$$

The issue nonlinearity vs. perturbation

- **Recall**: If we search for $\psi(x,t)$ such that, for any $p = 0, 1, \ldots, \kappa_0$

$$\psi''_{\mathfrak{m}}(y) = F_{p,\mathfrak{m}}(\psi_{\mathfrak{m}}(y)) \rightsquigarrow \Delta\psi(x,y) = F_{p,\mathfrak{m}}(\psi(x,y)), \quad (x,y) \in \mathbb{R} \times \mathtt{I}_p$$

then we immediately lose continuity at $\{|y| = \mathtt{y}_{p,\mathfrak{m}}\}$: in general

$$\lim_{|y| \to \mathtt{y}_{p,\mathfrak{m}}^-} F_{p-1,\mathfrak{m}}(\psi_{\mathfrak{m}}(y) + \varphi(x,y)) \neq \lim_{|y| \to \mathtt{y}_{p,\mathfrak{m}}^+} F_{p,\mathfrak{m}}(\psi_{\mathfrak{m}}(y) + \varphi(x,y))$$

unless $\varphi(x, \pm\mathtt{y}_{p,\mathfrak{m}}) = 0$ for any $x \in \mathbb{R}$ (too strict!)

- **Key idea**: replace $F_{p,\mathfrak{m}}$ with a perturbed version to "accomodate" the perturbation $\varphi(x,y) = O(\varepsilon)$:

$$F_{p,\eta}(\psi) = \begin{cases} \frac{1}{2}\big(F_{p-1,\mathfrak{m}}(\psi) + F_{p,\mathfrak{m}}(\psi)\big) = F_{p-1,\eta}(\psi) & |\psi - \psi_{\mathfrak{m}}(\mathtt{y}_{p,\mathfrak{m}})| \leqslant \eta\,, \\ F_{p,\mathfrak{m}}(\psi) & \begin{array}{l} |\psi - \psi_{\mathfrak{m}}(\mathtt{y}_{p,\mathfrak{m}})| \geqslant 2\eta \ \text{ and} \\ |\psi - \psi_{\mathfrak{m}}(\mathtt{y}_{p+1,\mathfrak{m}})| \geqslant 2\eta\,, \end{array} \\ \frac{1}{2}\big(F_{p+1,\mathfrak{m}}(\psi) + F_{p,\mathfrak{m}}(\psi)\big) = F_{p+1,\eta}(\psi) & |\psi - \psi_{\mathfrak{m}}(\mathtt{y}_{p+1,\mathfrak{m}})| \leqslant \eta\,, \end{cases}$$

where $\eta = \varepsilon^{1/S} \gg \varepsilon \in (0,1)$, with smooth connections in the remaining regions, so that $F_{p,\eta}(\psi) \to F_{p,\mathfrak{m}}(\psi)$ uniformly as $\eta \to 0$.

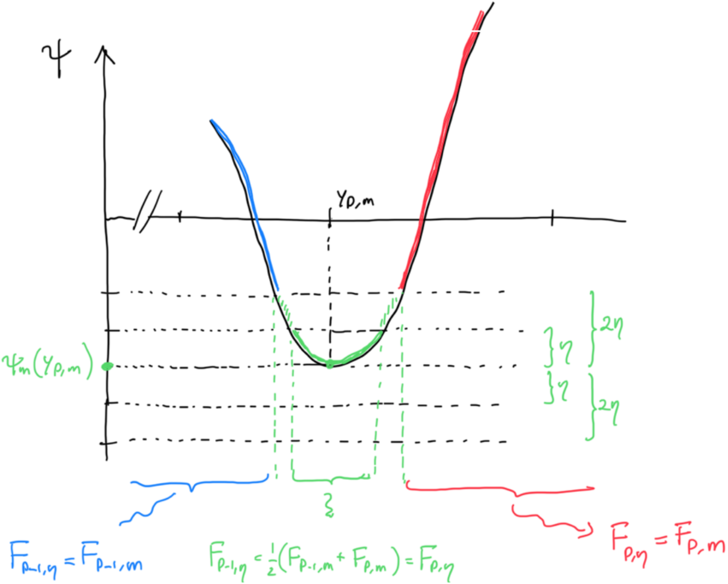


FIGURE: Where the nonlinearities $F_{p,\eta}$ are constructed

Spectral analysis of $\mathcal{L}_{\mathfrak{m}} = -\partial_y^2 + Q_{\mathfrak{m}}(y)$

Proposition

The Schrödinger operator $\mathcal{L}_{\mathfrak{m}} = -\partial_y^2 + Q_{\mathfrak{m}}(y)$ *is self-adjoint in* $L_0^2([-1,1])$ *on* $D(\mathcal{L}_{\mathfrak{m}}) := H^1_{0,\mathrm{even}}[-1,1]$ *with a countable* L^2*-basis* $(\phi_{j,\mathfrak{m}}(y))_{j\in\mathbb{N}} \subset \mathcal{C}^\infty[-1,1]$ *corresponding to the eigenvalues* $(\mu_{j,\mathfrak{m}})_{j\in\mathbb{N}}$*. Moreover, there exists* $\overline{\mathfrak{m}} = \overline{\mathfrak{m}}(\mathtt{E}_1, \mathtt{E}_2, \kappa_0) \gg 1$ *large enough such that, for any* $\mathfrak{m} \geqslant \overline{\mathfrak{m}}$,

$$\mu_{j,\mathfrak{m}} = \begin{cases} -\lambda_{j,\mathfrak{m}}^2 \in (-\mathtt{E}^2, 0) & j = 1, \ldots, \kappa_0 \,, \\ \lambda_{j,\mathfrak{m}}^2 > 0 & j \geqslant \kappa_0 + 1 \,. \end{cases}$$

In particular, for any $j = 1, \ldots, \kappa_0$*, we have that* $\lambda_{j,\mathfrak{m}}$ *is close to* $\lambda_{j,\infty}$*, with the latter being the* j*-th root out of* κ_0 *in the region* $\lambda \in (0, \mathtt{E})$ *of the equation*

$$\mathfrak{F}(\lambda) := \lambda \cos\left(\mathtt{r}\sqrt{\mathtt{E}^2 - \lambda^2}\right) \coth((1-\mathtt{r})\lambda) - \sqrt{\mathtt{E}^2 - \lambda^2} \sin\left(\mathtt{r}\sqrt{\mathtt{E}^2 - \lambda^2}\right) = 0 \,.$$

Lemma

For any $j = 1, \ldots, \kappa_0$*, we have the asymptotic expansion*

$$\lambda_{j,\infty}(\mathtt{E}) = \mathtt{E} \cos\Big(\pi\big(\alpha_0(j) + \alpha_2(j)\beta_j(\mathtt{E})^2 + o(\beta_j(\mathtt{E})^3)\big)\Big) \,, \quad \mathtt{E} \to +\infty \,,$$

$$\beta_j(\mathtt{E}) := \exp\left(((\kappa_0 + \tfrac{1}{4})\pi - \mathtt{E}) \cos(\pi\alpha_0(j))\right) \,, \quad \sin(\pi\alpha_0(j)) = \frac{j - \frac{1}{2} - \alpha_0(j)}{\kappa_0 + \frac{1}{4}} \,.$$

Thank you for your attention!