Integrable turbulence and Soliton gas

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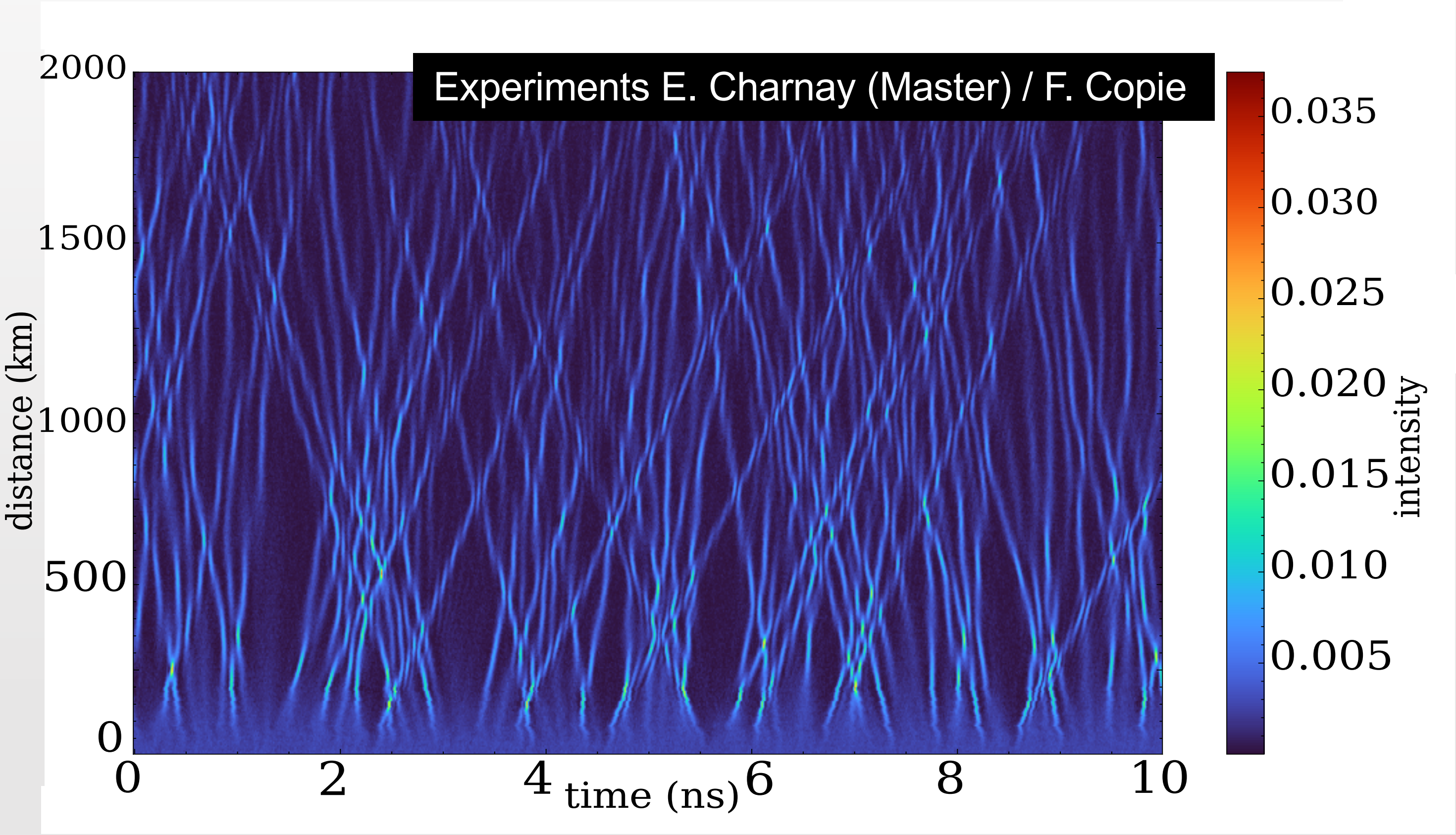


Workshop on Wave Dynamics: Turbulent vs Integrable Effects , Trieste 30th August 2023

Wave Turbulence – integrable turbulence

Wave turbulence = nonlinear dispersive random waves

hydrodynamics, optics, mechanics, plasma physics…

Integrable turbulence = nonlinear dispersive random waves **in integrable systems**

hydrodynamics, optics, mechanics, …

- Korteweg De Vries (KdV), 1D nonlinear Schrodinger (1DNLS), sine-Gordon (Universal equations)
- Inverse Scattering Transform
- Solitons, breathers solutions

> *"Nonlinear wave systems integrable by Inverse Scattering Method could demonstrate a complex behavior that demands the statistical description. The theory of this description composes a new chapter in the theory of wave turbulence -Turbulence in Integrable Systems"*
>
> *Turbulence in Integrable Systems*, V.E. Zakharov, Studies in Applied Mathematics, **122**, 219 (2009)

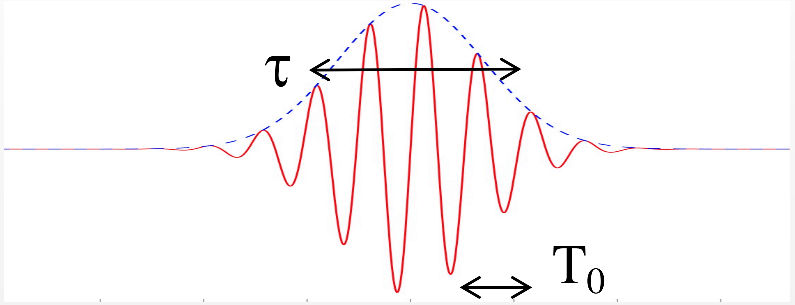
D.S. Agafontsev and V.E. Zakharov, Nonlinearity, (2015)
P. Walczak et al., Phys. Rev. Lett., **114**, 143903, (2015)
J. Soto-Crespo et al., Phys. Rev. Lett., 2016
S. Randoux et al, Physica D : Nonlinear Phenomena, **333**, (2016)
P. Suret *et al.* Nat. Commun. **7,** 13136 (2016).

A. Tikan, *et al.*, Nat. Photon., **12**, 228 (2018)
A. Gelash *et al.,* 123 (23), 234102, Phys. Rev. Lett., (2019)
F. Copie *et al.*, Reviews in Physics 5, 100037 (2020)
A. Tikan *et al.*, Scientific reports 12 (1), 10386 (2022)

1D Nonlinear Schrodinger Equation (NLS)
Experiments in optical fibers and in water tank

$$\frac{\partial \psi}{\partial z} = i\frac{1}{2}\frac{\partial^2 \psi}{\partial t^2} + i|\psi|^2\psi$$

$$t = t_{laboratory} - \frac{z}{v_g(\omega_0)}$$



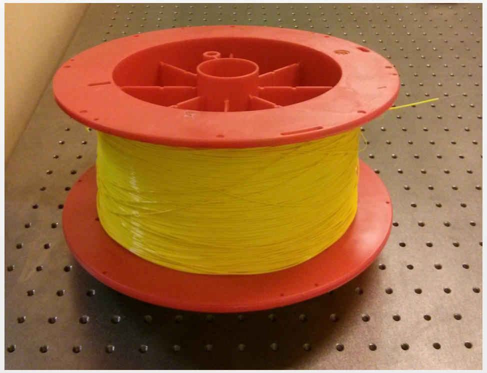
- **Optical fibers** *(Phlam, Univ Lille)*

$$E(x,y,z,t) = \Re\big(A(x,y)\,\psi(z,t)\,e^{i(k_0 z-\omega_0 t)}\big)$$

$T_0 \sim 5$ fs
τ ~ ps
L ~ 0.1-1 km

or

$T_0 \sim 5$ fs
τ ~ 100 ps
L ~ 1000 km



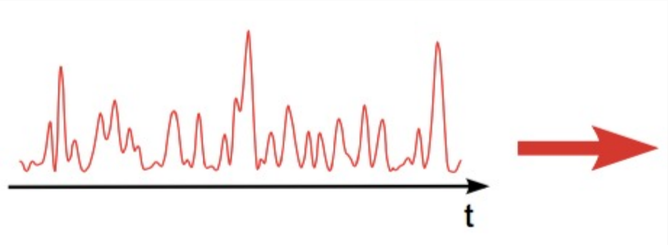
- **Deep water waves** *(Ecole Centrale de Nantes)*

$$\eta(z,t) = \Re\big(\psi(z,t)\,e^{i(k_0 z-\omega_0 t)}\big)$$

$T_0 \sim$ s
τ ~ 5s
L ~ 0.1 km

1. Integrable Turbulence

Random initial conditions + integrable system **(focusing 1D-NLSE)**



$$i\psi_z + \psi_{tt} + 2|\psi|^2\psi = 0, \quad \psi(z,t) \in \mathbb{C}$$

Hamiltonian PDE with an infinite number of conserved quantities

- Conservations laws $\boxed{(p_j[\psi])_z + (q_j[\psi])_t = 0 (j \geq 1)}$
- Ex: $p_1 = |\psi|^2, \quad p_2 = \psi^* \psi_t, \quad p_3 = |\psi|^4 - |\psi_t|^2, \quad q_2 = |\psi|^4 - 2|\psi_t|^2, \ldots$
- Conserved quantities (with z) $\boxed{\int p_j \, dt = P_j}$

$\mathit{Hamiltonian}: H{=}P_3$

$$H = H_L + H_{NL} = -\int \left|\frac{\partial \psi}{\partial t}\right|^2 dt + \int |\psi|^4 dt$$

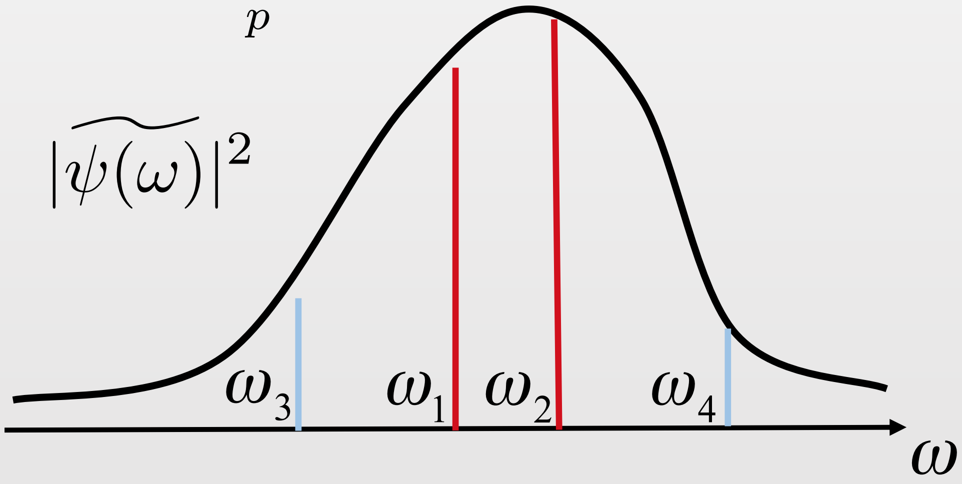
- Conserved quantities (with t) $\boxed{\int q_j \, dz = Q_j}$

1. “standard” Wave Turbulence

- ✓ **Nonlinear dispersive random waves**
- ✓ **Resonant four waves mixing (FWM)**

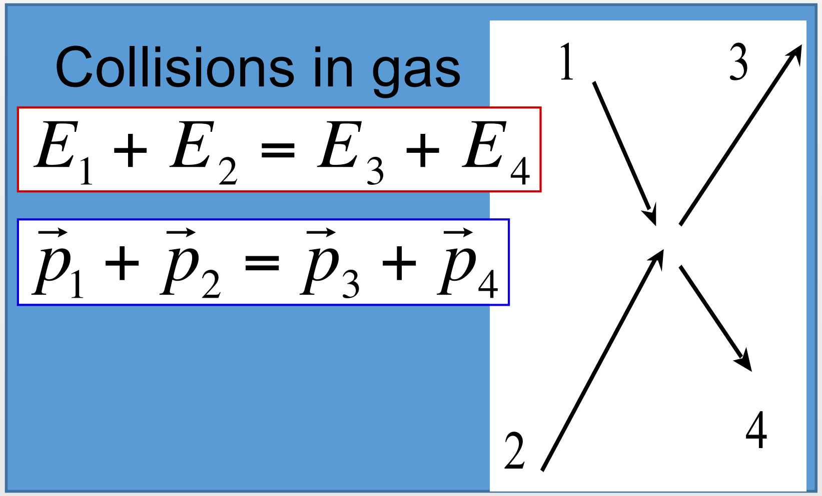
$$\boxed{nonlinearity \propto |\psi(t)|^2\psi(t)}$$

$$\psi(t) = \sum_p \widetilde{\psi(\omega_p)}\, e^{i\omega_p t}$$



$$\boxed{k(\omega_1) + k(\omega_2) = k(\omega_3) + k(\omega_4)}$$

$$\boxed{\omega_1 + \omega_2 = \omega_3 + \omega_4}$$



Collisions in gas

$$\boxed{E_1 + E_2 = E_3 + E_4}$$

$$\boxed{\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4}$$

Wave Turbulence, S. Nazarenko, Springer Science 2011

Kolmogorov spectra of turbulence I: Wave turbulence
VE Zakharov, VS L'vov, G Falkovich Springer, 2012

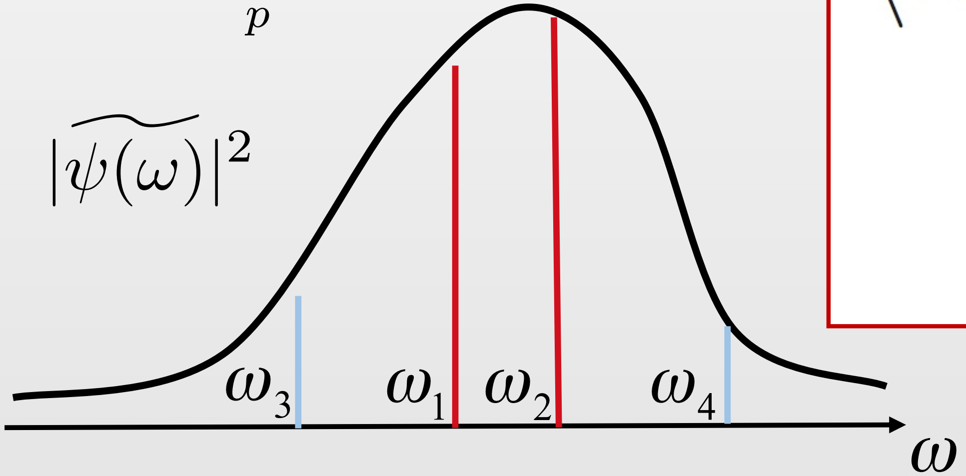
1. Integrable Turbulence

- ✓ **Nonlinear dispersive random waves**
- ✓ **Non resonant four waves mixing (FWM)**

$$k(\omega) = \frac{\beta_2}{2}\omega^2$$

$$\mathrm{i}\frac{\partial\psi}{\partial z} = \frac{\beta_2}{2}\frac{\partial^2\psi}{\partial t^2} - \gamma|\psi|^2\psi$$

$$\psi(t) = \sum_p \widetilde{\psi(\omega_p)}\, e^{i\omega_p t}$$



$$\xcancel{\boxed{k(\omega_1) + k(\omega_2) = k(\omega_3) + k(\omega_4)}}$$

$$\boxed{\omega_1 + \omega_2 = \omega_3 + \omega_4}$$

$$\left\langle \tilde{\psi}(z,\omega)\, \tilde{\psi}^*(z,\omega') \right\rangle = n_\omega(z)\, \delta(\omega - \omega')$$

$$\frac{dn}{dz} = 0 \quad !!!$$

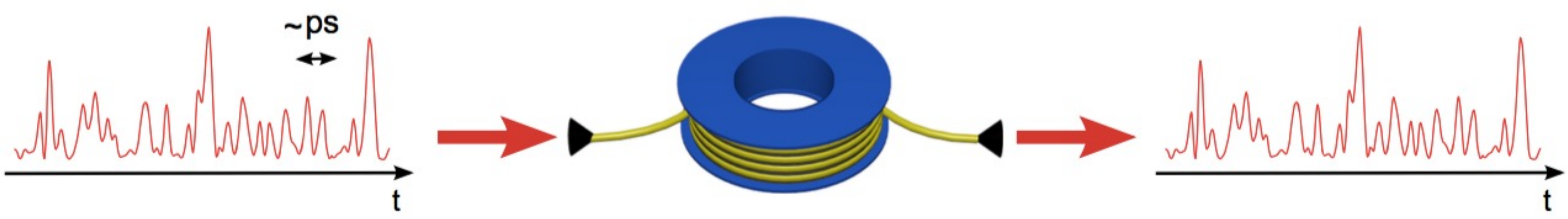
usual Wave Turbulence theory

A. Picozzi *et al.*, Physics Reports, (2014)

P Suret, A Picozzi, S Randoux Opt. Express **19,** (2011)

PAEM Janssen, Journal of Physical Ocean. 33 (4), (2003)

1. Integrable Turbulence: example of partially coherent waves



Initial partially coherent waves

- ✓ Linear superposition of independent waves

$$\psi(t) = \sum_p \widetilde{\psi(\omega_p)}\, e^{i\omega_p t}$$

- ✓ Probability Density Functions (PDF)
  - ➢ Field : Gaussian statistics (central limit theorem)

$$PDF\big[\Re(\psi)\big] = \exp\big[-\Re(\psi)^2\big]$$

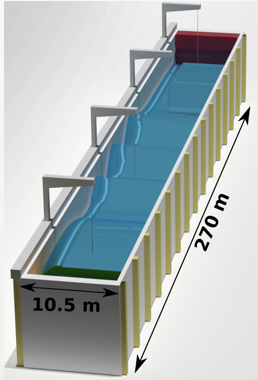
  - ➢ Power : exponential

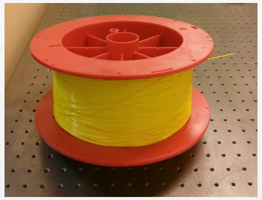
$P = |\psi|^2$

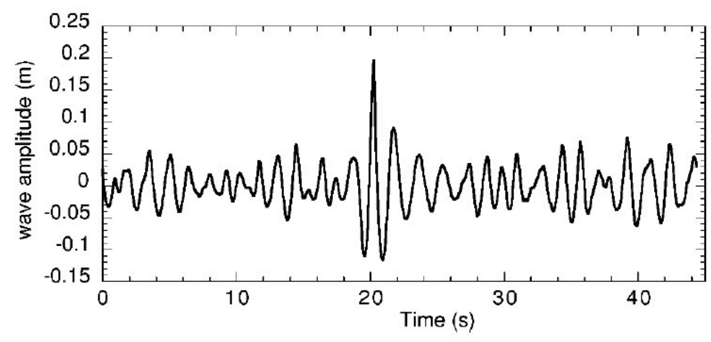
$$PDF(P/<P>) = \exp(-P/<P>)$$

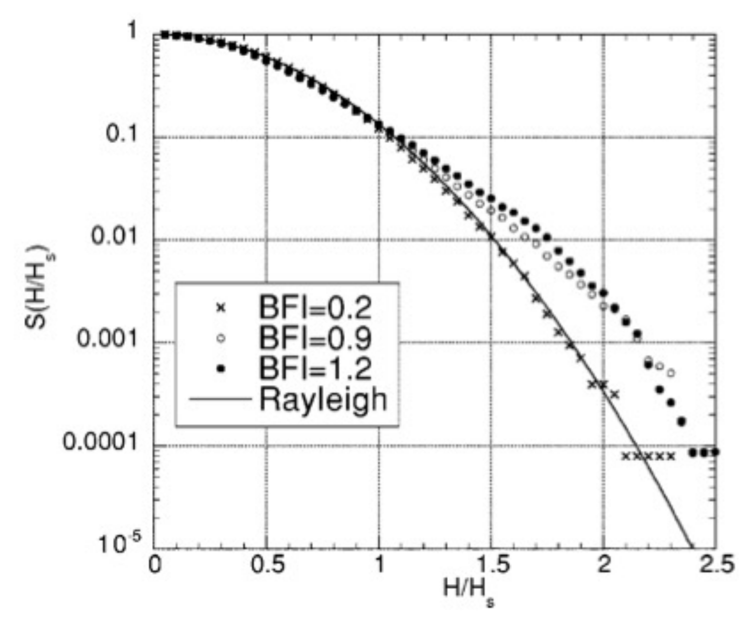
1D deep water waves + optical fiber experiments (rogue waves)

- ✓ From Gaussian to heavy-tailed statistics (surface elevation)
- ✓ Initial conditions = JONSWAP spectrum + Random phases





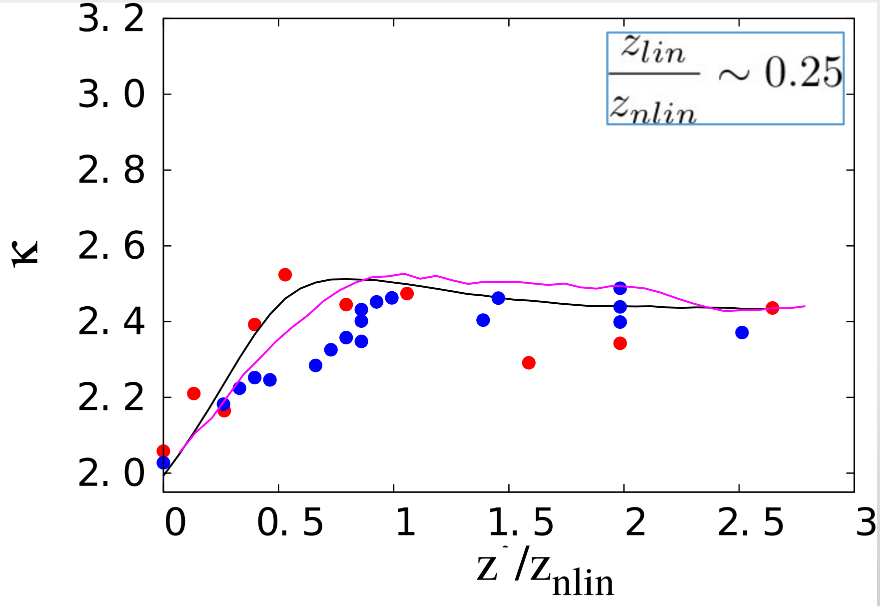




$$\eta = \Re(\psi\, e^{i(k_0 z - \omega_0 t)})$$

$$\boxed{\kappa = \frac{\langle|\psi|^4\rangle}{\langle|\psi|^2\rangle^2}}$$

Gaussian statistics : *k* =2



Experiments

Water waves ●

Optical fibers ●

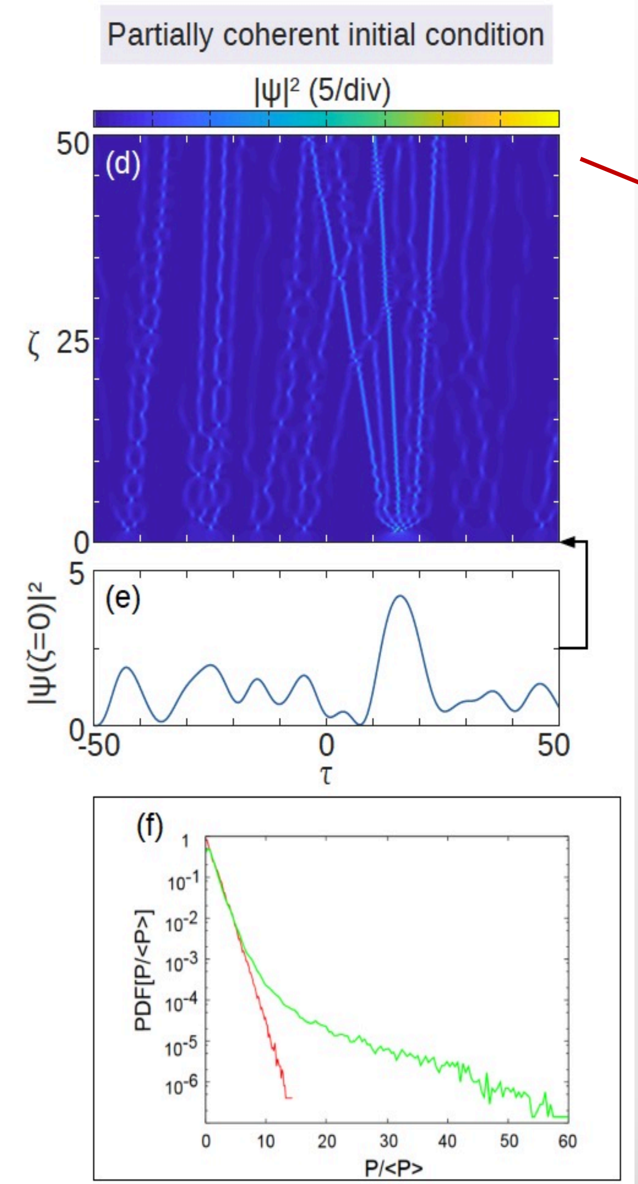
Numerical simulations 1DNLSE

Euler eqs. (higher order spectral method)

M. Onorato *et al.*, Phys. Rev. E, **70**, 067302 (2004)

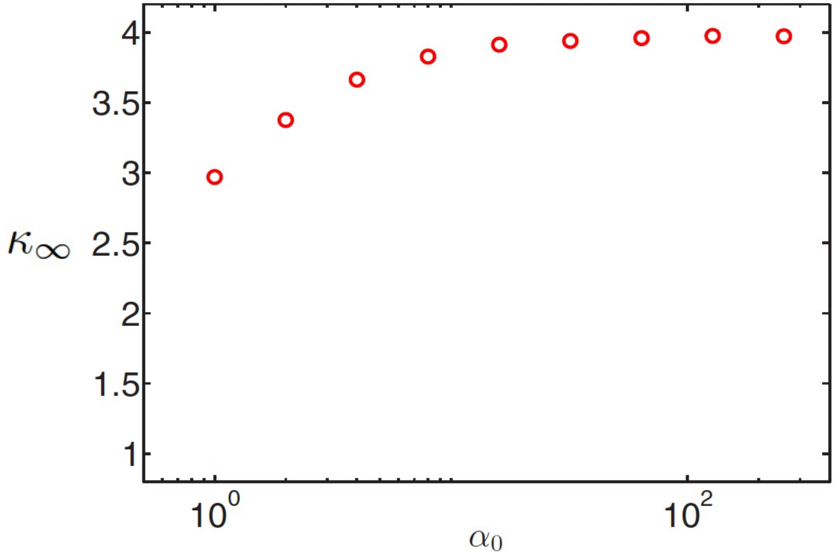
R El Koussaifi *et al.*, Phys. Rev. E, **97** (1), 012208 (2018)

1. Integrable turbulence: kurtosis (exemple focusing NLS)



$$\boxed{\kappa = \frac{\langle|\psi|^4\rangle}{\langle|\psi|^2\rangle^2}}$$

$$\max(\kappa_\infty) = 4$$



$$\alpha_0 = \frac{|\langle H_{nl}\rangle|}{H_l}\bigg|_{z=0} = \frac{\langle|\psi|^2\rangle}{\Delta\omega^2}\bigg|_{z=0} \qquad \Delta\omega^2 = \frac{\int \omega^2 \left|\widetilde{\psi}_\omega(z=0)\right|^2 d\omega}{\int \left|\widetilde{\psi}_\omega\right|^2 d\omega}$$

D. Agafontsev, S. Randoux and P. Suret, Phys. Rev. E, 103, 032209 (2021)

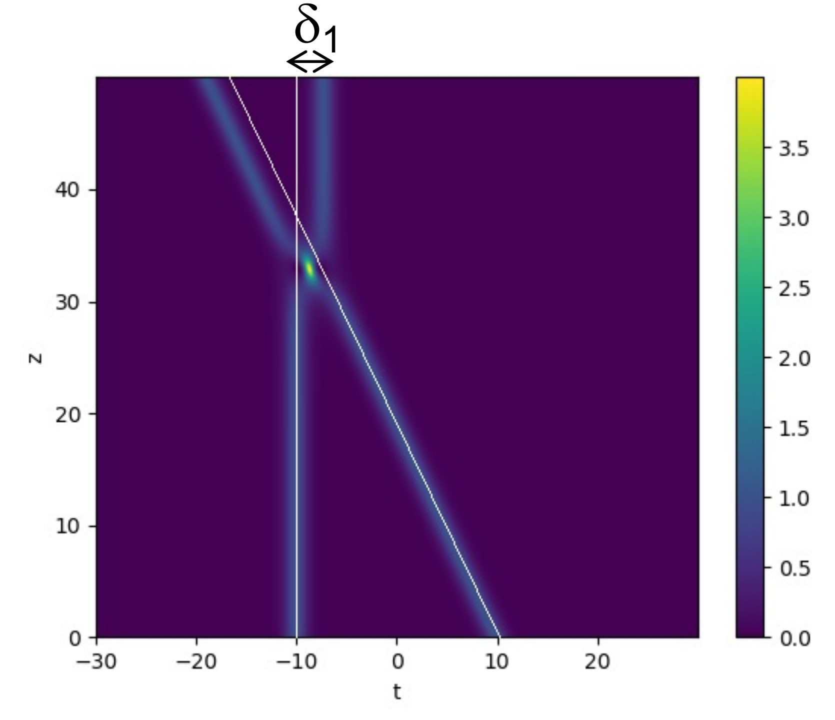
1. Solitons in integrable systems

- Integrable equations
  Korteweg De Vries (KdV), 1D nonlinear Schrödinger (NLS), Sine Gordon…

- Inverse Scattering Transform (IST) –nonlinear Fourier transform-

  V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP 34, 62 (1972)

- Solitons
  - ✓ Exact and stable solutions
  - ✓ Elastic collision
    (phase and space shifts)



NLS simulations, Elias Charnay (master)

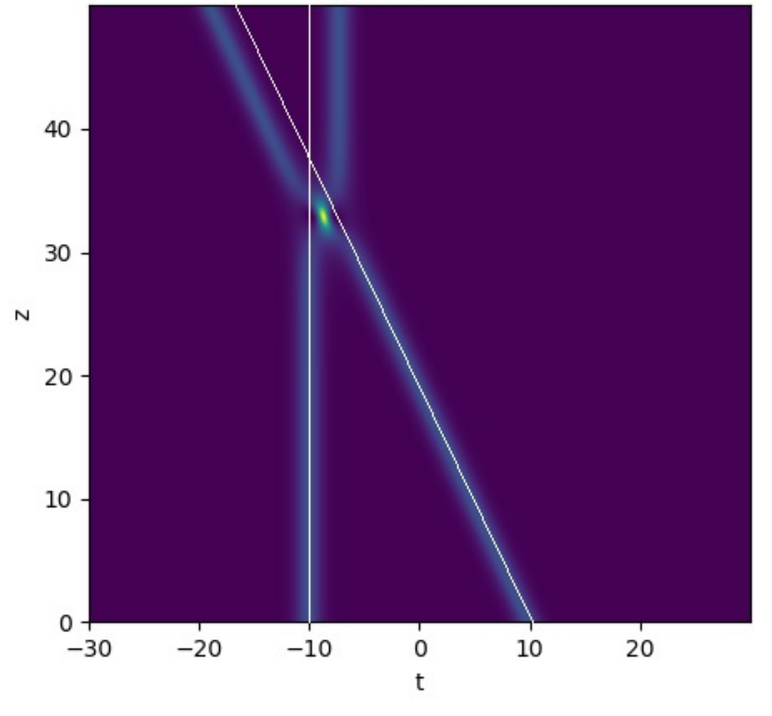
Particle like behavior => statistical mechanics of solitons ?

1. Inverse Scattering Transform (IST)

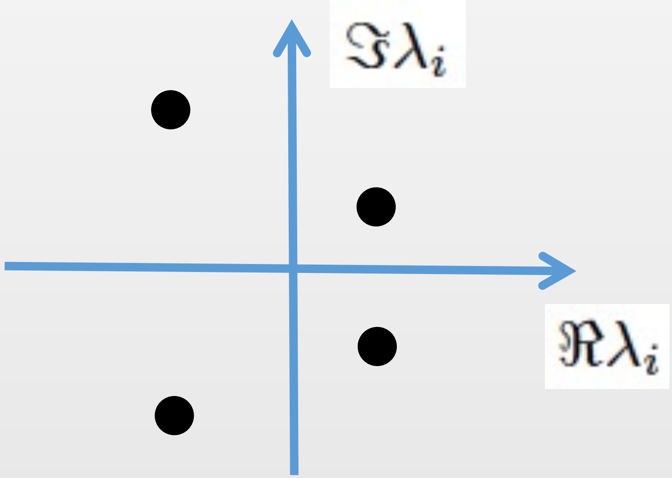
Zakharov, V. E., Sov. Phys. JETP, 33(3), 538-540, (1971)

- **Focusing nonlinear Schrödinger equation** $\frac{\partial \psi}{\partial z} = i\frac{1}{2}\frac{\partial^2 \psi}{\partial t^2} + i|\psi|^2\psi$

✓ Discrete spectrum (=solitons) : constant of motion



$$\boxed{\lambda_n = const}$$



✓ Continuous spectrum (=nonlinear dispersive waves)

1. Regimes of integrable turbulence

- **Focusing nonlinear Schrödinger equation**

$$\frac{\partial \psi}{\partial z} = i\frac{1}{2}\frac{\partial^2 \psi}{\partial t^2} + i|\psi|^2\psi$$

  - ✓ Discrete spectrum (=solitons)
  - ✓ Continuous spectrum (=nonlinear dispersive waves)

- **Hamiltonian structure**

$$H = H_L + H_{NL} = -\frac{1}{2}\int \left|\frac{\partial \psi}{\partial t}\right|^2 dt + \frac{1}{2}\int |\psi|^4 dt$$

- **(empirical) rule of thumb (partially coherent waves)**

at z=0, $\boxed{\frac{H_{NL}}{H_L} \nearrow \Longrightarrow \frac{\text{Discrete spectrum}}{\text{continuous spectrum}} \nearrow}$

- **Nonlinearity strength in integrable turbulence**

$\boxed{\frac{H_{NL}}{H_L} \ll 1}$ ➡ $\boxed{\frac{H_{NL}}{H_L} \gg 1}$

Fourier components
Wave turbulence

Nonlinear basis (solitons)
Soliton gas

2. Wave Turbulence theory of integrable turbuence

Wave turbulence / Transcient regime

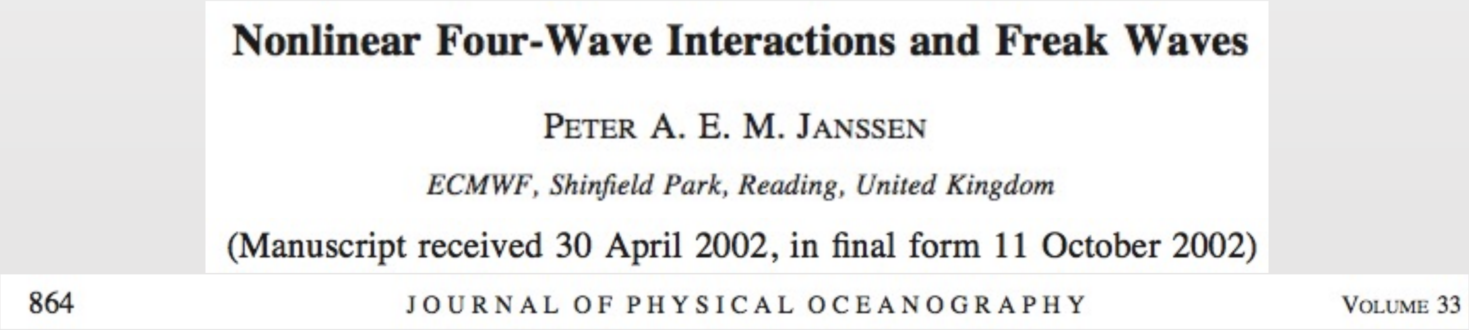
$$\boxed{\partial_z \, n(\omega, z) \neq 0}$$

Fiber Optics

D.B.S. Soh *et al.*, Opt. Express **18**, 22393-22405 (2010)

P. Suret *et al.*, Opt. Express **19**, 17852-17863 (2011)

Previously in hydrodynamics



Nonlinear Four-Wave Interactions and Freak Waves

PETER A. E. M. JANSSEN

ECMWF, Shinfield Park, Reading, United Kingdom

(Manuscript received 30 April 2002, in final form 11 October 2002)

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2. Wave Turbulence theory of integrable turbuence

1D NLS

$$i\,\partial_z\psi(z,t) = -\sigma\,\partial_t^2\psi(z,t) + |\psi(z,t)|^2\,\psi(z,t)$$

$\sigma = \pm 1$

t unit $1/\Delta\omega$
z unit $L_D = 2/(\beta_2\Delta\omega^2)$

Kinetic equation

$$\partial_z\, n_\omega(z) = Coll = ?$$

$$\left\langle \tilde{\psi}(z,\omega)\,\tilde{\psi}^*(z,\omega')\right\rangle = n_\omega\,(z)\,\delta(\omega-\omega')$$

Wave Turbulence theory : gaussian statistics ($H_{NL} << H_L$))

Closure of the moments equation

$$\left\langle \tilde{\psi}(\omega_1)\,\tilde{\psi}(\omega_2)\,\ldots\,\tilde{\psi}^*(\omega_m)\,\tilde{\psi}^*(\omega_n)\right\rangle = \mathcal{F}\left[n_1, n_2,\ \ldots\, n_m n_n\right]$$

P. Suret *et al.*, Opt. Express **19**, 17852-17863 (2011)

2. Wave Turbulence theory of integrable turbuence

$$\left\langle \tilde{\psi}(z,\omega)\,\tilde{\psi}^*(z,\omega')\right\rangle = n_\omega\,(z)\,\delta(\omega-\omega')$$

$$\left\langle \tilde{\psi}(z,\omega_1)\,\tilde{\psi}(z,\omega_2)\,\tilde{\psi}^*(z,\omega_3)\,\tilde{\psi}^*(z,\omega_4)\right\rangle = J_{1,2}^{3,4}\,(z)\,\delta(\omega_1+\omega_2-\omega_3-\omega_4)$$

$$\frac{\partial n_{\omega_1}(z)}{\partial z} = \frac{1}{\pi}\int\!\!\int\!\!\int d\omega_{2-4}\,Im\left[J_{1,2}^{3,4}(z)\right]\delta(\omega_1+\omega_2-\omega_3-\omega_4),$$

$$\frac{\partial J_{1,2}^{3,4}\,(z)}{\partial z} - i\,\Delta k\;J_{1,2}^{3,4}\,(z) = \frac{i}{\pi}\mathcal{N}(z)$$

$$\mathcal{N}(z) = n_{\omega_1}(z)n_{\omega_2}(z)n_{\omega_3}(z)n_{\omega_4}(z)\left[\frac{1}{n_{\omega_2}(z)}+\frac{1}{n_{\omega_1}(z)}-\frac{1}{n_{\omega_4}(z)}-\frac{1}{n_{\omega_3}(z)}\right]$$

$$J_{1,2}^{3,4}\,(z) = J_{1,2}^{3,4}\,(z=0)e^{i\,\Delta kz} + \frac{i}{\pi}\int_0^z dz'\;\mathcal{N}(z')\,e^{-i\,\Delta k(z'-z)}$$

2. Wave Turbulence theory of integrable turbuence

$$J_{1,2}^{3,4}(z) = J_{1,2}^{3,4}(z=0)e^{i\,\Delta k z} + \frac{i}{\pi}\int_0^z dz'\, \mathcal{N}(z')\, e^{-i\,\Delta k(z'-z)}$$

Oscillatory terms neglected in the usual treatment

$$\boxed{z >> \frac{1}{\Delta k}}$$

$$\frac{\partial n_{\omega_1}(z)}{\partial z} = \frac{1}{\pi}\int\int\int d\omega_{2-4}\,\mathcal{N}(z)\,\delta(\omega_1+\omega_2-\omega_3-\omega_4)\,\delta(\Delta k)$$

$$\boxed{\begin{gathered}\Delta k = k(\omega_1)+k(\omega_2)-k(\omega_3)-k(\omega_4)\\[1em] k(\omega)=\sigma\omega^2\end{gathered}}$$

1DNLS : NO Phase matched interactions

$$\boxed{\partial_z\, n_\omega(z) = 0}$$

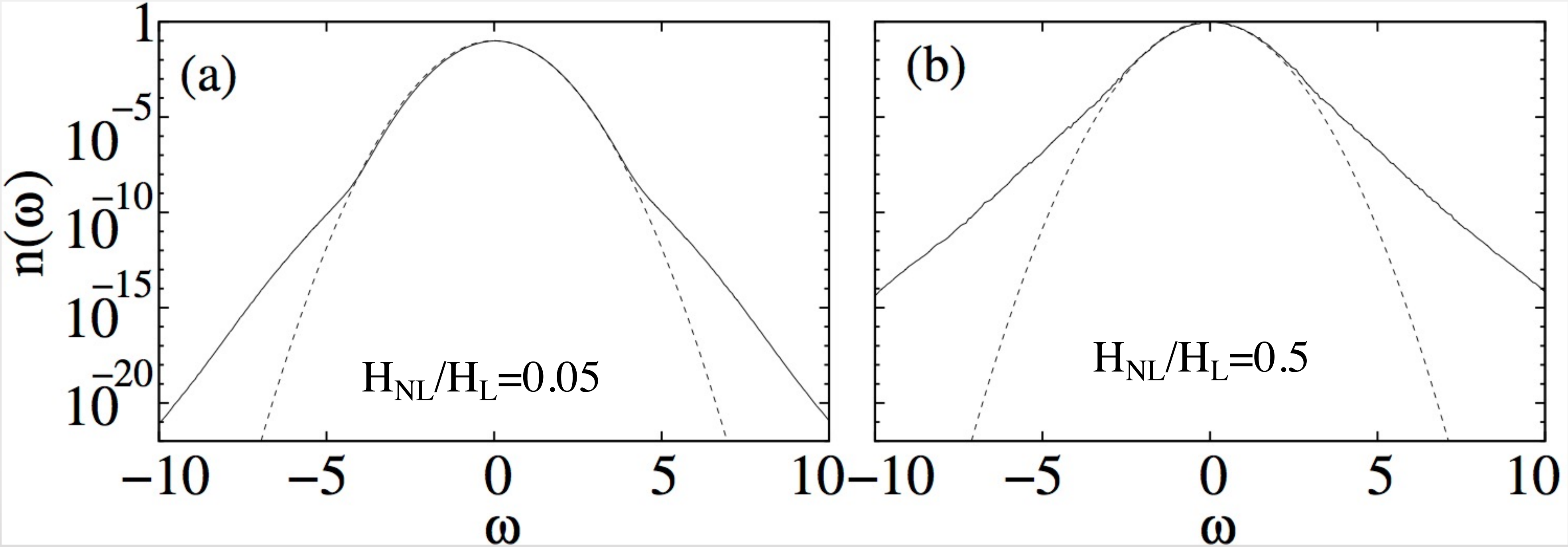
Oscillatory terms ?
Transient regime ?

P. Suret *et al.*, Opt. Express **19**, 17852-17863 (2011)

2. WTT of IT : numerical simulations

$$i\,\partial_z\psi(z,t) = -\sigma\,\partial_t^2\psi(z,t) + |\psi(z,t)|^2\,\psi(z,t)$$

$$\boxed{n_\omega^0 = n_\omega(z=0) = n_0\,\exp\left[-\left(\frac{\omega}{\Delta\omega}\right)^2\right]}$$



P. Suret *et al.*, Opt. Express **19**, 17852-17863 (2011)

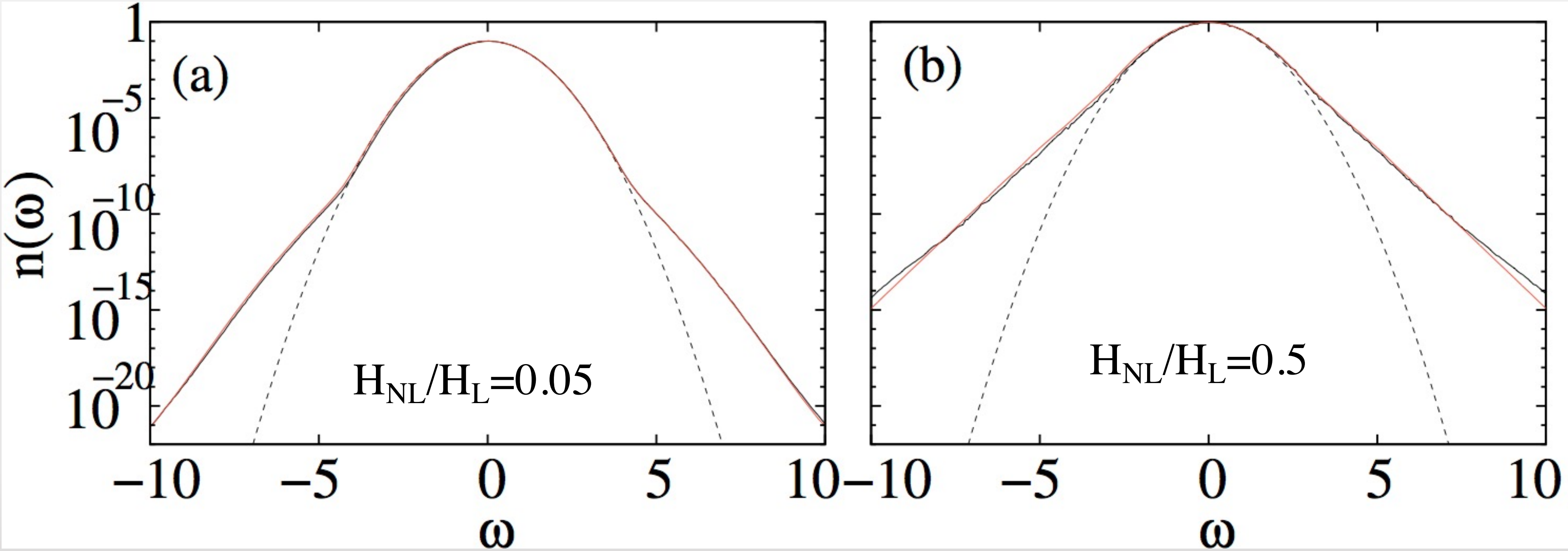
2. WTT of IT : numerical simulations

$$\frac{\partial n_{\omega_1}(z)}{\partial z} = \frac{1}{\pi} \int\int\int d\omega_{2-4}\, Im\left[J^{3,4}_{1,2}(z)\right] \delta(\omega_1+\omega_2-\omega_3-\omega_4),$$

$$\frac{\partial J^{3,4}_{1,2}(z)}{\partial z} - i\,\Delta k\; J^{3,4}_{1,2}(z) = \frac{i}{\pi}\mathcal{N}(z)$$

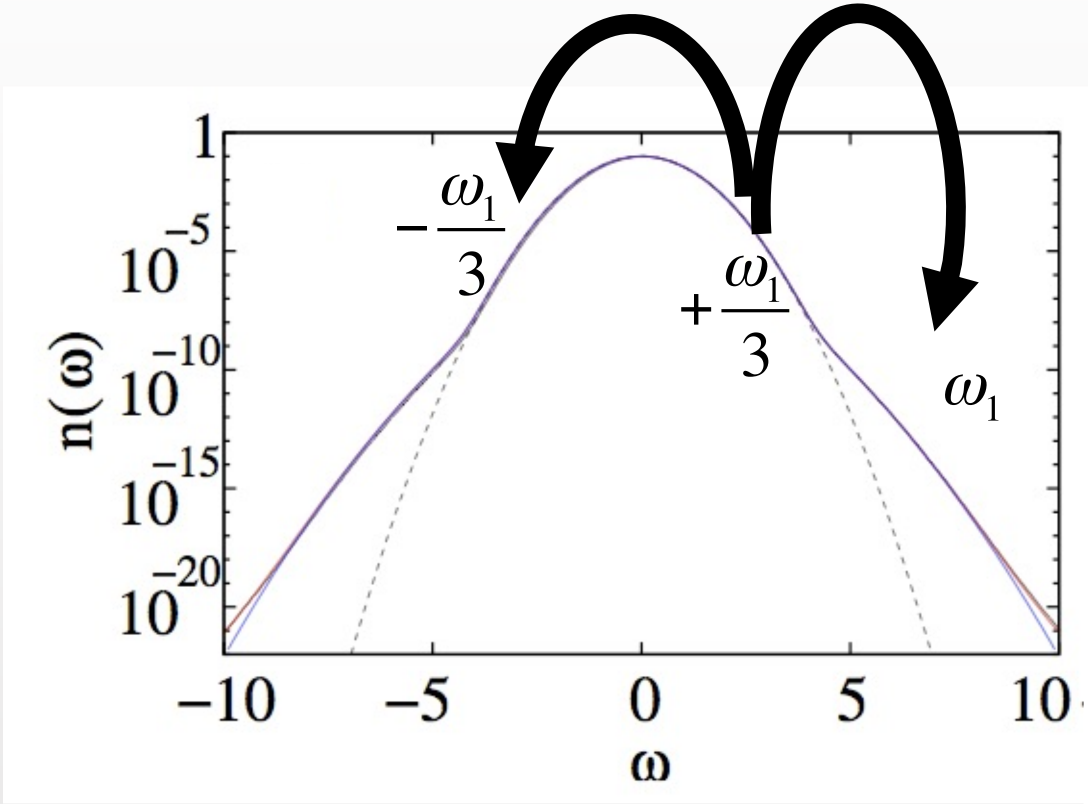
$$\boxed{n^0_\omega = n_\omega(z=0) = n_0\,\exp\left[-\left(\frac{\omega}{\Delta\omega}\right)^2\right]}$$

$$\mathcal{N}(z) = n_{\omega_1}(z)n_{\omega_2}(z)n_{\omega_3}(z)n_{\omega_4}(z)\left[\frac{1}{n_{\omega_2}(z)} + \frac{1}{n_{\omega_1}(z)} - \frac{1}{n_{\omega_4}(z)} - \frac{1}{n_{\omega_3}(z)}\right]$$



P. Suret *et al.*, Opt. Express **19**, 17852-17863 (2011)

2. WTT of IT : no cascade !



$$\Delta k = \left[\left(\frac{\omega_1}{3}\right)^2 + \omega_1^2\right] - \left[\left(\frac{\omega_1}{3}\right)^2 + \left(\frac{\omega_1}{3}\right)^2\right]$$

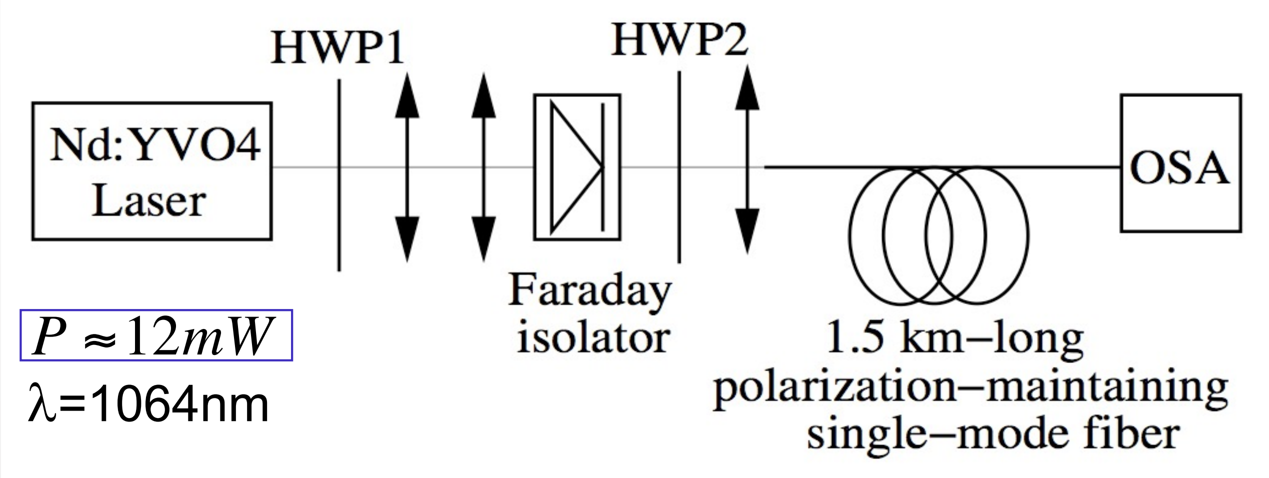
$$\boxed{\overline{\Delta k} = \frac{8}{9}\omega_1^2}$$

$$\boxed{\frac{\partial n_{\omega_1}(z)}{\partial z} \simeq \frac{n_0^3}{\sqrt{3}\pi} \frac{9}{8\omega_1^2} \exp\left(\frac{-\omega_1^2}{3}\left(1 + \frac{8\,z^2}{9}\right)\right) \sin\left(\frac{8\,\omega_1^2\,z}{9}\right)}$$

P. Suret *et al.*, Opt. Express **19**, 17852-17863 (2011)

2. WTT of IT : experiments

P. Suret *et al.*, Opt. Express **19**, 17852-17863 (2011)

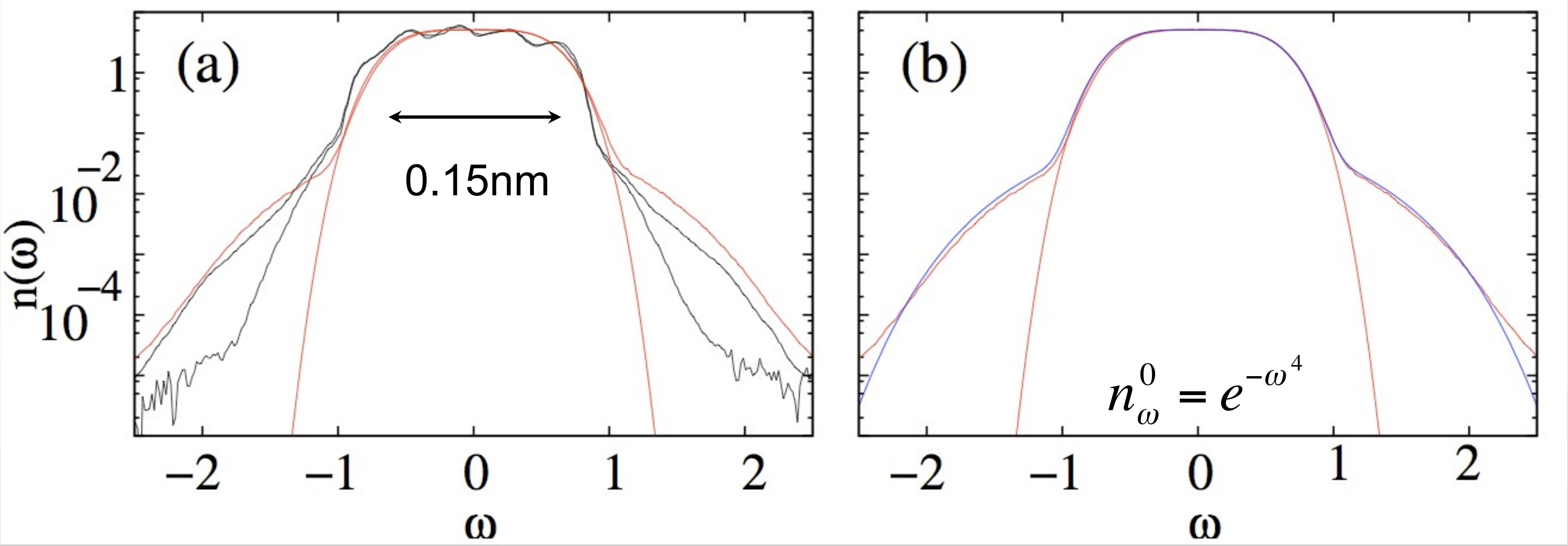


$\boxed{P \approx 12mW}$

λ=1064nm

Experiments / numerical simulations (NLS)

Simulations (NLS / Kinetic equations)

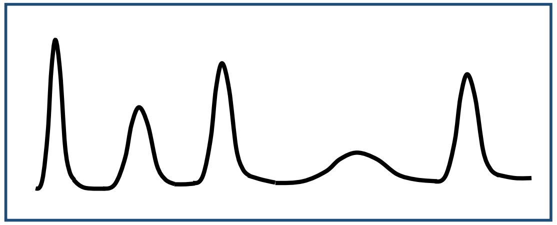


3. Soliton gas (integrable systems)

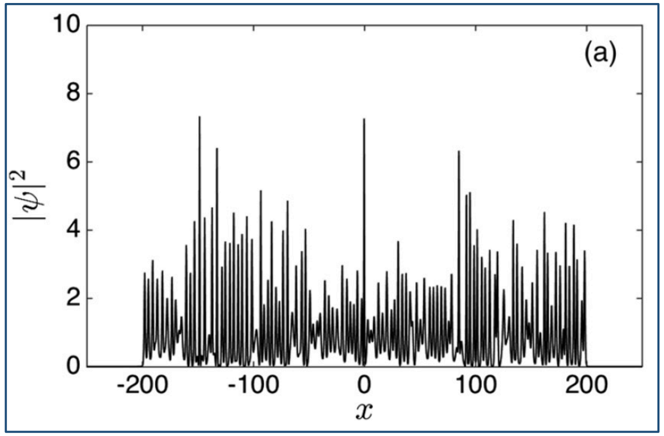
- 1971 DILUTED Soliton gas in KdV

V. E. Zakharov, *Kinetic equation for solitons*, Sov. Phys. JETP 33, 538 (1971)

  - ✓ Large ensemble of weakly interacting solitons
  - ✓ Random parameters



- 2003 Dense soliton gas (KdV+NLS)



El. G, Phys. Lett A, **311**, 374 (2003)
El G. and Kamchantov, Phys. Rev. Lett. **95**, 204101 (2005)
El G. and Tovbis A., Phys. Rev. E **101,** 052207 (2020)

- 1980 Solitons in optical fibers

Mollenauer, L. F., Stolen, R. H., & Gordon, J. P. *Experimental observation of picosecond pulse narrowing and solitons in optical fibers*. Physical Review Letters, *45*(13), 1095 (1980)

- Recent soliton gas experiments (water waves + optical fibers)

Redor I. *et al.*, Experimental evidence of a hydrodynamic soliton gas. *Phys. Rev. Lett.*, **122***,* 21, (2019)
Suret P., et al. "Nonlinear spectral synthesis of soliton gas in deep-water surface gravity waves." *Phys. Rev. Lett.* **125**. 26 (2020)
Suret, Pierre, et al. "Soliton refraction through an optical soliton gas." *arXiv preprint arXiv:2303.13421* (2023)

3. Soliton gas (integrable systems)

Dense SG (focusing 1DNLS)

El G. and Tovbis A., Phys. Rev. E **101,** 052207 (2020)

- ✓ **Isospectrality.** λ : spectral parameter (IST discrete eigenvalue)
- ✓ **Density of state (DOS) : $\rho(\lambda,z,t)$**

$\rho(\lambda,z,t)\ d\lambda\ dt$ is the number of soliton states found at length z in the element of the phase space $[\lambda,\lambda+d\lambda]\mathrm{x}[t,t+dt]$

z : evolution coordinate
t equivalent to space !

- ✓ Theory of Inhomogeneous soliton gas (kinetic equation)

$$\rho_z + (s\rho)_t = 0$$

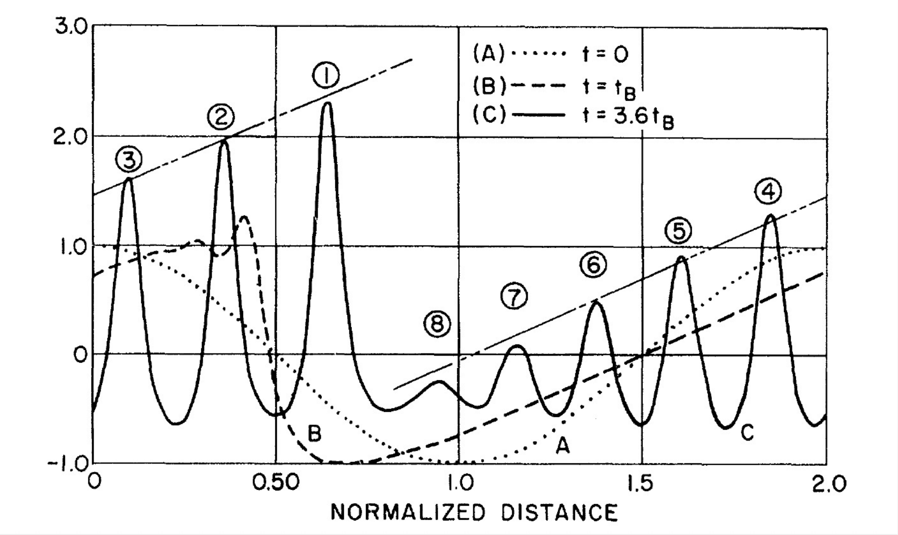
- ✓ **Non trivial Effective velocity** ⟺ **space shift**

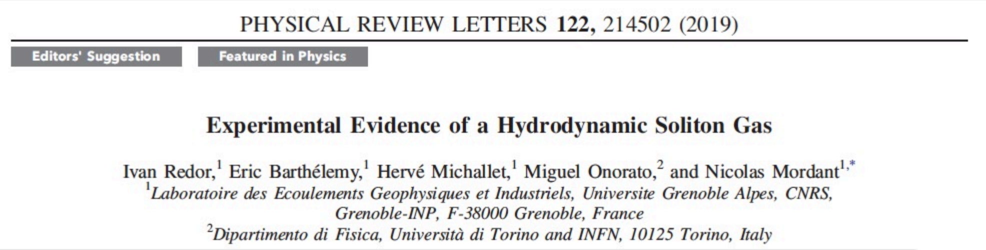
$$s(\lambda) = s_0(\lambda) + \int_0^{\infty} \Delta(\lambda,\mu)\rho(\mu)[s(\lambda) - s(\mu)]d\mu$$

- ✓ In this talk *:* Homogeneous gas : $\rho(\lambda, z, t) = \rho(\lambda)$

3. Soliton gas in shallow water experiments

- ➢ Shallow water
- ➢ Generation : Zabusky-Kruskal (1965)
- ➢ Analysis: dispersion relation (Fourier)





PHYSICAL REVIEW LETTERS **122,** 214502 (2019)

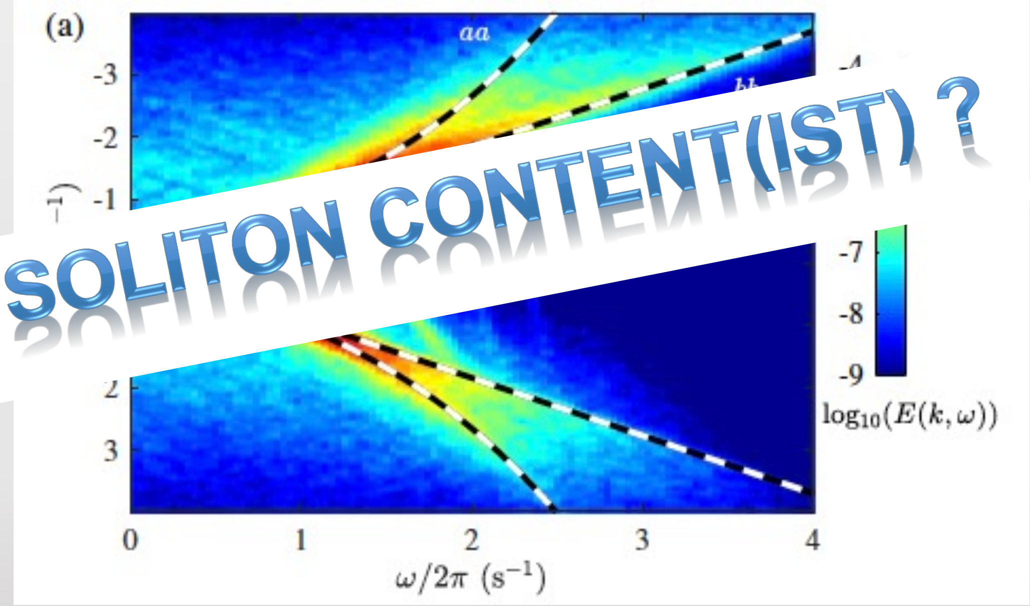
Editors' Suggestion **Featured in Physics**

Experimental Evidence of a Hydrodynamic Soliton Gas

Ivan Redor,[1] Eric Barthélemy,[1] Hervé Michallet,[1] Miguel Onorato,[2] and Nicolas Mordant[1,*]

[1]*Laboratoire des Ecoulements Geophysiques et Industriels, Universite Grenoble Alpes, CNRS, Grenoble-INP, F-38000 Grenoble, France*

[2]*Dipartimento di Fisica, Università di Torino and INFN, 10125 Torino, Italy*



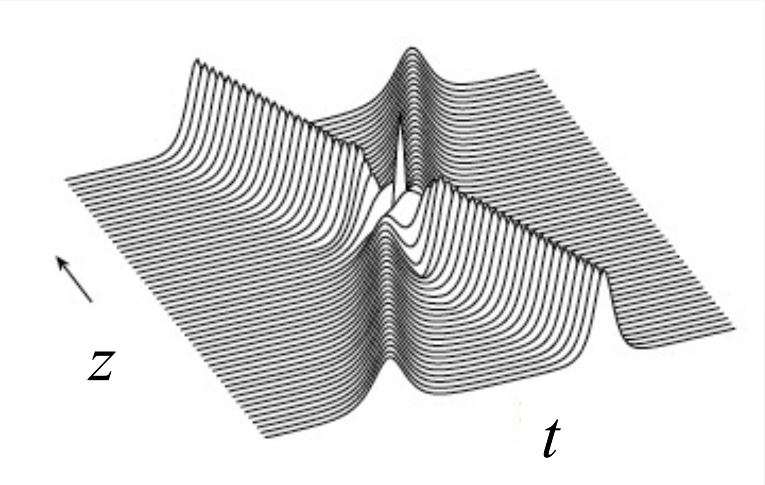
HOW TO CONTROL THE SOLITON CONTENT(IST) ?

3. Soliton Gas : model with N-solitons

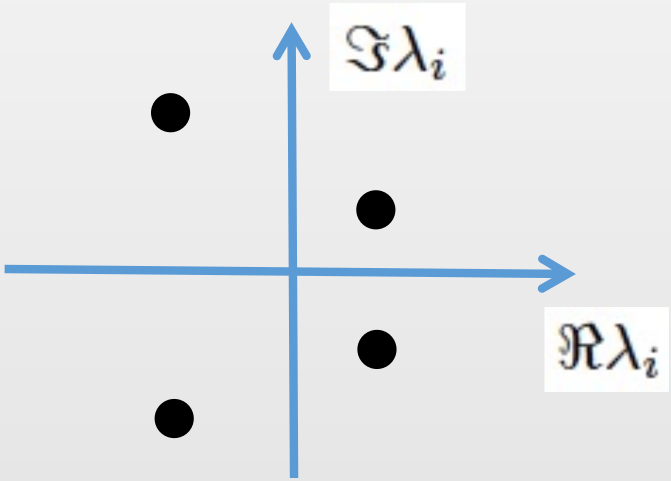
- **Focusing nonlinear Schrödinger equation**

$$\frac{\partial \psi}{\partial z} = i\frac{1}{2}\frac{\partial^2 \psi}{\partial t^2} + i|\psi|^2\psi$$

- **Inverse scattering transform (nonlinear Fourier transform)**
  - ✓ Discrete spectrum (=solitons) : constant of motion



$$\boxed{\lambda_n = const}$$



  - ✓ Norming constants C_n

$$\boxed{C_n(z) = C_n(0)e^{-2i\lambda_n^2 z}}$$

- **Soliton gas : random phases of C_n**

Zakharov, V. E., Sov. Phys. JETP, 33(3), 538-540, (1971).

El, G. A., & Kamchatnov, A. M., Phys. Rev. Lett. 95(20), 204101, (2005)

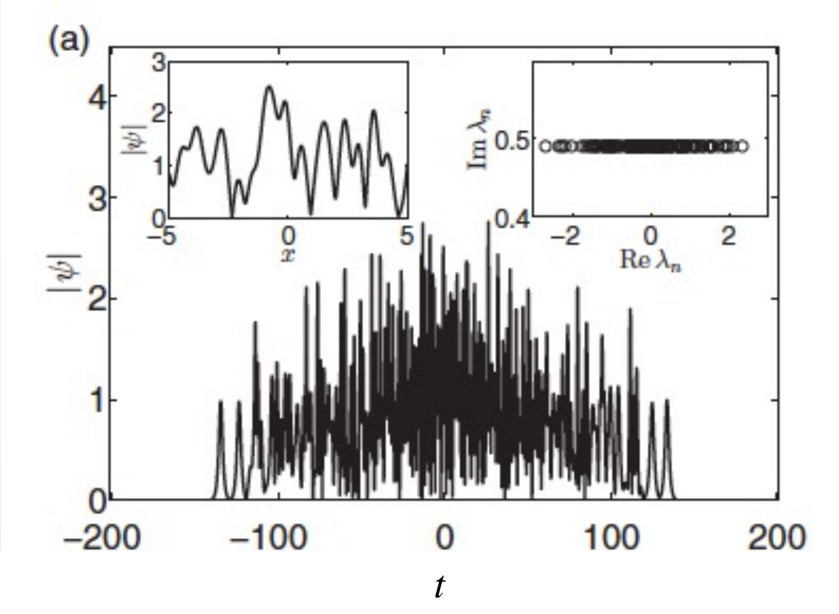
3. How to “build” an experimental **dense** soliton gas ?

- ➢ **Focusing 1D NLS equation** $$\frac{\partial\psi}{\partial z} = i\frac{1}{2}\frac{\partial^2\psi}{\partial t^2} + i|\psi|^2\psi$$

- ➢ **Model of a *dense* (and stationary) SG**
  - ✓ Zero boundary conditions
  - ✓ N-soliton solution with N>>1 V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP 34, 62 (1972)
  - ✓ Density of state $\rho(\lambda)$: λ_n (n=1..N)
  - ✓ Random phases (norming constant) and $|C_n(0)| = 1$

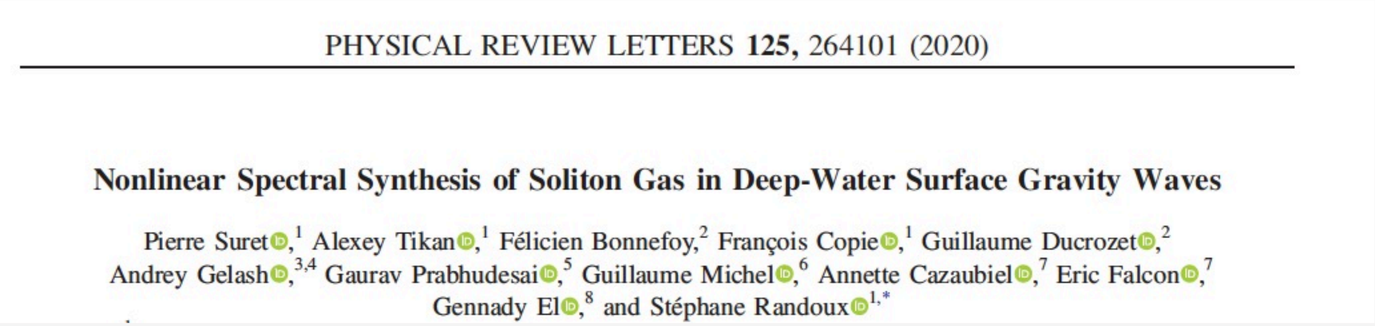
- ➢ **Arbitrary precision + dressing method N~200**

Gelash, A. A., & Agafontsev, D. S.,
Physical Review E, 98(4), 042210 (2018)



≠ rarefied SG

3.Water Tank experiments

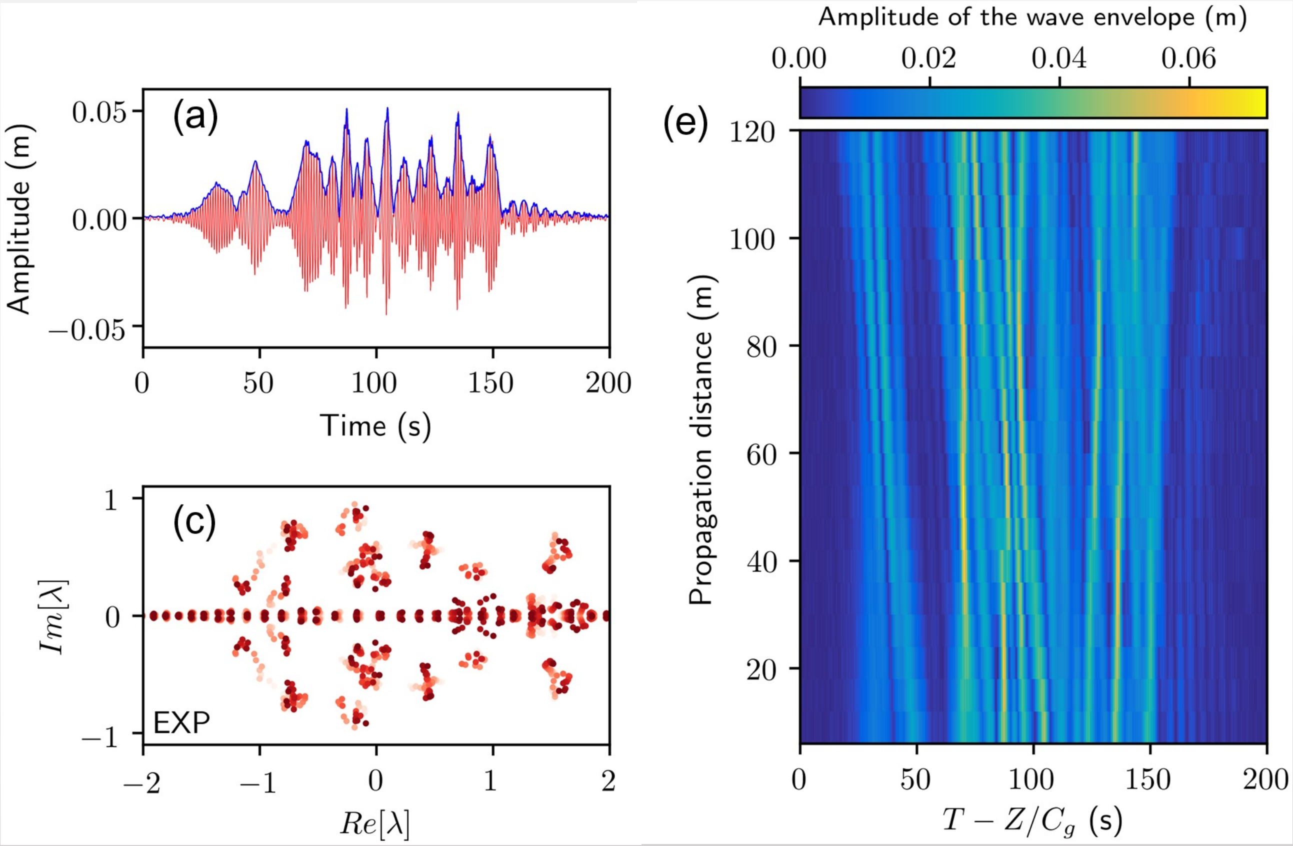


PHYSICAL REVIEW LETTERS **125,** 264101 (2020)

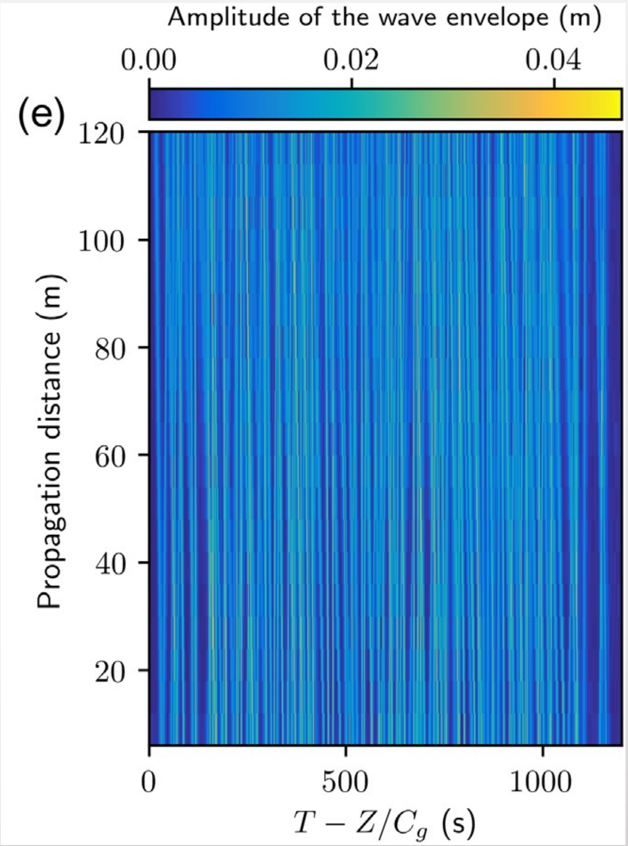
Nonlinear Spectral Synthesis of Soliton Gas in Deep-Water Surface Gravity Waves

Pierre Suret,[1] Alexey Tikan,[1] Félicien Bonnefoy,[2] François Copie,[1] Guillaume Ducrozet,[2]
Andrey Gelash,[3,4] Gaurav Prabhudesai,[5] Guillaume Michel,[6] Annette Cazaubiel,[7] Eric Falcon,[7]
Gennady El,[8] and Stéphane Randoux[1,*]

- N=16, random phase



- N=128, random phase



4. Modulation Instability

- Benjamin-Feir instability (1967)

Benjamin, T. Brooke; Feir, J.E. (1967). Journal of Fluid Mechanics. 27 (3) p.417–430
Benjamin, T.B. (1967). Proceedings of the Royal Society of London. A. 299 (1456) p.59–76

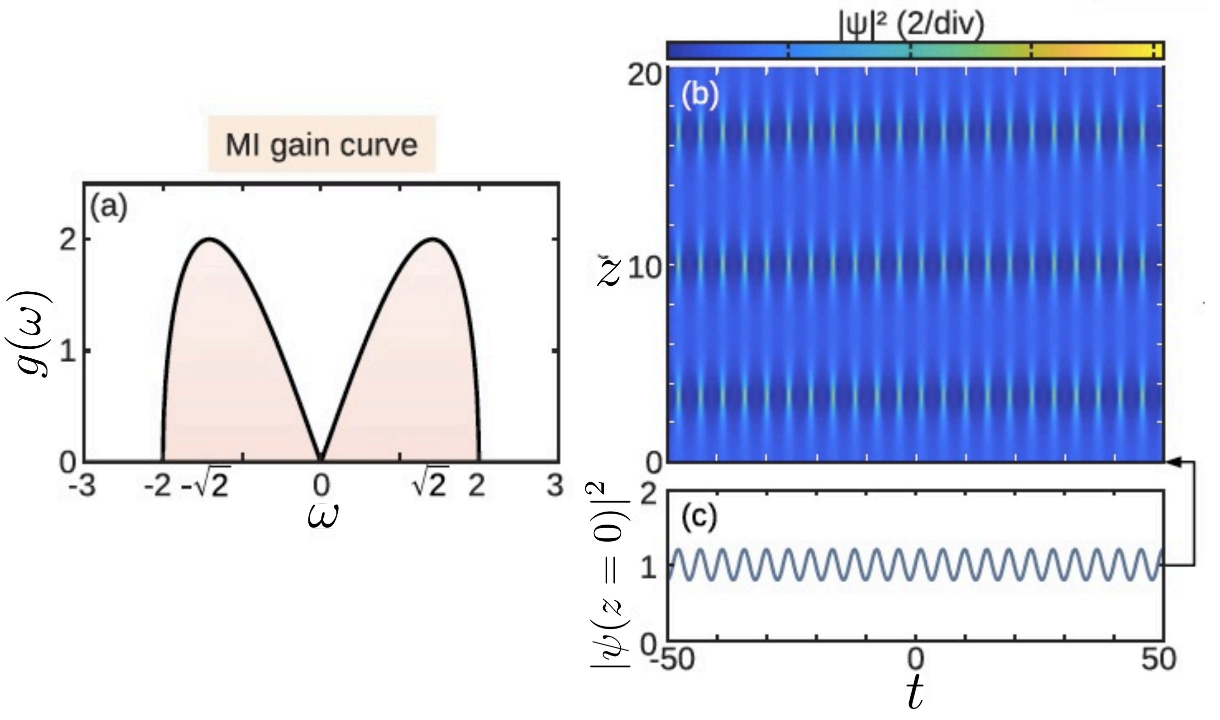
- deep Water waves, optical fibers (anomalous dispersion)…
- 1D focusing NLS

$$\frac{\partial \psi}{\partial z} = i\frac{1}{2}\frac{\partial^2 \psi}{\partial t^2} + i|\psi|^2\psi$$

N. Akhmediev *et al.*,, Sov. Phys. JETP 62, 894 (1985).
N. Akhmediev and V. Korneev,, Theor. Math. Phys. 69, 1089 (1986).
N. Akhmediev, *et al.* Phys. Lett. A 373, 675 (2009).

- Harmonic perturbation

$$\psi_{(z=0,t)} = 1 + \epsilon_1.e^{i\omega t} + \epsilon_2.e^{-i\omega t}$$



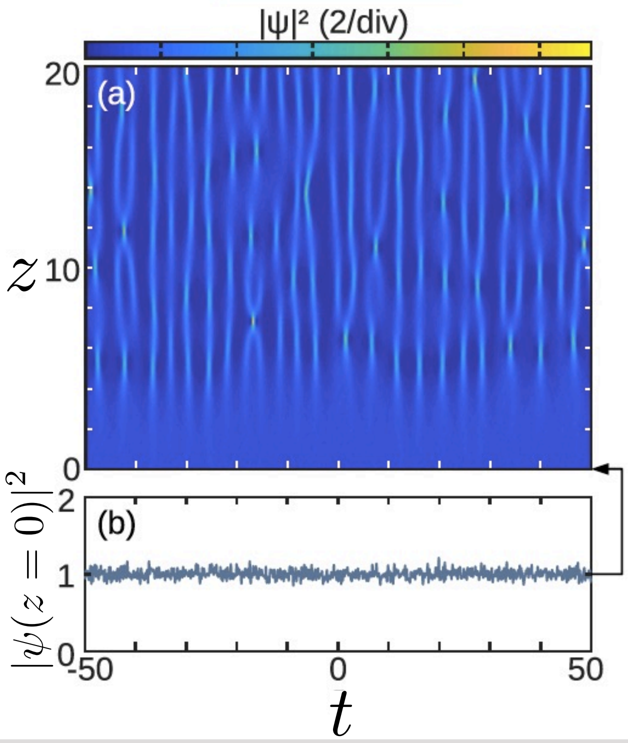
4. Spontaneous (noise induced) Modulation Instability

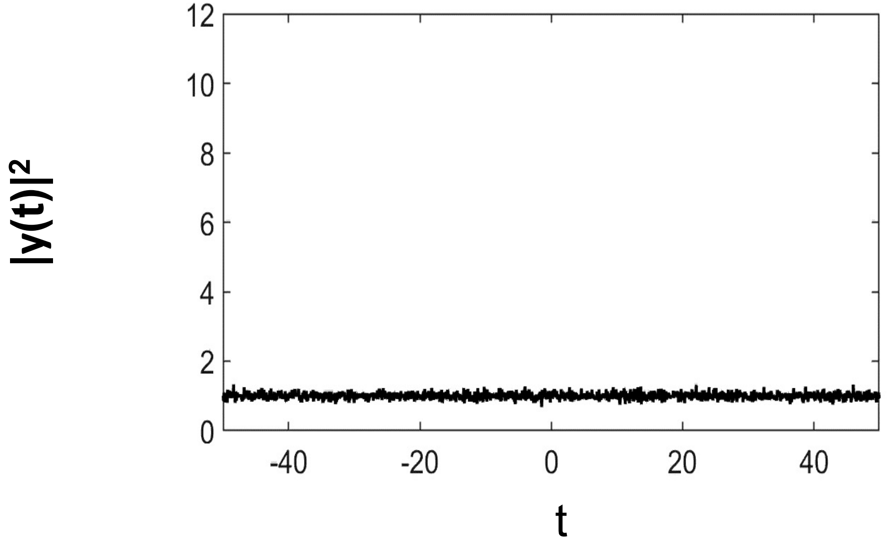
- Numerical simulations (focusing 1DNLS)

$$\frac{\partial\psi}{\partial z} = i\frac{1}{2}\frac{\partial^2\psi}{\partial t^2} + i|\psi|^2\psi$$

$$t = t_{laboratory} - \frac{z}{v_g(\omega_0)}$$

$$\psi(z=0,t) = 1 + \epsilon(t)$$





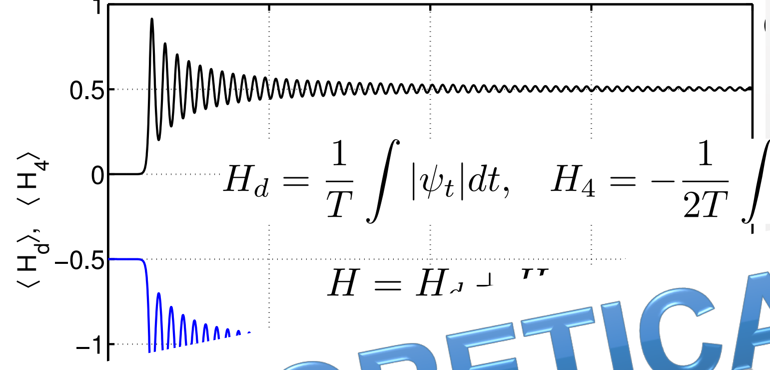
Copie, F., Randoux, S., & Suret, P. *Reviews in Physics*, *5*, 100037 (2020)
Toenger, S., Godin, T., Billet, C., Dias, F., Erkintalo, M., Genty, G., & Dudley, J. M. *Scientific reports*, *5*(1), 1-8 (2015)

4. Spontaneous modulation instability: statistics

Agafontsev, D. S., & Zakharov, V. E. *Integrable turbulence and formation of rogue waves,* Nonlinearity, **28**,(8), 2791. (2015)

$$\psi_z = i\psi_{tt} + i|\psi|^2\psi$$

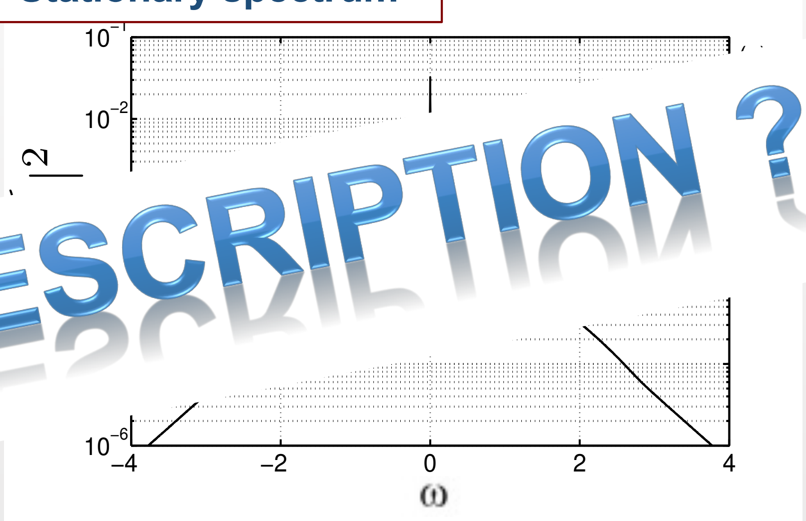
- **Transient regime: oscillations**



$$H_d = \frac{1}{T}\int |\psi_t|dt, \quad H_4 = -\frac{1}{2T}\int |\psi|^4 dt$$

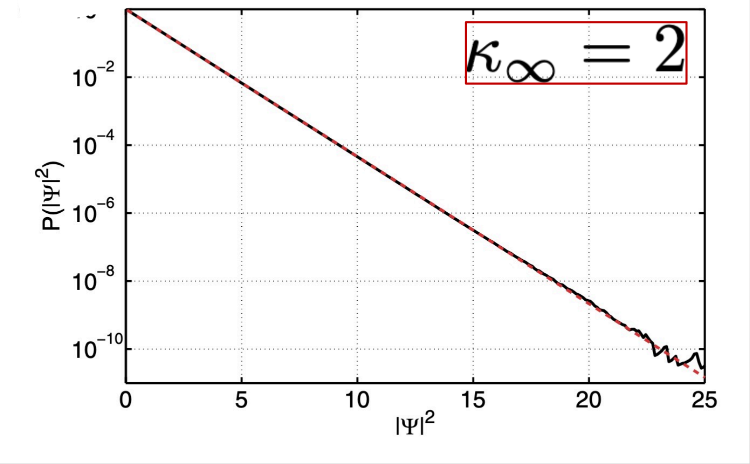
$$H = H_d + H$$

- **Stationary spectrum**



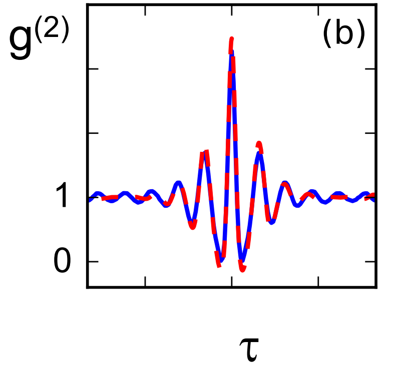
THEORETICAL DESCRIPTION ?

- **normal law !**



- **Two points statistics**

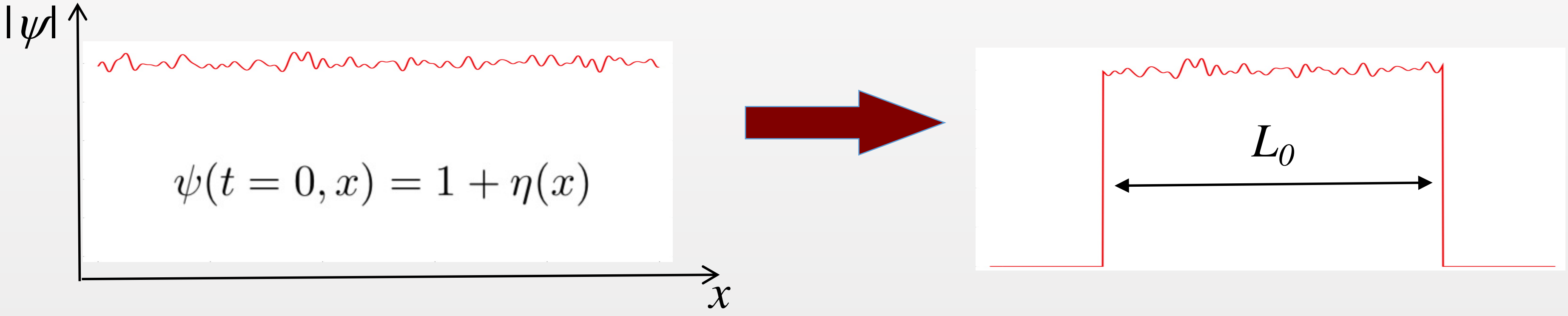
$$g^{(2)}(\tau) = \frac{\langle|\psi(t)|^2|\psi(t-\tau)|^2\rangle}{\langle|\psi(t)|^2\rangle^2}$$



A. E. Kraych *et al.*, Phys. Rev. Lett. **123**, 093902 (2019)

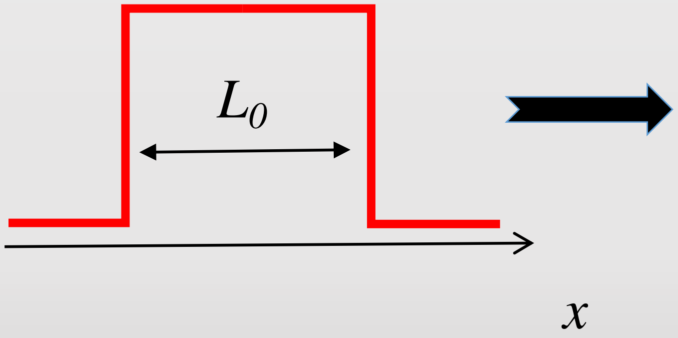
MI modeled by soliton gas ?
Soliton gas modeled by N soliton ?

Boundary conditions



$$\psi(t=0,x) = 1 + \eta(x)$$

IST eigenvalues (limit $\eta \rightarrow 0$)



$$\lambda_n = i\,\beta_n = i\sqrt{1-\left[\frac{\pi(n-\frac{1}{2})}{L_0}\right]^2}, \quad n = 1, 2, \ldots, N$$

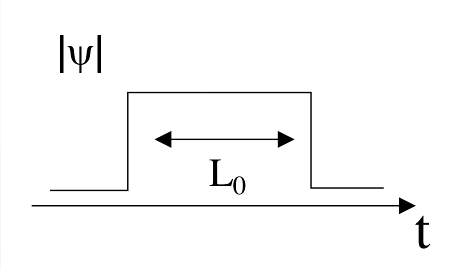
$$N = \mathrm{int}[L_0/\pi]$$

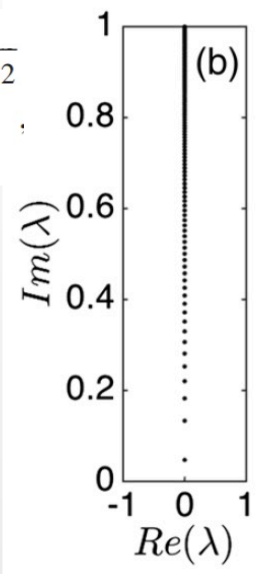
Bohr-Sommerfeld (semi-classical approximation)
Exact formula : S. V. Manakov, Zh. Eksp. Toor. Fiz. 65, 1392-1398 (1973)

4. Spontaneous MI modeled by Soliton Gas

- eigenvalues λ_n of a box (IST)

$$\lambda_n = i\beta_n = i\sqrt{1 - \left(\frac{\pi(n-\frac{1}{2})}{L_0}\right)^2},$$

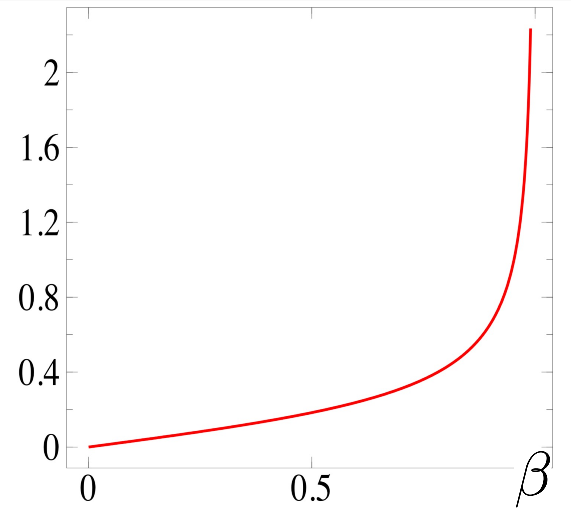




- Density of state : Weyl distribution

$$\boxed{\rho(\beta) = \frac{1}{\pi}\frac{\beta}{\sqrt{1-\beta^2}}}$$

$$\lambda = i\beta$$



$$\rho\, d\beta\, dt = \text{number of solitons in}[\beta : \beta + d\beta][t : t + dt]$$

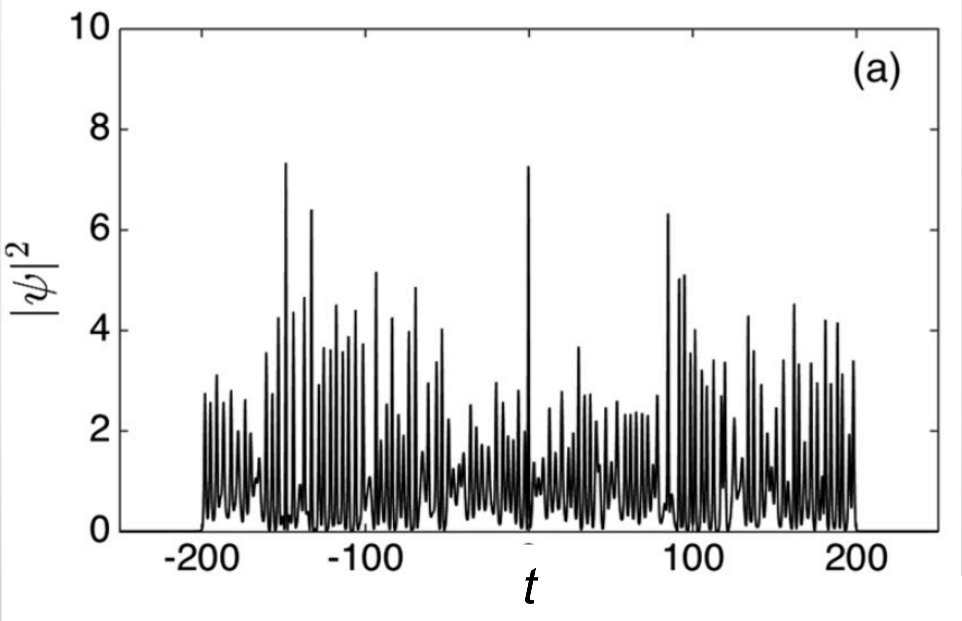
Norming constants : random phases

$$\lambda_n = const, \quad C_n(z) = C_n(0)e^{-2i\lambda_n^2 z}$$

Gelash, A. , Agafontsev, D., Zakharov, V., El, G., Randoux, S., & Suret, P.

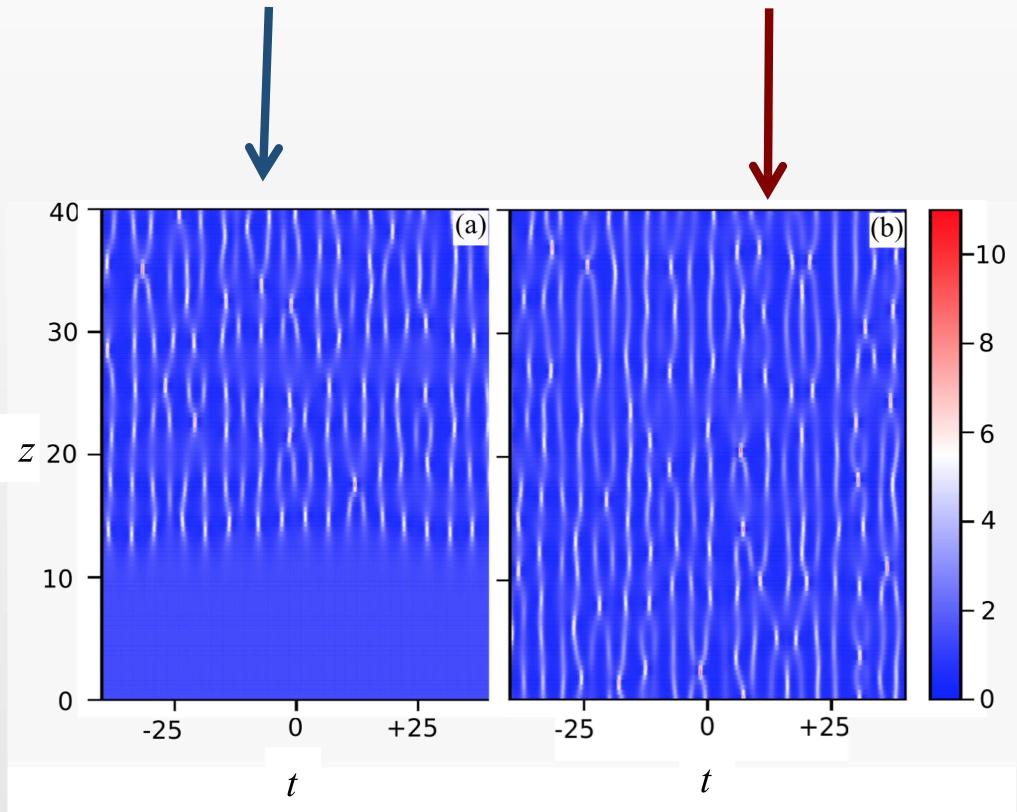
Bound state soliton gas dynamics underlying the spontaneous modulational instability

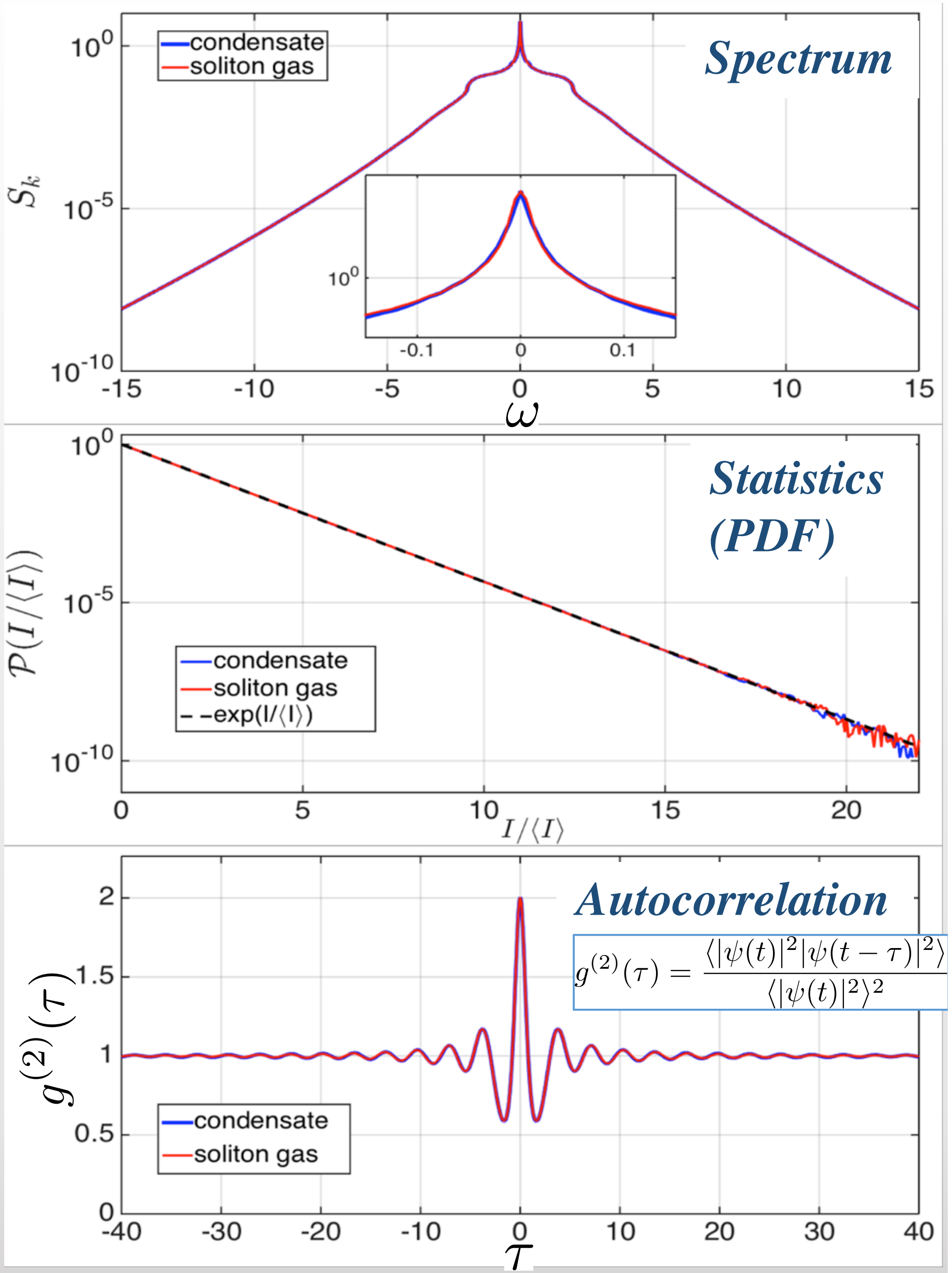
Phys. Rev. Lett (2019)



Spontaneous MI
(condensate)

Soliton Gas
(N soliton)



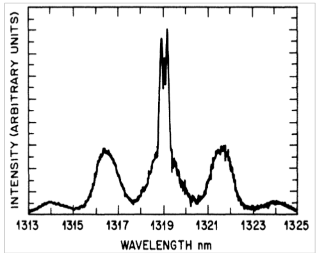


Gelash, A. , Agafontsev, D., Zakharov, V., El, G., Randoux, S.
& Suret, P.

Bound state soliton gas dynamics
underlying the spontaneous modulational instability

Phys. Rev. Lett (2019)

4. Modulation Instability in optical fiber experiments



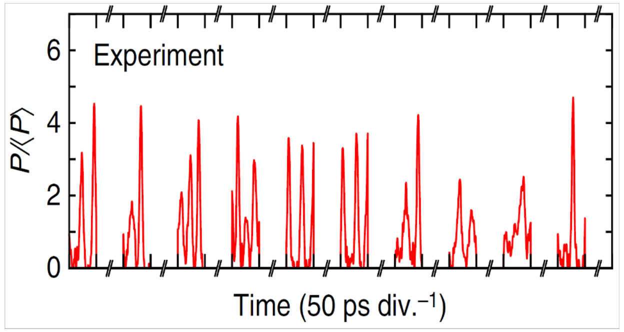
Tai *et al.* PRL 56 (2) (1986)

First observation of MI in optical fiber (spectrum)

A. Mussot *et al.* Nature photonics 12 (5) (2018)
B. Kibler *et al*. Physical Review X, 5 (4) (2015)
D. R. Solli, *et al*. Nature Photonics, 6(7) (2012)
M. Erkintalo *et al*. Phys. Rev. Lett. *107* (25) (2011)
Dudley, J. M. *et al*. *Opt. Express*, **17**(24) (2009)

…

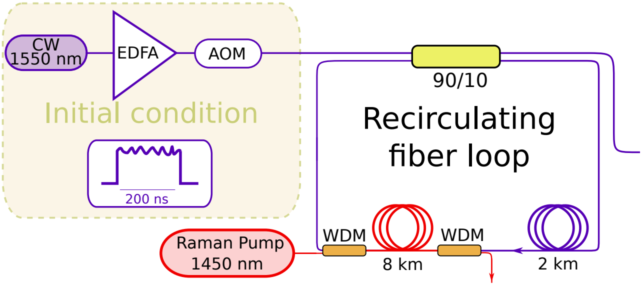
… 10^X papers !

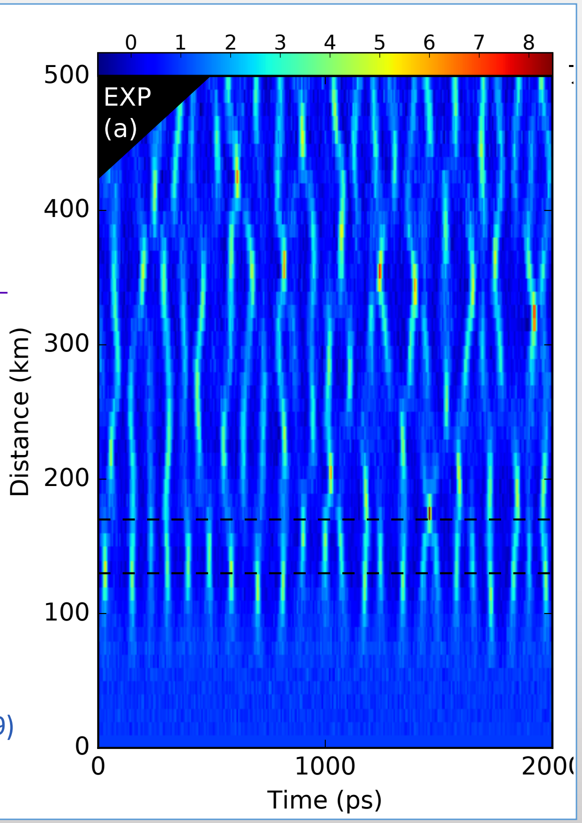


Time Lens (amplitude)

Närhi *et al.* Nat. Comm. 7 (2016)

Spatio-Temporal dynamics



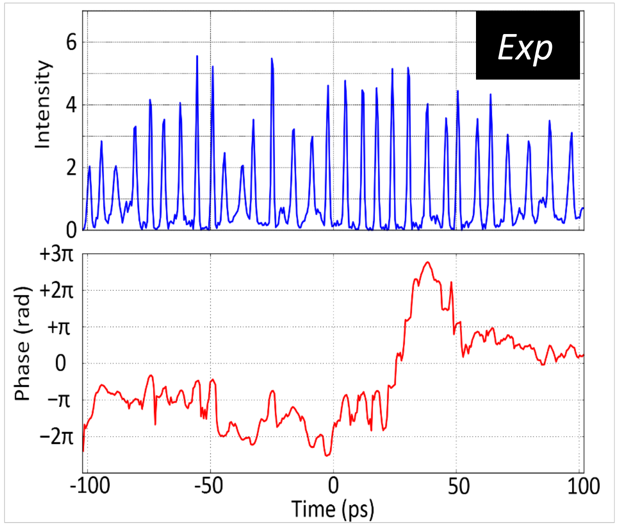


Recirculating loop fiber

A. E. Kraych *et al.*, Phys. Rev. Lett. **123**, 093902 (2019)

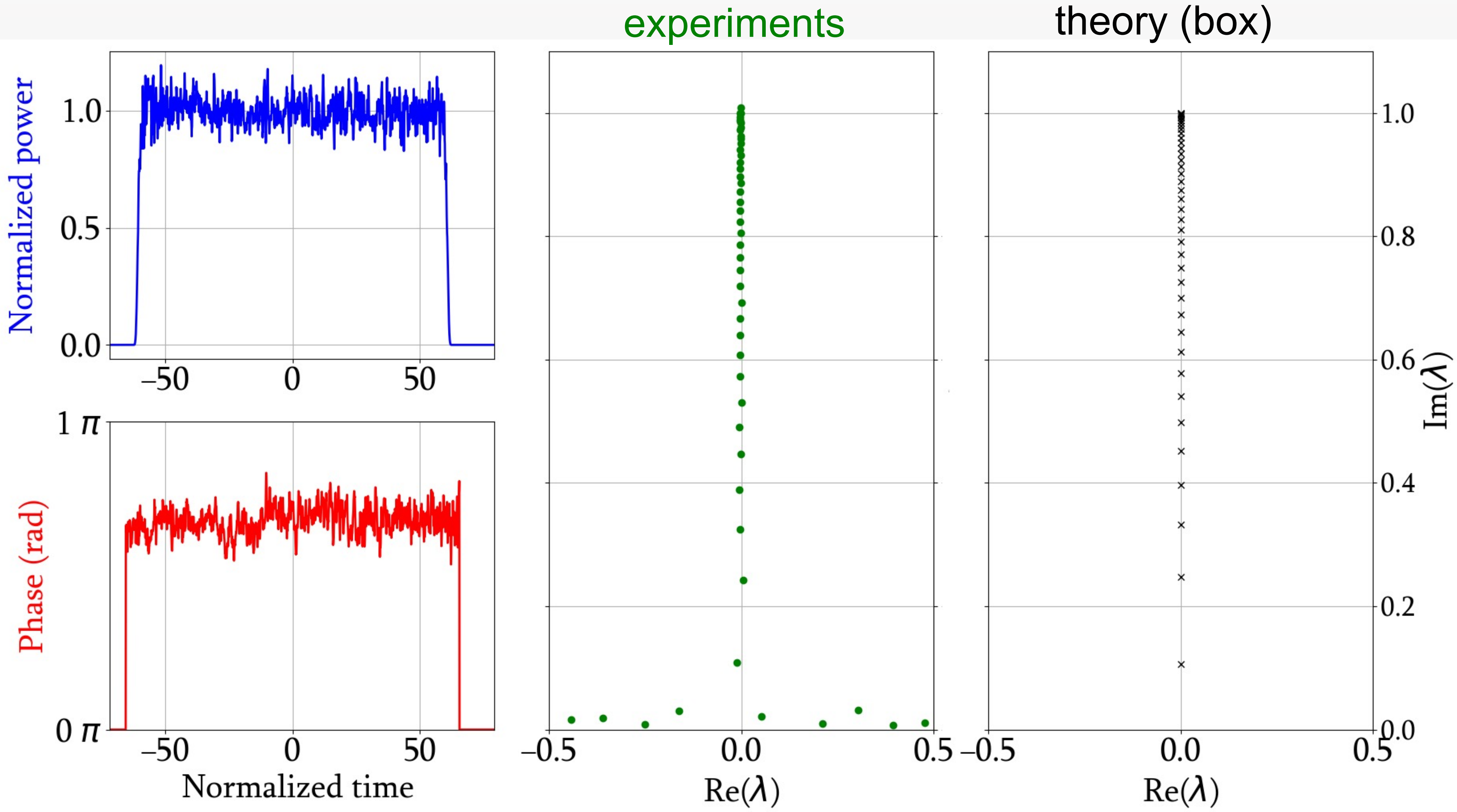
Heterodyne temporal imaging phase & amplitude (SEAHORSE)

P. Suret *et al.*, Nat. Commun. **7**, 13136 (2016)
A. Tikan *et al.*, Nat. Photon. **12** (2018)
A Lebel *et al.* Opt. Lett. 46 (2), 298-301 (2021)



4. Density of state in optical fiber experiments (new results)

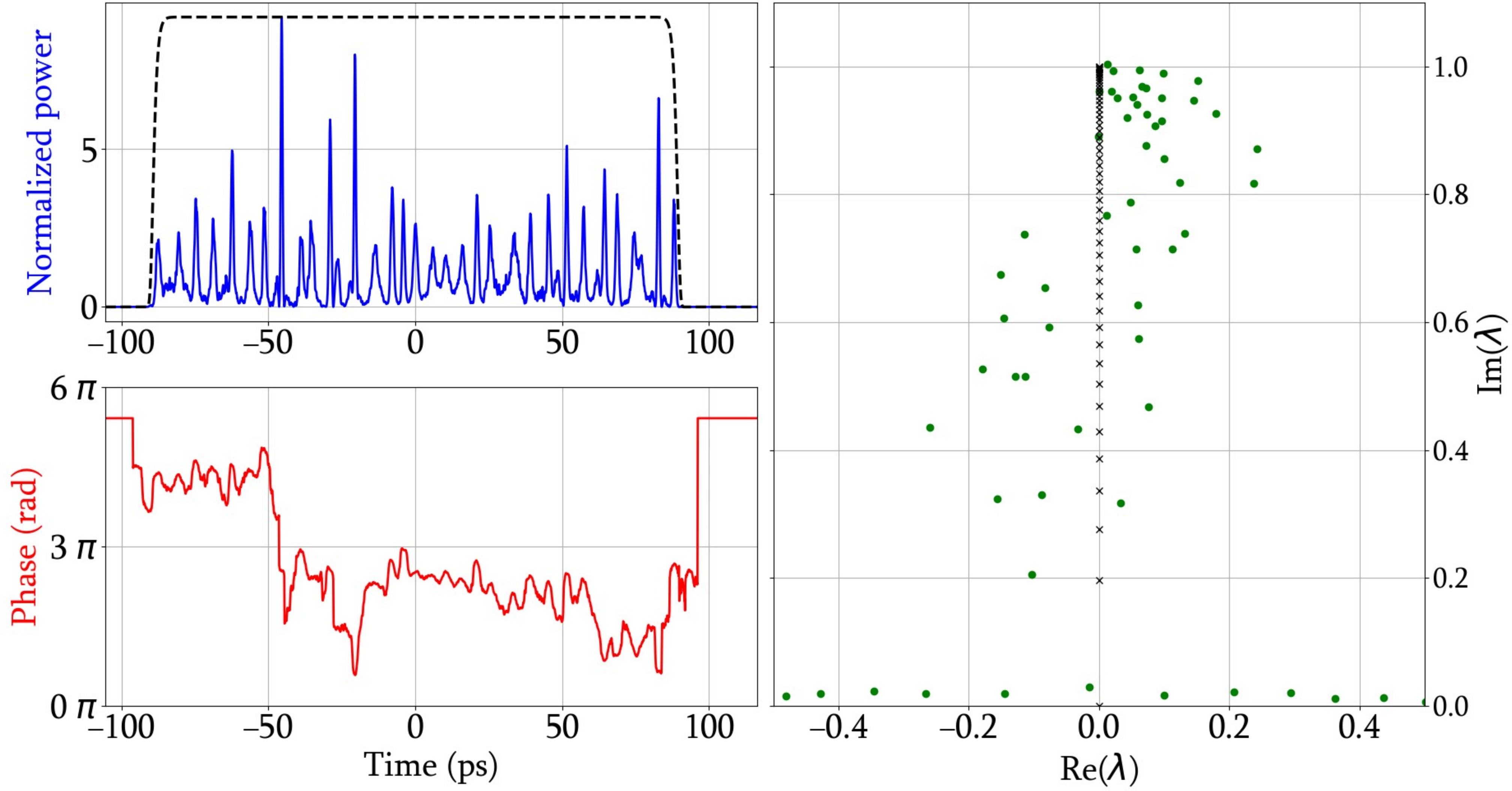
- *Initial condition (z=0)*



4. Density of state in optical fiber experiments

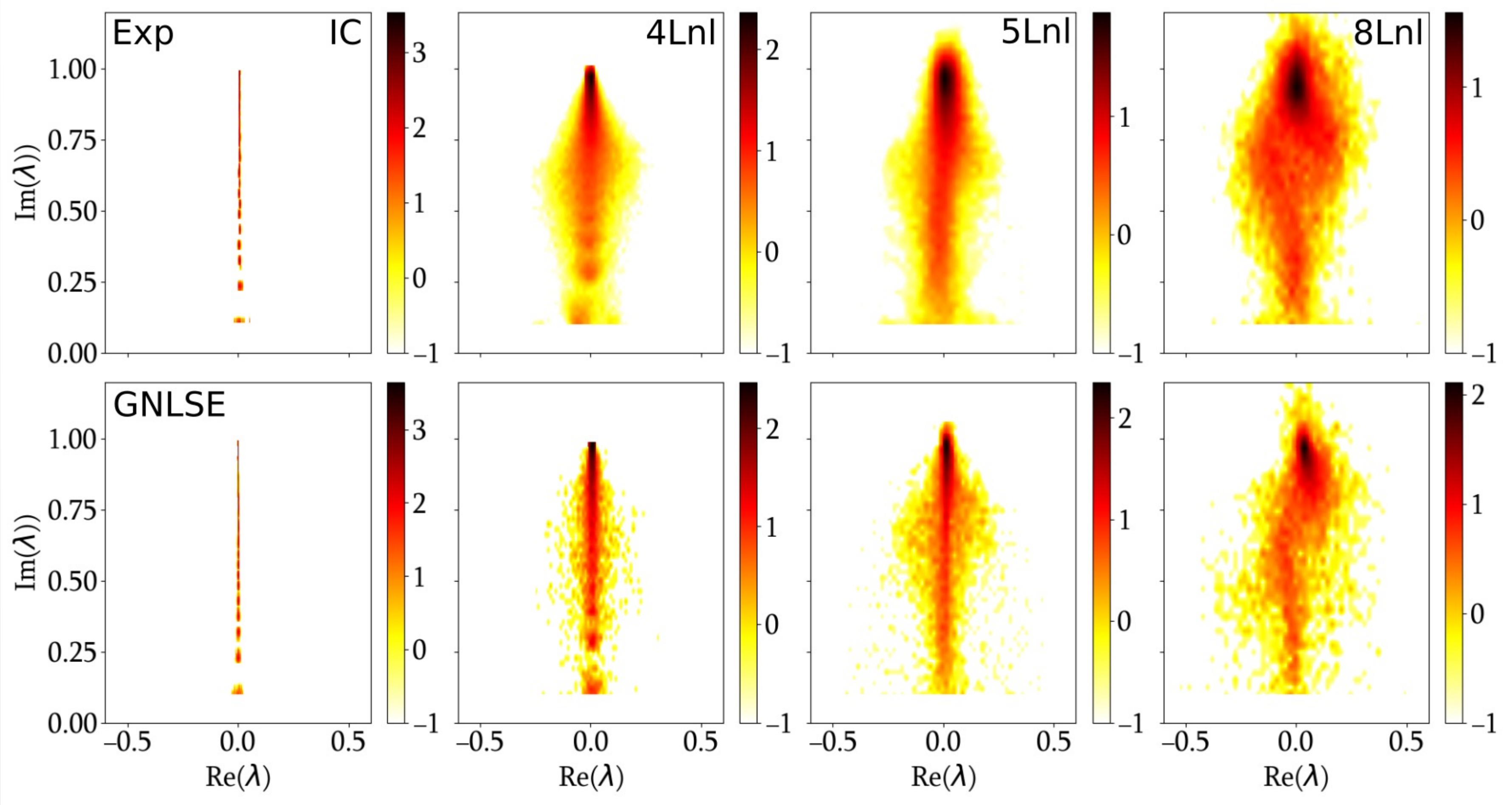
PMF, P = 6.6W
β_2 = -20.6ps^2/km
$\chi^{(3)}$ = 2.4W^{-1}km^{-1}
L=400m

- *Output*



4. Density of state : Experiments + Numerical simulations

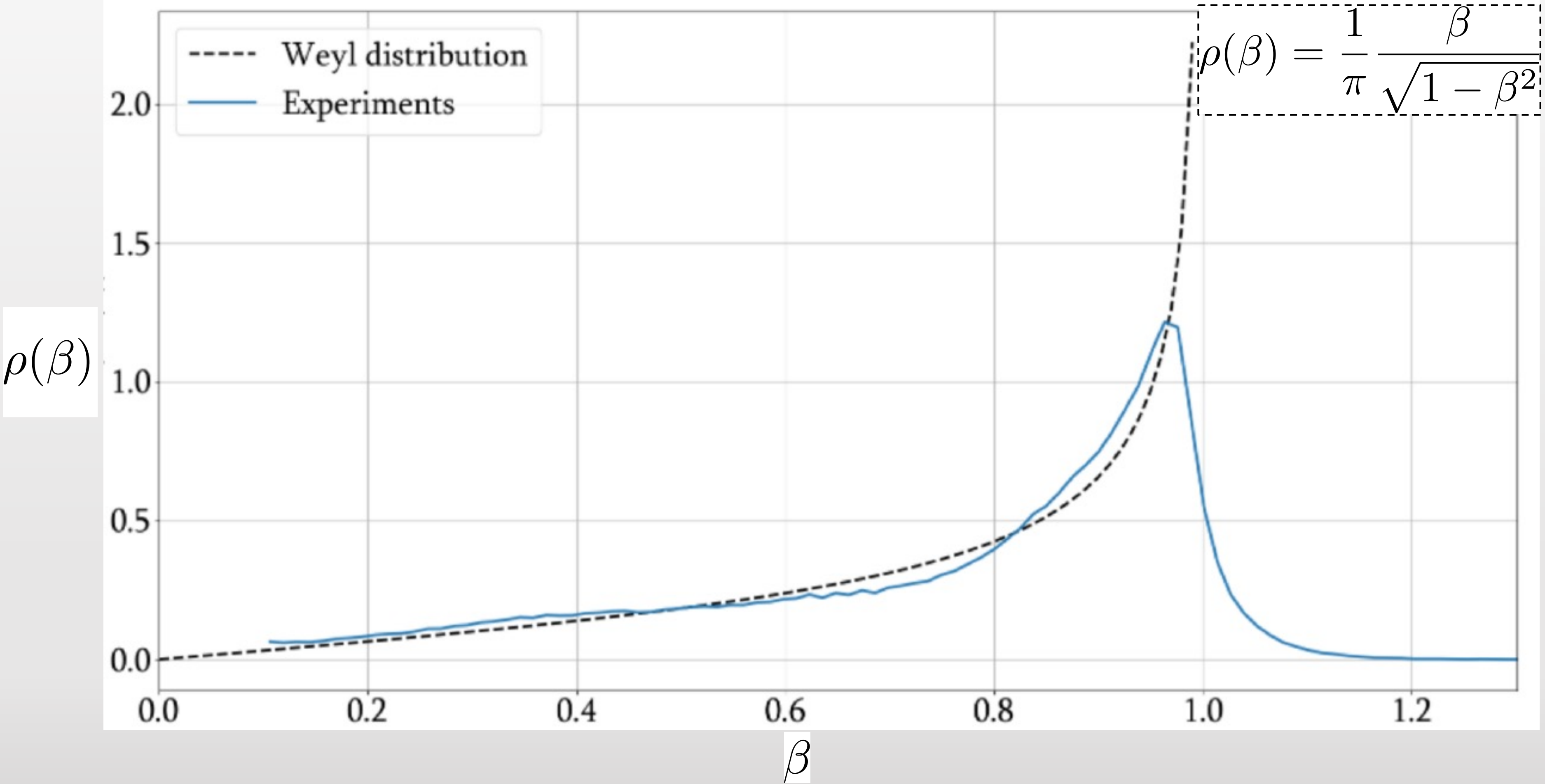
$$\frac{\partial \psi}{\partial z} = -i\frac{\beta_2}{2}\frac{\partial^2 \psi}{\partial t^2} + \frac{\beta_3}{6}\frac{\partial^3 \psi}{\partial t^3} + i\gamma(|\psi|^2 - T_r\frac{\partial |\psi|^2}{\partial t})\psi - \alpha\psi$$



4. Density of state in optical fiber experiments

- *Output (PMF, z=400m)*

PMF, P = 4W
$\beta_2 = -20.6 ps^2/km$
$\chi^{(3)} = 2.4 W^{-1} km^{-1}$
L=400m



$$\rho(\beta) = \frac{1}{\pi} \frac{\beta}{\sqrt{1-\beta^2}}$$

$$\lambda = i\beta$$

5. Soliton Gas : **statistics**

1D focusing NLSE $i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0.$

- Conservation laws $(p_j[\psi])_t + (q_j[\psi])_x = 0 \ (j \geq 1)$

$$p_1 = |\psi|^2, \quad p_3 = |\psi|^4 - |\psi_x|^2, \quad q_2 = |\psi|^4 - 2|\psi_x|^2$$

- Finite gap theory / heuristic derivation

$$\langle p_1 \rangle = 4\,\mathrm{Im}(\overline{\lambda}), \quad \langle p_3 \rangle = -\frac{16}{3}\,\mathrm{Im}(\overline{\lambda^3}),$$

$$\langle q_2 \rangle = 4\,\mathrm{Im}(\overline{\lambda^2 s(\lambda)}),$$

where the spectral average $\overline{h(\lambda)} = \int_{\Gamma^+} h(\lambda)\rho(\lambda)|\mathrm{d}\lambda|$

- Kurtosis

$$\boxed{\kappa = \frac{\langle |\psi|^4 \rangle}{\langle |\psi|^2 \rangle^2} = -\frac{\mathrm{Im}(\frac{2}{3}\overline{\lambda^3} + \frac{1}{4}\overline{\lambda^2 s(\lambda)})}{\mathrm{Im}(\overline{\lambda})^2}}$$

The knowledge of the DOS is required…

T. Congy, G. A. El, G. Roberti, A. Tovbis, S. Randoux, P. Suret, *Statistics of extreme events in integrable turbulence*

arXiv:2307.08884

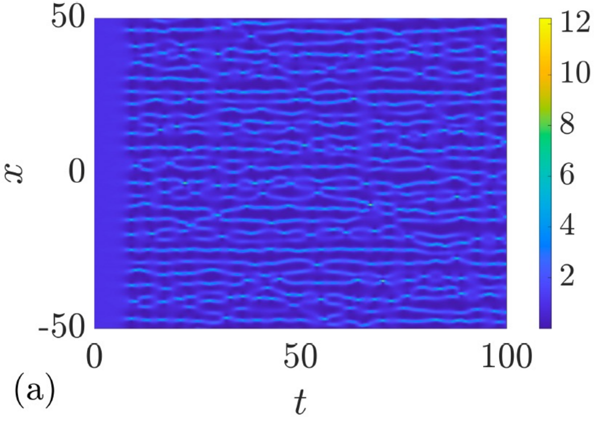
5. Spontaneous MI and PCW modeled by Soliton Gas

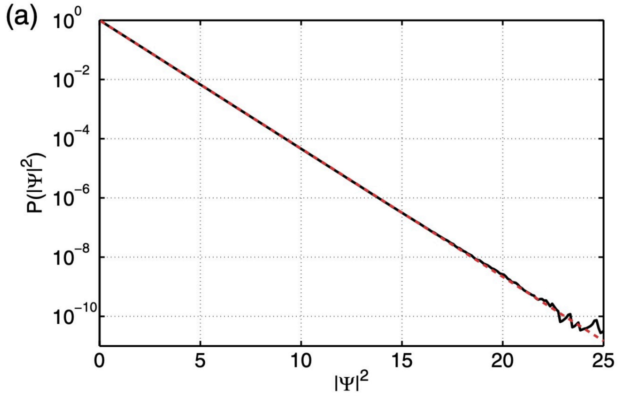
Bound state (no velocity)

$$\kappa = \frac{2}{3}\overline{\eta^3}/\overline{\eta}^2$$

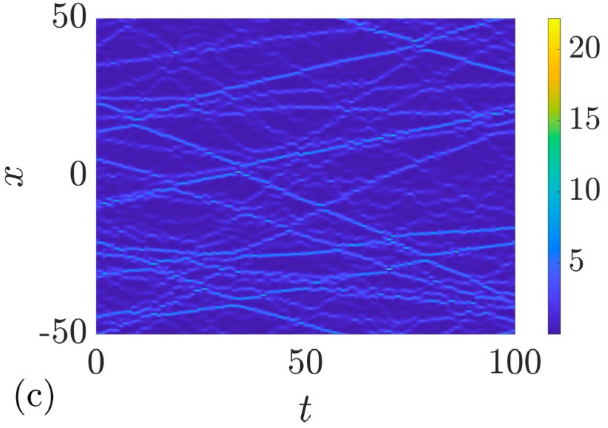
$$\kappa_\infty = 2\kappa_0, \quad \text{where} \quad \kappa_0 = \frac{\langle|\psi(x,0)|^4\rangle}{\langle|\psi(x,0)|^2\rangle^2}$$

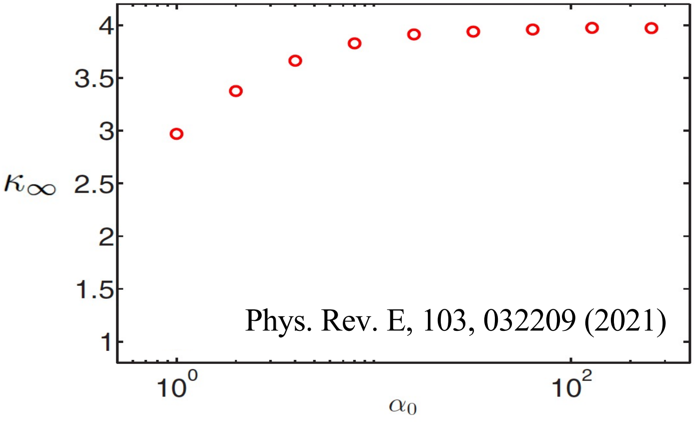
- **MI : κ =2**





- **PCW : κ =4**

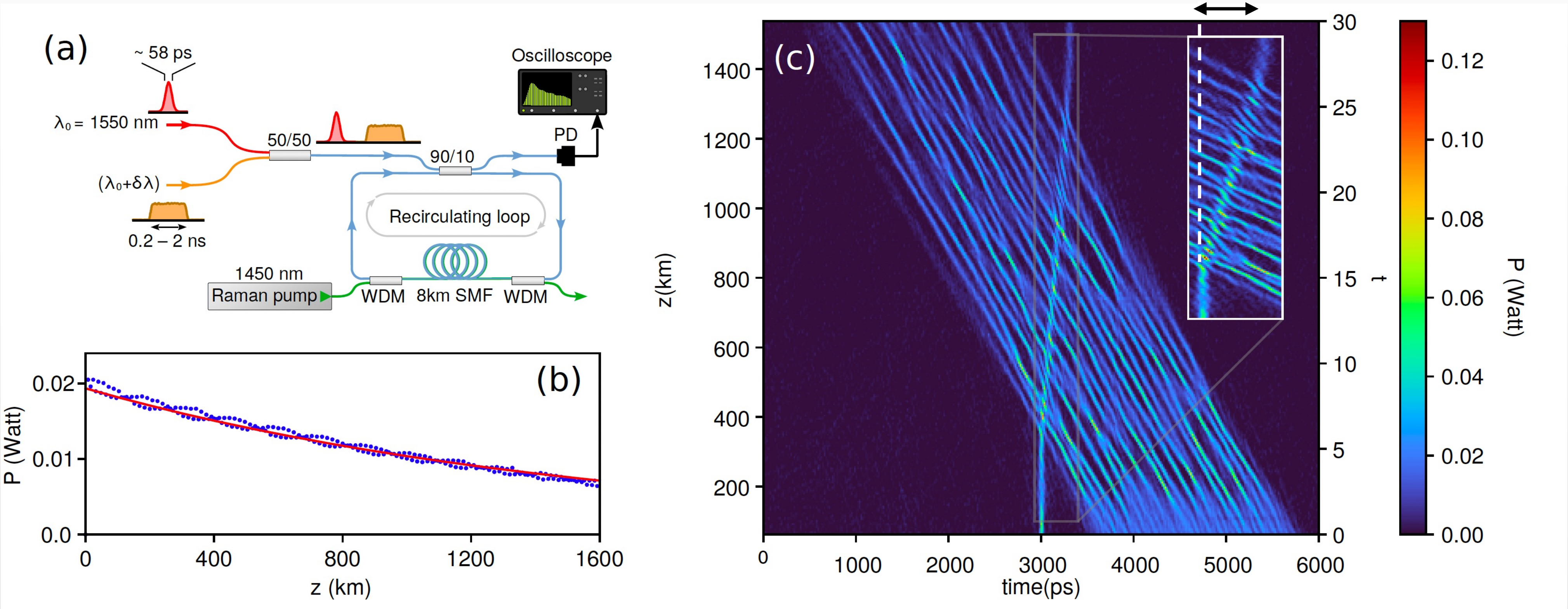




Phys. Rev. E, 103, 032209 (2021)

T. Congy, G. A. El, G. Roberti, A. Tovbis, S. Randoux, P. Suret, *Statistics of extreme events in integrable turbulence,* arXiv:2307.08884 (2023)

6. Refraction of a soliton by a soliton gas

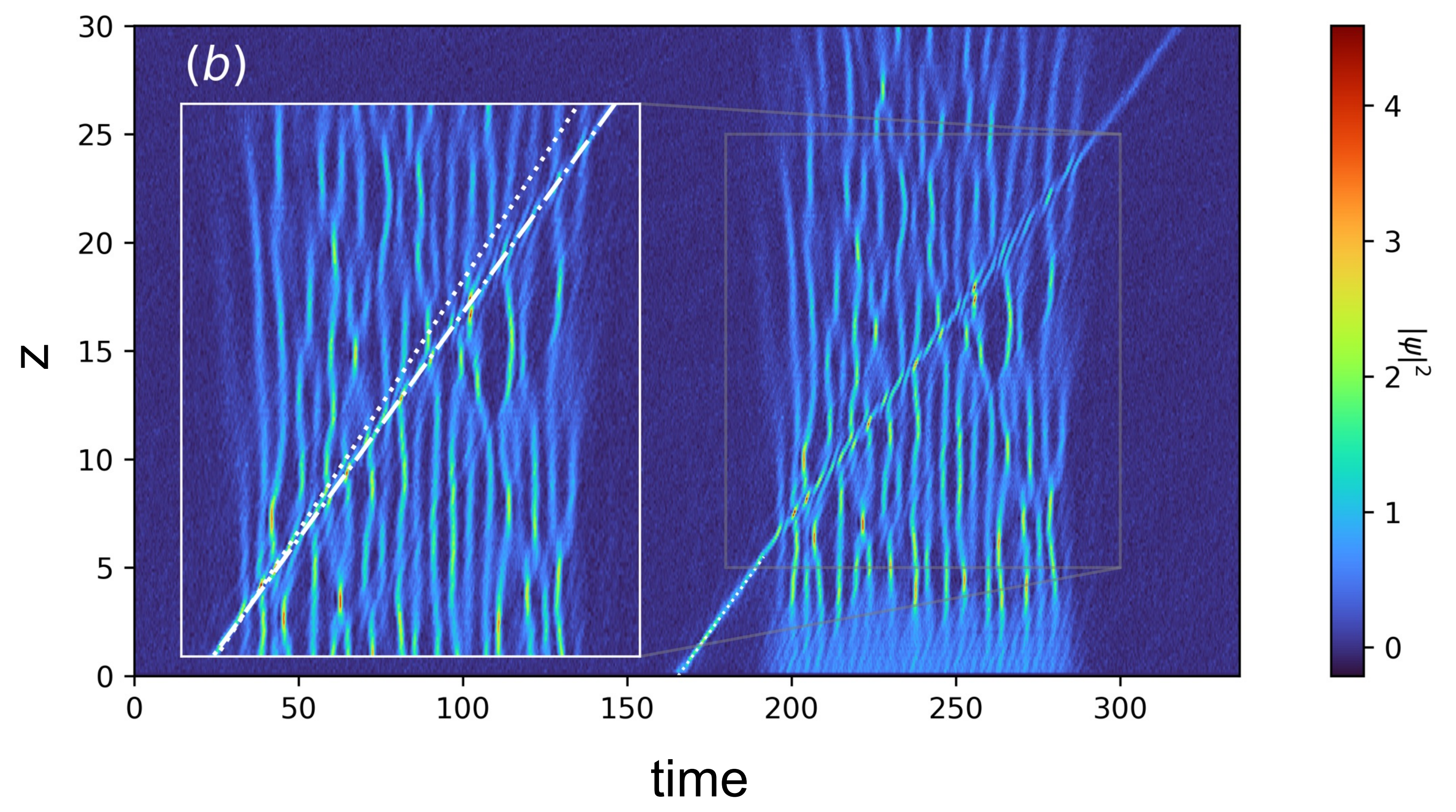


Shift : ~ 280 ps

Pulse duration : ~ 58 ps
Size (duration) of the optical soliton gas : ~ 2 ns

"Soliton refraction through an optical soliton gas", P. Suret et al, arXiv:2303.13421 [nlin.PS] (2023)

6. Quantitative test of the theory of soliton gas (velocity)



$$s(\lambda) = s_0 + \frac{1}{\beta}\int_0^\infty \ln\left|\frac{\lambda-\mu^*}{\lambda-\mu}\right| \rho(\mu)[s(\lambda)-s(\mu)]d\mu$$

$$s_0 = -4\alpha$$

$$\lambda = \alpha + i\beta$$

$$\lambda = i\beta$$

$$\rho(\beta) = \frac{1}{\pi}\frac{\beta}{\sqrt{1-\beta^2}}$$

$$s(\beta) = \frac{s_0\beta}{\Im\sqrt{1-\beta^2+\frac{s_0^2}{16}+i\frac{\beta s_0}{2}}}$$

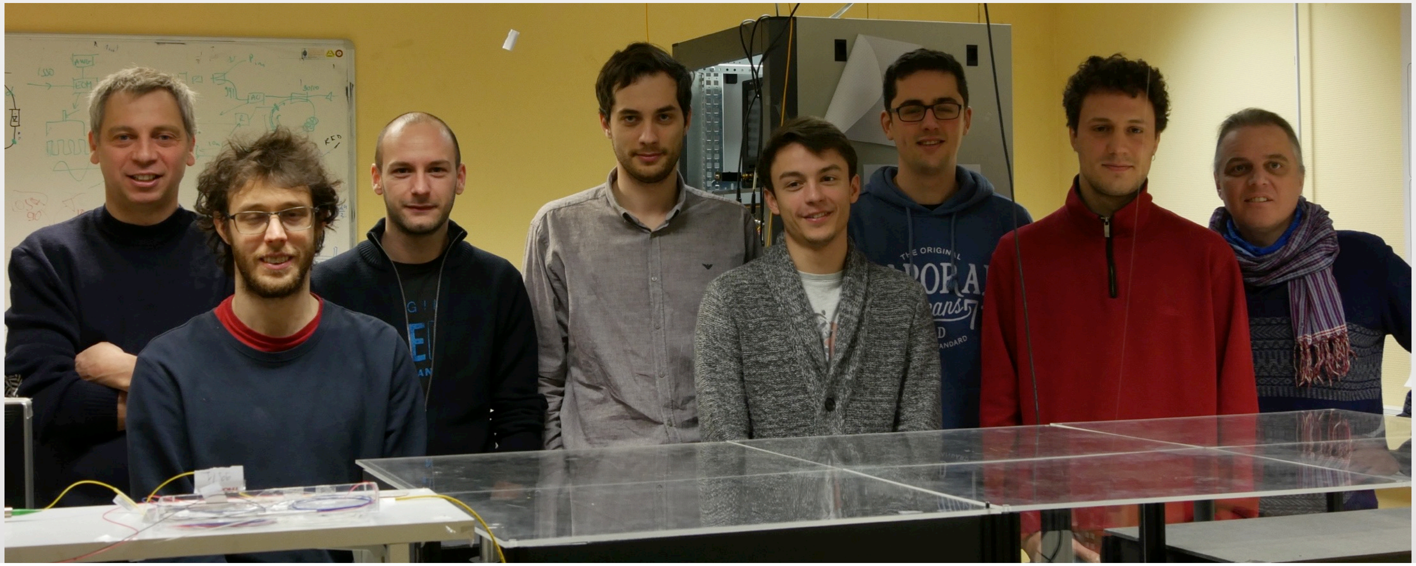
"Soliton refraction through an optical soliton gas", P. Suret et al, arXiv:2303.13421 [nlin.PS] (2023)

Pierre Suret, Stéphane Randoux, François Copie, Loic Fache (PhD)

Former PhD students : Alexey Tikan, Rebecca El Koussaifi, Adrien Kraych, Alexandre Lebel

Christophe Szwaj, Clément Evain, Serge Bielawski
Laboratoire de Physique des Lasers, Atomes et Molécules (Phlam), Univ. de Lille, France

Gennady El, Thibault Congy, Giacomo Roberti, Dmitry Agafontsev, *Northumbria univ., UK*
Andrey Gelash, Univ Bourgogne Franche-Comté, France
Alexander Tovbis , *Univ of central Florida, USA*
Antonio Picozzi, Univ. Bourgogne Franche-Comté, France
Miguel Onorato, *Univ. Torino, Italy*
Eric Falcon, MSC, *Univ. Paris Diderot, France,* **Guillaume Michel,** *Ecole Norm. Sup., France*
Félicien Bonnefoy, Guillaume Ducrozet, *Ecole Centrale de Nantes*



Conclusion and perspectives

- ✓ Observation of a soliton gas designed by using IST

  P. Suret, *et al.*, PRL, **125**, (2020)

- ✓ MI = soliton gas

  Gelash, A. *et al* , *Phys. Rev. Lett.*, **123**, 234102, (2019)

- ✓ Measurement of IST spectrum (MI)
- ✓ Refraction of soliton

  P. Suret, *et al.*, arXiv:2303.13421 (2023)

- ✓ Kurtosis = f (DOS)

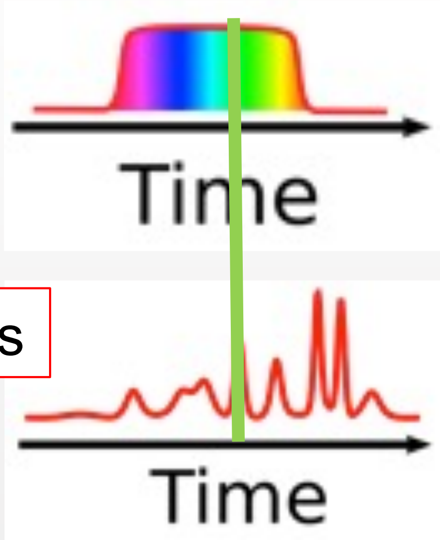
  T. Congy *et al.*, arXiv:2307.08884 (2023)

Open questions

- ✓ Experimental study of inhomogenous soliton gas
- ✓ Finite gap SG: a complete model of integrable turbulence (DWs+solitons) ?
- ✓ Soliton gas with perturbative effects (breaking integrability) ?
- ✓ Generalized Hydrodynamics

SEAHORSE(phase+amplitude)

Chirped pump

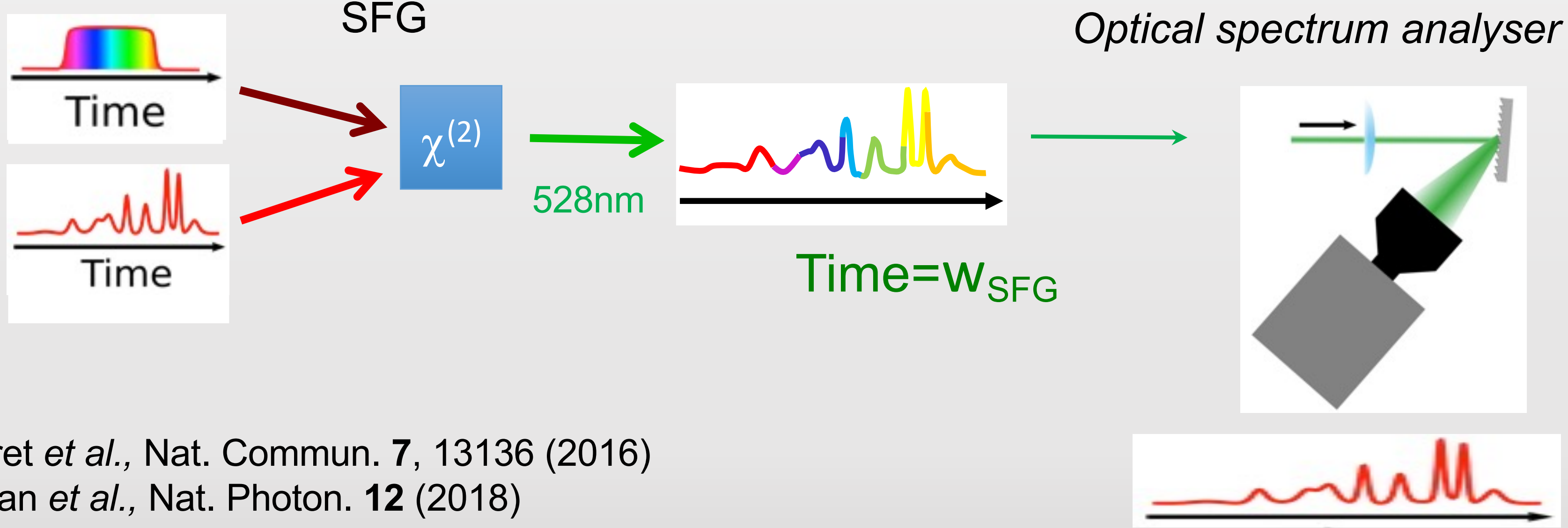


Time scale : ~1ps

Signal

Sum frequency generation

$$w_{pump}+w_{signal}=w_{SFG}$$



800nm

1550nm

SFG

528nm

Time=w_{SFG}

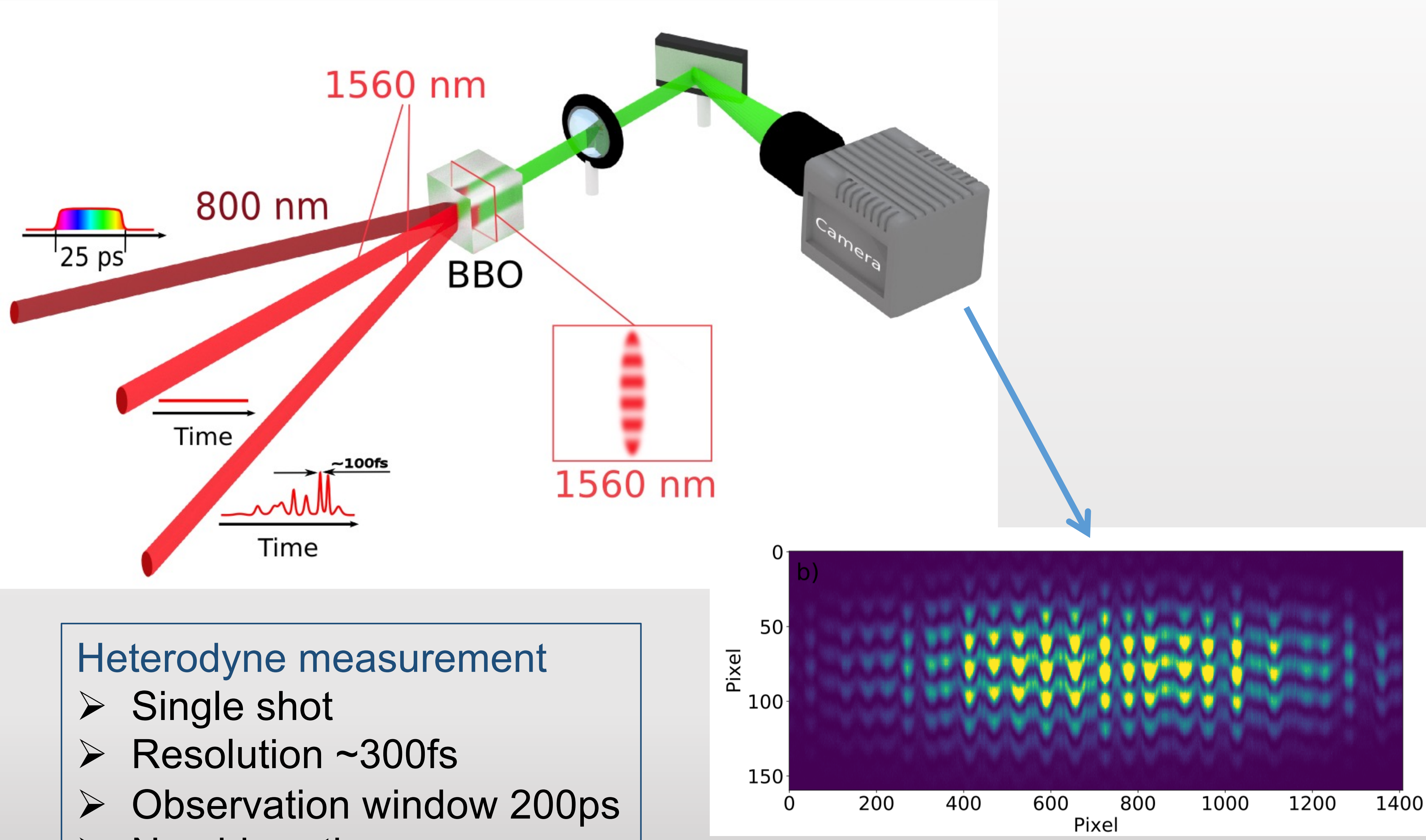
Single-shot
Optical spectrum analyser

P. Suret *et al.,* Nat. Commun. **7**, 13136 (2016)
A. Tikan *et al.,* Nat. Photon. **12** (2018)

Space !

SEAHORSE(phase+amplitude)

A. Tikan *et al.,* Nat. Photon. **12** (2018)



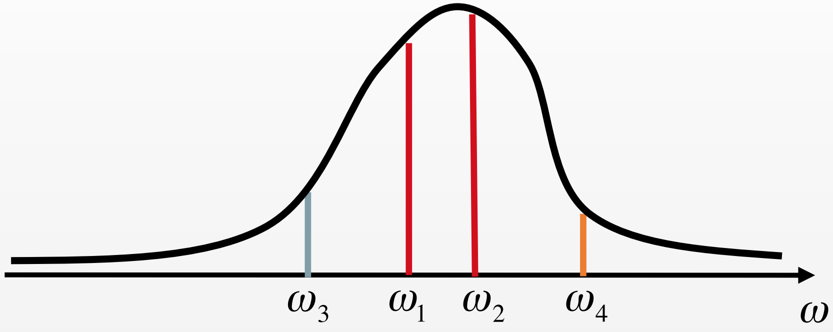
Heterodyne measurement

- ➢ Single shot
- ➢ Resolution ~300fs
- ➢ Observation window 200ps
- ➢ No abberations

1D nonlinear Schrödinger equation

$\chi^{(3)}$ / 4 waves interaction

$$i\,\partial_z\ A(z,t) = -\frac{1}{2}\beta_2\,\partial_t^2\ A(z,t) + \chi^{(3)}\left|A(z,t)\right|^2)\ A(z,t)$$



$$\omega_1 + \omega_2 = \omega_3 + \omega_4$$

$$k(\omega_1) + k(\omega_2) = k(\omega_3) + k(\omega_4) \Rightarrow \omega_1 = \omega_3$$

$$k(\omega) = \frac{1}{2}\beta_2\omega^2$$

Trivial interaction !

NLS1D : integrable equation

infinity of motion constants
quasi-periodic behavior

usual Wave Turbulence theory

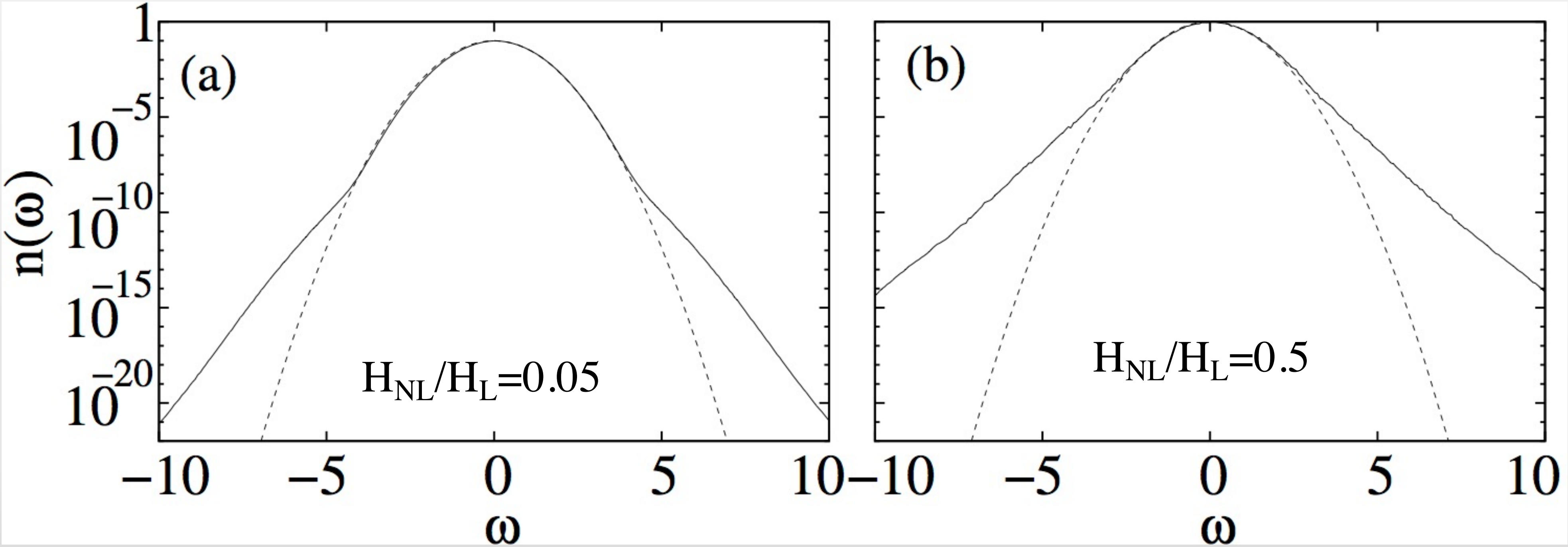
$$\boxed{\partial_z\ n(\omega,z) = 0 \quad !!!}$$

Zakharov

Numerical simulations

$$i\,\partial_z\psi(z,t) = -\sigma\,\partial_t^2\psi(z,t) + |\psi(z,t)|^2\,\psi(z,t)$$

$$\boxed{n_\omega^0 = n_\omega(z=0) = n_0\,\exp\left[-\left(\frac{\omega}{\Delta\omega}\right)^2\right]}$$



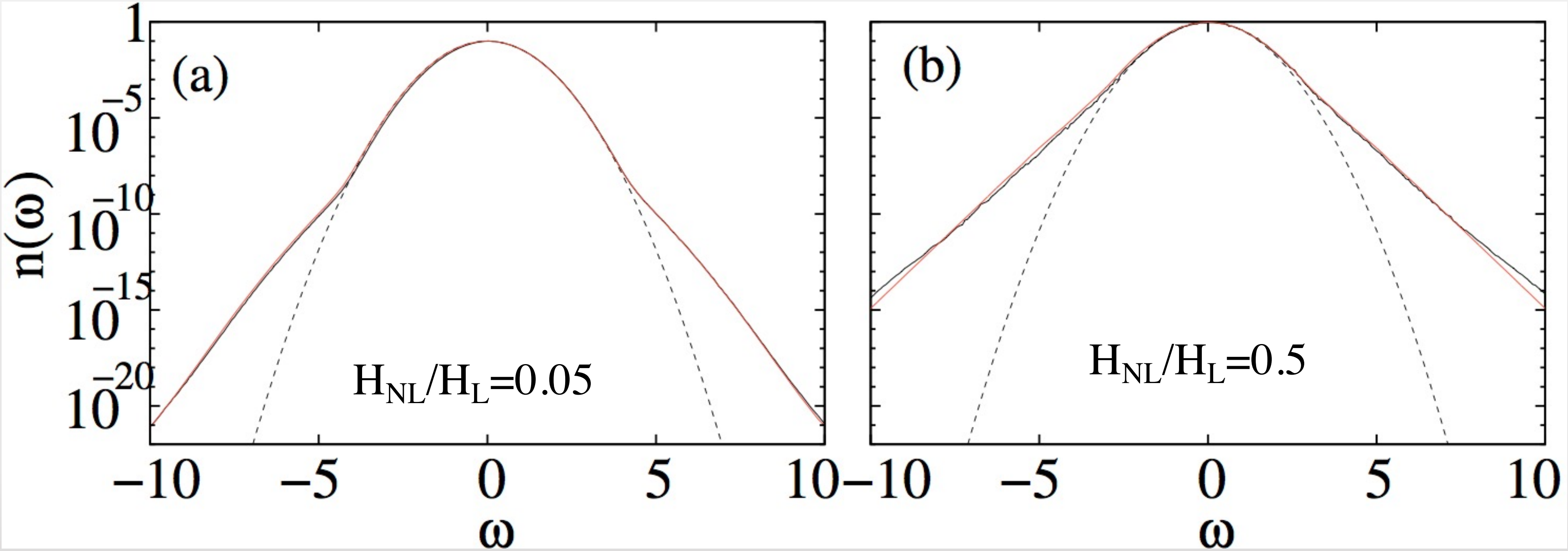
Numerical simulations

$$\frac{\partial n_{\omega_1}(z)}{\partial z} = \frac{1}{\pi} \int\int\int d\omega_{2-4}\, Im\left[J_{1,2}^{3,4}(z)\right] \delta(\omega_1+\omega_2-\omega_3-\omega_4),$$

$$\frac{\partial J_{1,2}^{3,4}(z)}{\partial z} - i\,\Delta k\; J_{1,2}^{3,4}(z) = \frac{i}{\pi}\mathcal{N}(z)$$

$$\boxed{n_\omega^0 = n_\omega(z=0) = n_0\,\exp\left[-\left(\frac{\omega}{\Delta\omega}\right)^2\right]}$$

$$\mathcal{N}(z) = n_{\omega_1}(z)n_{\omega_2}(z)n_{\omega_3}(z)n_{\omega_4}(z)\left[\frac{1}{n_{\omega_2}(z)} + \frac{1}{n_{\omega_1}(z)} - \frac{1}{n_{\omega_4}(z)} - \frac{1}{n_{\omega_3}(z)}\right]$$



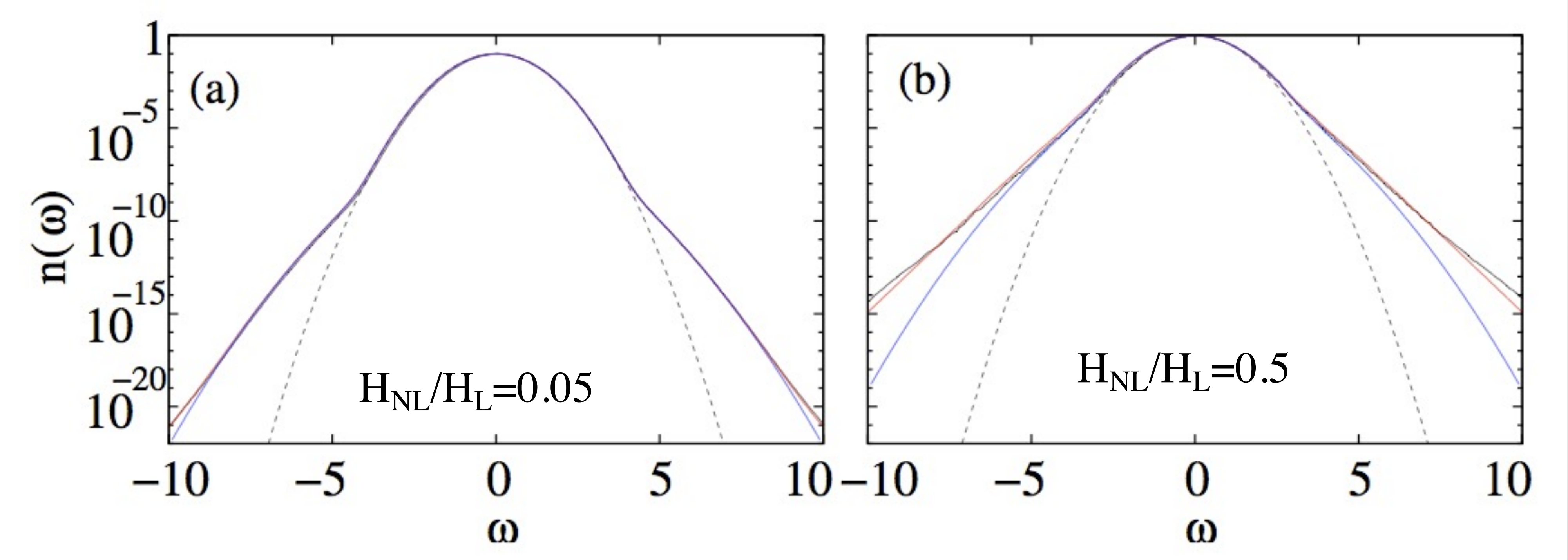
Numerical simulations

$$\frac{\partial n_{\omega_1}(z)}{\partial z} = \frac{1}{\pi}\int\int\int d\omega_{2-4}\, Im\left[J^{3,4}_{1,2}(z)\right]\delta(\omega_1+\omega_2-\omega_3-\omega_4),$$

$$\frac{\partial J^{3,4}_{1,2}(z)}{\partial z} - i\,\Delta k\; J^{3,4}_{1,2}(z) = \frac{i}{\pi}\mathcal{N}(z)$$

$$\boxed{n^0_\omega = n_\omega(z=0) = n_0\,\exp\left[-\left(\frac{\omega}{\Delta\omega}\right)^2\right]}$$

$$\mathcal{N}(z) = n_{\omega_1}(z)n_{\omega_2}(z)n_{\omega_3}(z)n_{\omega_4}(z)\left[\frac{1}{n_{\omega_2}(z)}+\frac{1}{n_{\omega_1}(z)}-\frac{1}{n_{\omega_4}(z)}-\frac{1}{n_{\omega_3}(z)}\right]$$



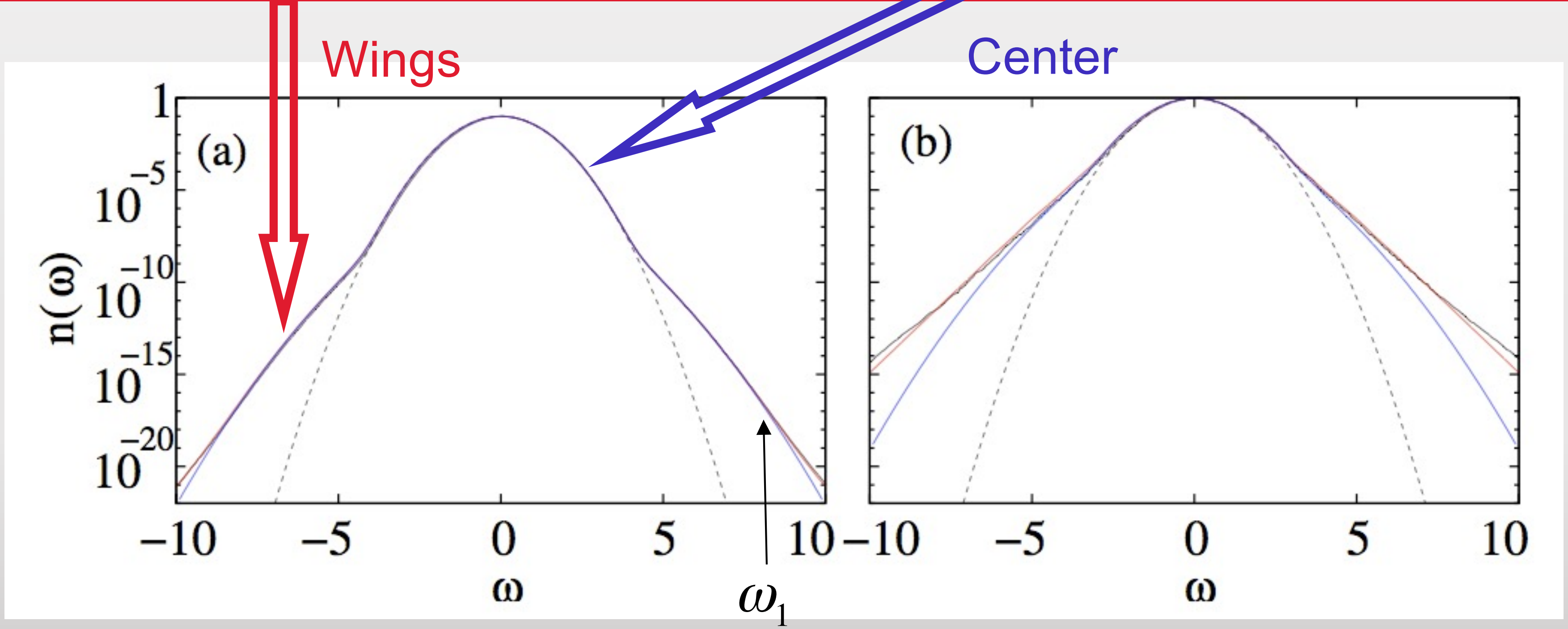
Good approximation N(*z*) = N(*z=0*) !!

Where initial conditions play a crucial role

$$\frac{\partial n_{\omega_1}(z)}{\partial z} = \frac{1}{\pi^2}\int_0^z dz' \int\int\int d\omega_{2-4}\,\mathcal{N}(z')\,\cos(\Delta k(z'-z))\;\delta(\omega_1+\omega_2-\omega_3-\omega_4)$$

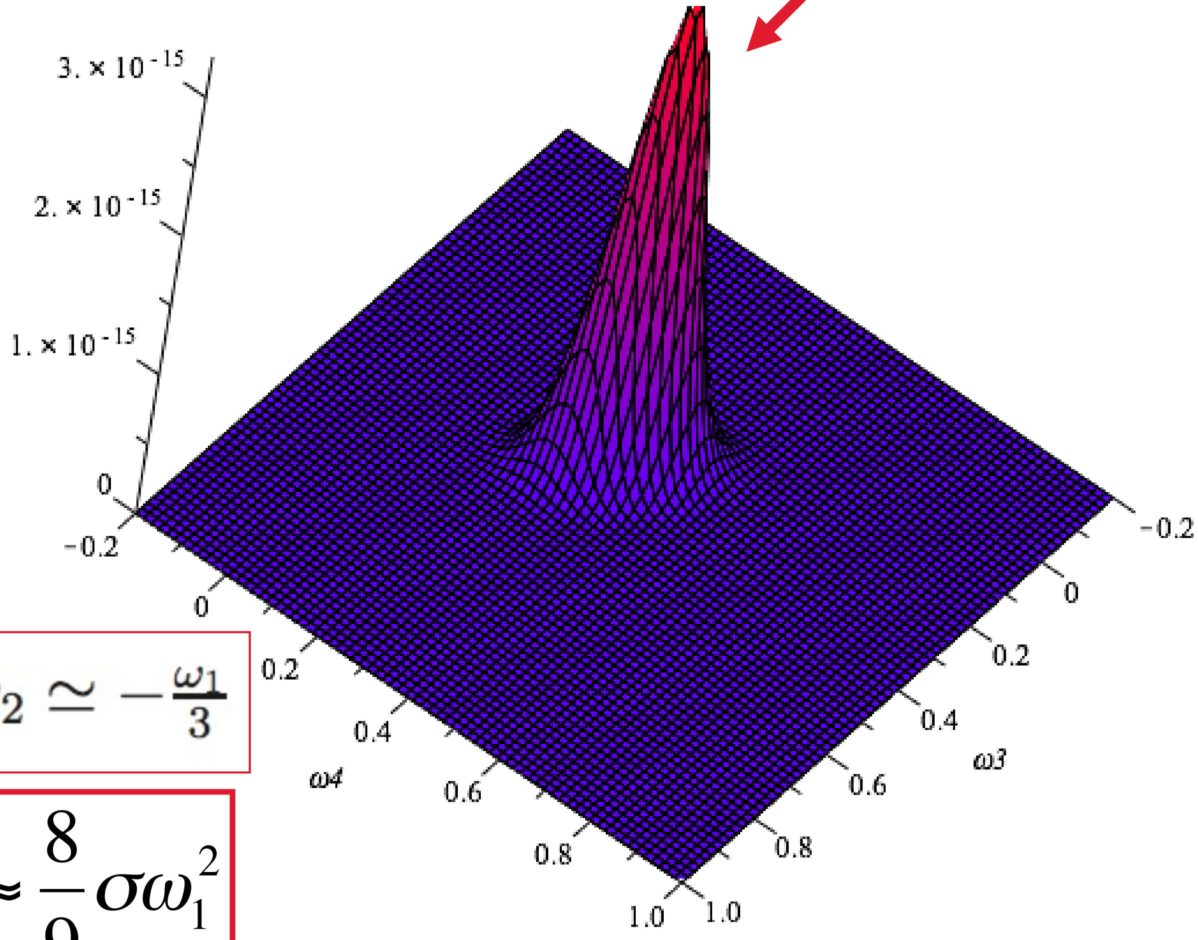
N(z') = **N**$(z'=0)$ / integration

$$\frac{\partial n_{\omega_1}(z)}{\partial z} = \frac{1}{\pi^2}\int\int d\omega_{3-4}\, n^0_{\omega_3} n^0_{\omega_4} n^0_{\omega_3+\omega_4-\omega_1}\frac{\sin(\Delta k\, z)}{\Delta k} - \frac{n^0_{\omega_1}}{\pi^2}\int\int d\omega_{3-4}\, n^0_{\omega_3} n^0_{\omega_4}\frac{\sin(\Delta k\, z)}{\Delta k}$$



A last approximation…

$$\frac{\partial n_{\omega_1}(z)}{\partial z} \simeq \frac{1}{\pi^2}\int\int d\omega_{3-4}\, n^0_{\omega_3}\, n^0_{\omega_4}\, n^0_{\omega_3+\omega_4-\omega_1}\, \frac{\sin(\Delta k\, z)}{\Delta k}$$



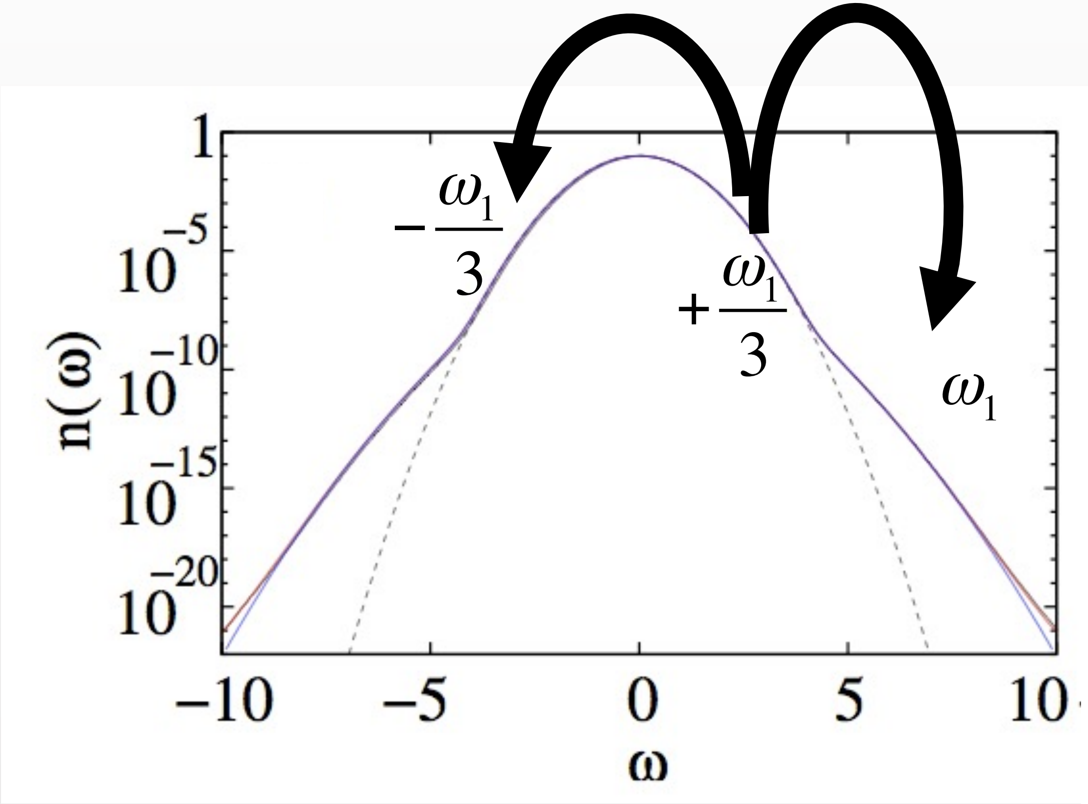
Dominant

contributions

$$\boxed{\omega_3 \simeq \omega_4 \simeq \frac{\omega_1}{3},\ \omega_2 \simeq -\frac{\omega_1}{3}}$$

$$\boxed{\Rightarrow \quad \overline{\Delta k} \approx \frac{8}{9}\sigma\omega_1^2}$$

Dominant contribution



$$\Delta k = \left[\left(\frac{\omega_1}{3}\right)^2 + \omega_1^2\right] - \left[\left(\frac{\omega_1}{3}\right)^2 + \left(\frac{\omega_1}{3}\right)^2\right]$$

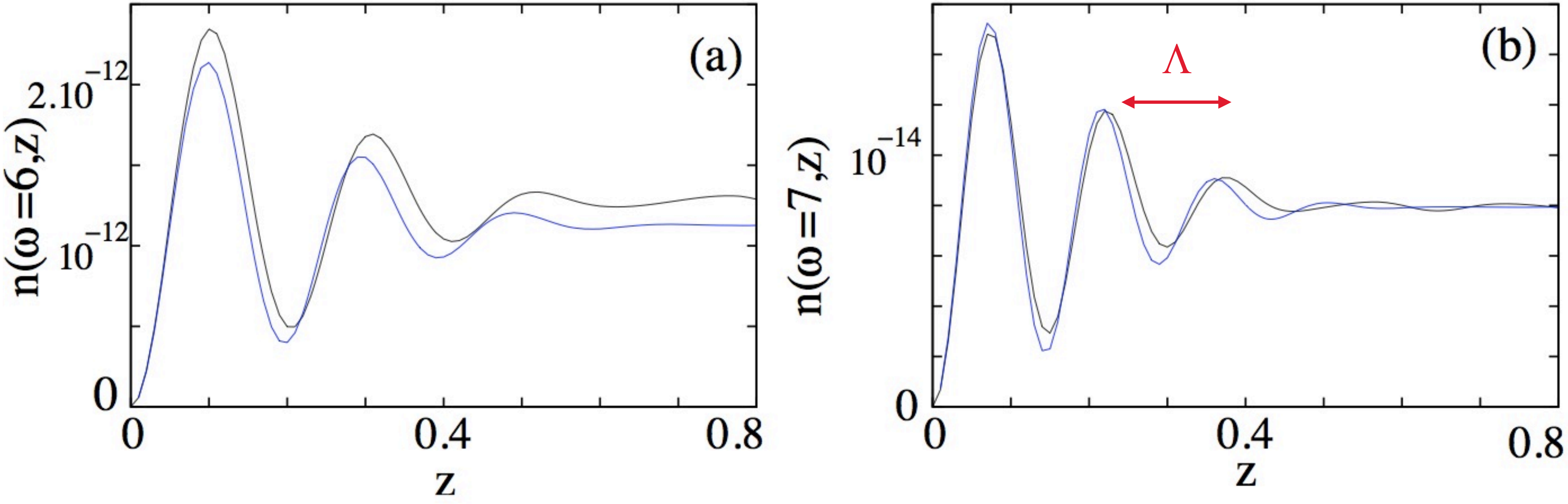
$$\boxed{\overline{\Delta k} = \frac{8}{9}\omega_1^2}$$

$$\boxed{\frac{\partial n_{\omega_1}(z)}{\partial z} \simeq \frac{n_0^3}{\sqrt{3}\pi}\frac{9}{8\omega_1^2}\exp\left(\frac{-\omega_1^2}{3}\left(1+\frac{8\,z^2}{9}\right)\right)\sin\left(\frac{8\,\omega_1^2\,z}{9}\right)}$$

Damped oscillations toward steady state

$$\frac{\partial n_{\omega_1}(z)}{\partial z} \simeq \frac{n_0^3}{\sqrt{3}\pi}\frac{9}{8\omega_1^2}\exp\left(\frac{-\omega_1^2}{3}\left(1+\frac{8\,z^2}{9}\right)\right)\sin\left(\frac{8\,\omega_1^2\,z}{9}\right)$$

comparison with numerical integration 1D NLS



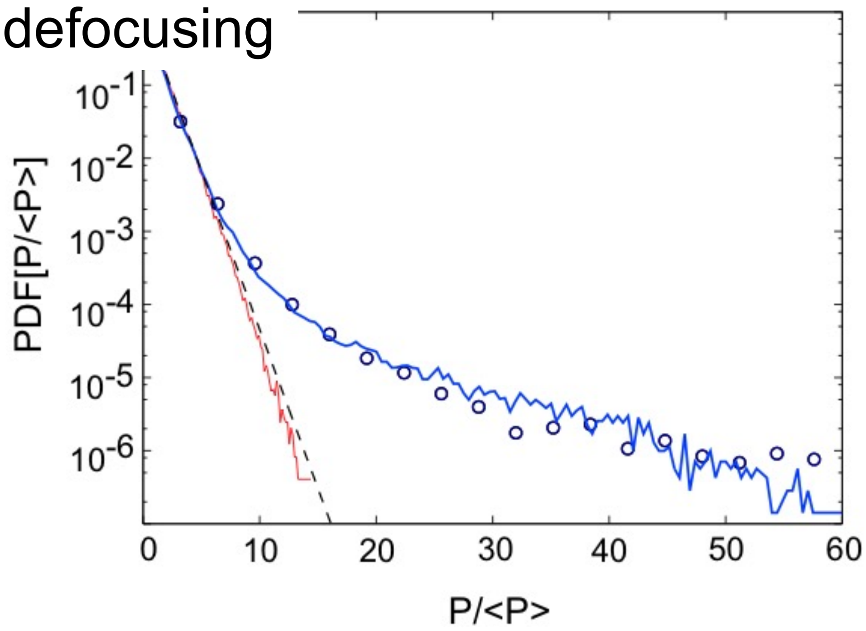
The period of oscillations

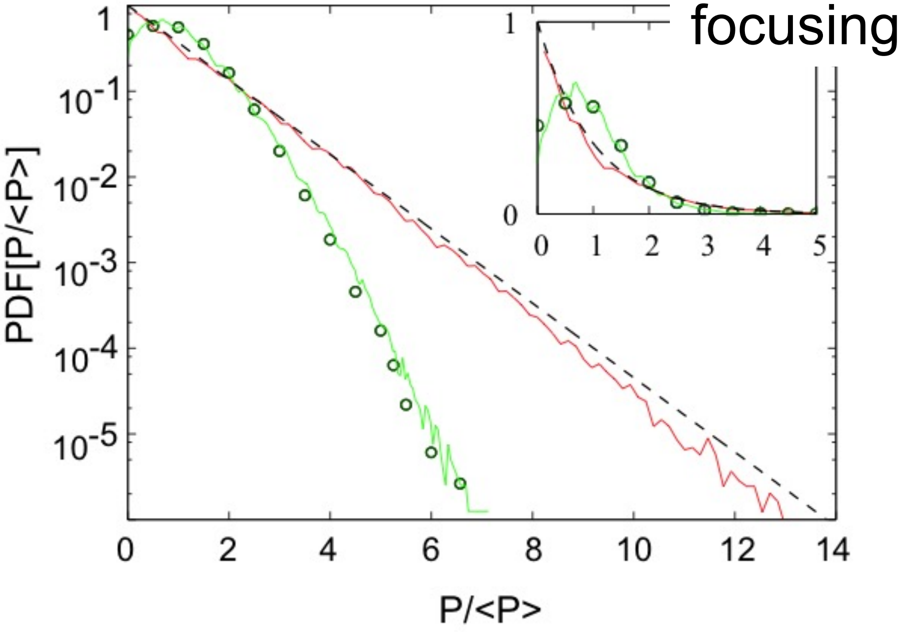
is given by the dominant contribution

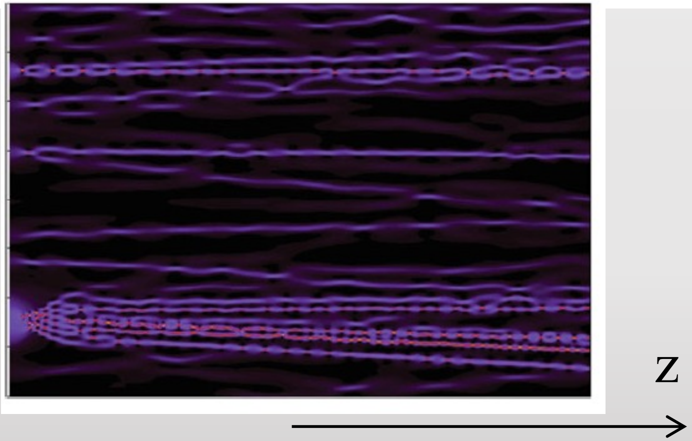
$$\Lambda = \frac{2\pi}{\Delta k} \approx \frac{2\pi}{\frac{8}{9}\omega_1^2}$$

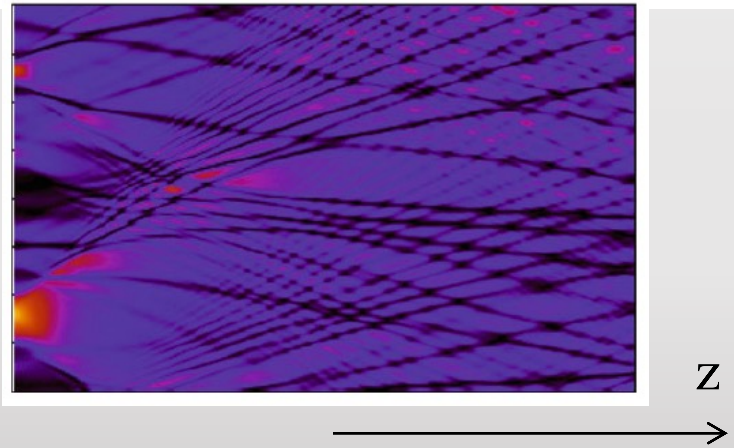
Example of integrable turbulence

- ✓ **Existence of a stationary state**
- ✓ Focusing / defocusing NLS : heavy/ low tail (PDF)





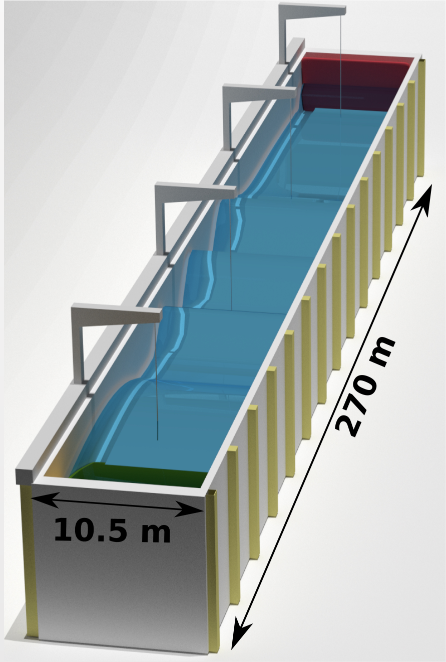


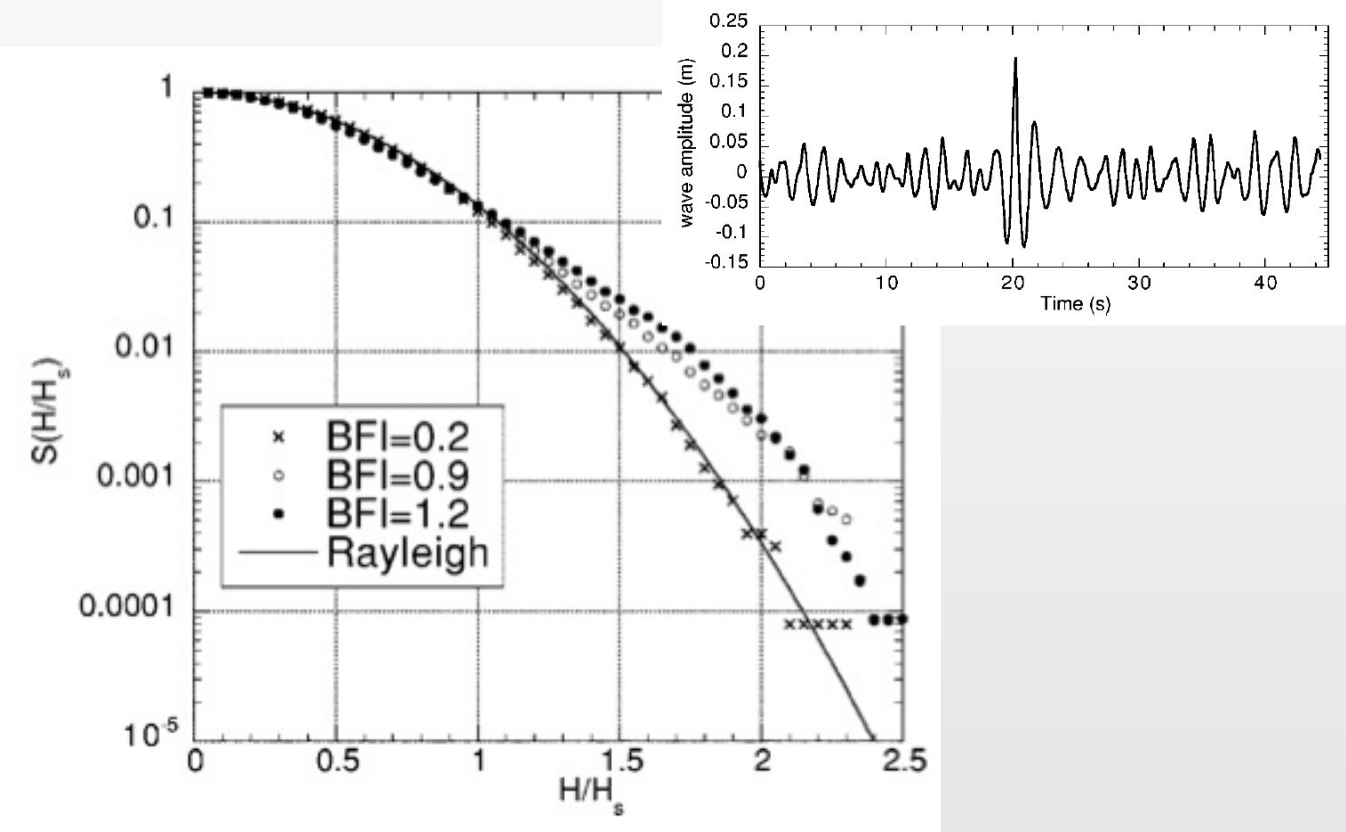


Randoux and Suret (2016), in “Rogue and shock waves in nonlinear dispersive media”, Springer

1D deep water waves Experiments (freak/rogue waves)

- ✓ **From Gaussian to heavy-tailed statistics (surface elevation)**
- ✓ **Initial conditions = JONSWAP spectrum + Random phases**





Benjamin-Feir index (BFI) = wave steepness / spectral bandwidth $BFI \sim \frac{\epsilon}{\Delta f/f_0} \sim \frac{k_0 H_s}{\Delta f/f_0}$

M. Onorato *et al.*, PRE, **70**, 067302 (2004)

Scaling : focusing 1D-NLSE

Optical fibers

$$i\frac{\partial\psi}{\partial z} = \frac{1}{2}\beta_2\frac{\partial^2\psi}{\partial t^2} - \chi^{(3)}|\psi|^2\psi$$

R El Koussaifi *et al.*, Phys. Rev. E, **97** (1), 012208 (2018)

Deep water waves

$$i\frac{\partial\psi}{\partial z} = \frac{1}{g}\frac{\partial^2\psi}{\partial t^2} + k_0^3|\psi|^2\psi$$

Surface elevation $\eta = \Re(\psi\, e^{i(k_0 z - \omega_0 t)})$

Optical power / Significant wave height $P = \langle|\psi|^2\rangle = 2\langle|\eta|^2\rangle = 2\sigma^2 = H_s^2/8$

One to one correspondence

$$\frac{z}{z_{nlin}} = z_{fiber} \times \chi^{(3)}P = z_{watertank} \times k_0^3 H_s^2/8$$

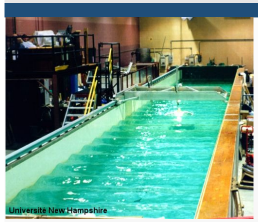
Strength of nonlinearity $$\frac{z_{lin}}{z_{nlin}} = \frac{2\chi^{(3)}P}{\beta_2\Delta\omega_{opt}^2} = \frac{gk_0^3H_s^2}{8\Delta\omega_{ww}^2}$$

Frequency scales

$$\Delta f_{optics} = \sqrt{\frac{2\chi^{(3)}P}{\beta_2}} \sim THz$$

$$\Delta f_{waterW} = \sqrt{gk_0^3H_s^2/8} \sim Hz$$

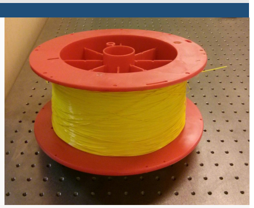
Water tank vs Optical fibers experiments

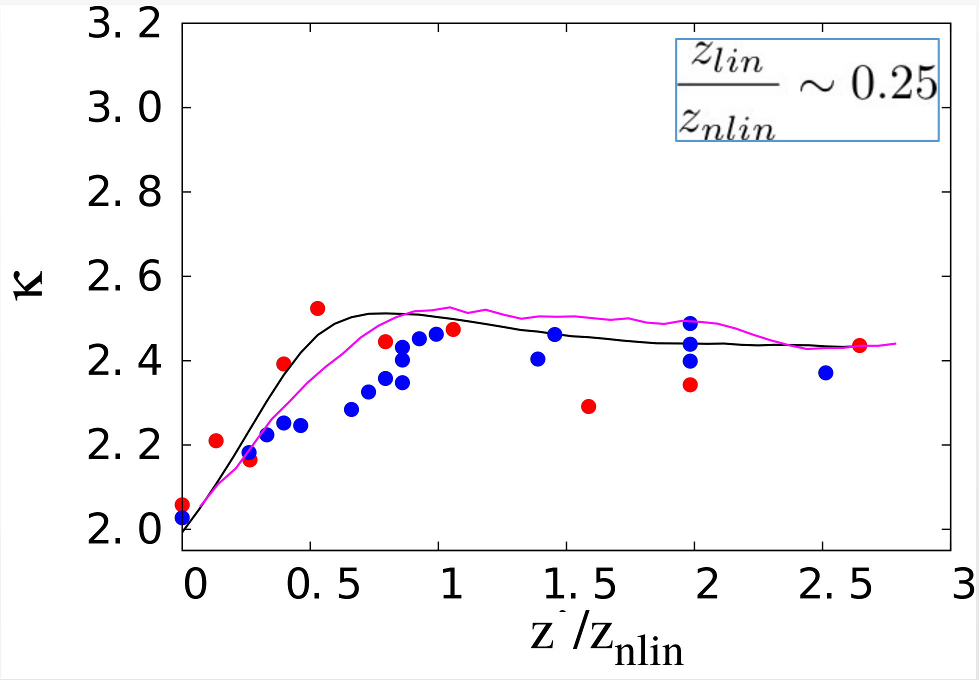


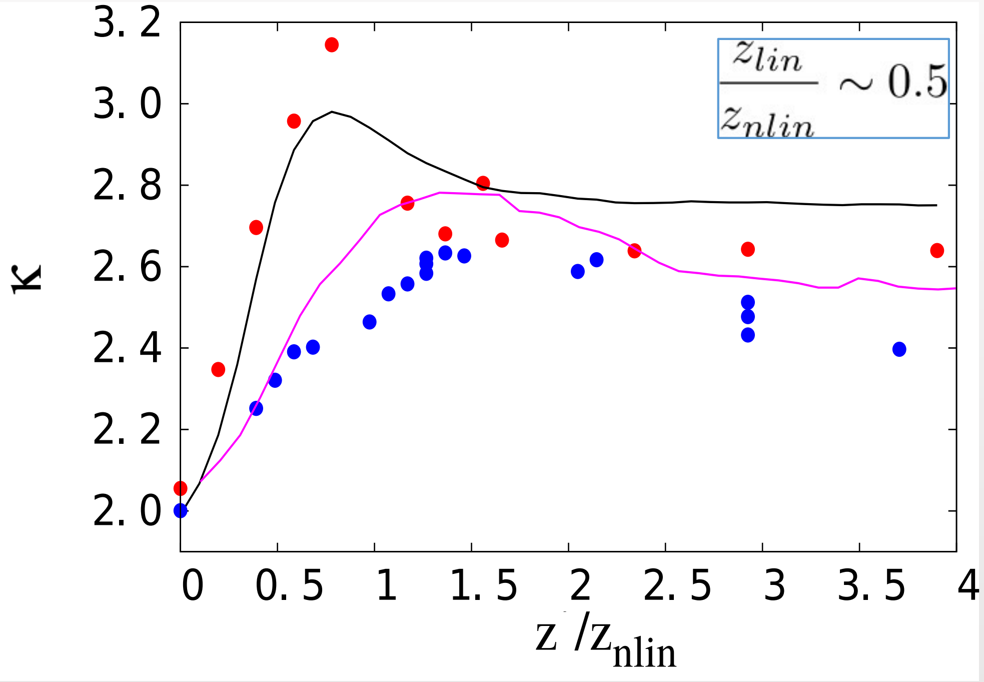
Fourth order moment

Gaussian statistics : k=2

$$\kappa = \frac{\langle|\psi|^4\rangle}{\langle|\psi|^2\rangle^2}$$







Experiments

Water waves ●

Optical fibers ●

Numerical simulations

1DNLSE ———

Euler eqs. (higher order spectral method, A. Toffoli) ———

PhD of Rebecca El Koussaifi

R El Koussaifi *et al.*, Phys. Rev. E, **97** (1), 012208 (2018)

Hamiltonian structure

Relation between spectrum and statistics

M. Onorato, *et al. On the origin of heavy-tail statistics in equations of the Nonlinear Schrödinger type* Phys. Lett. A, **380** , 39, (2016)

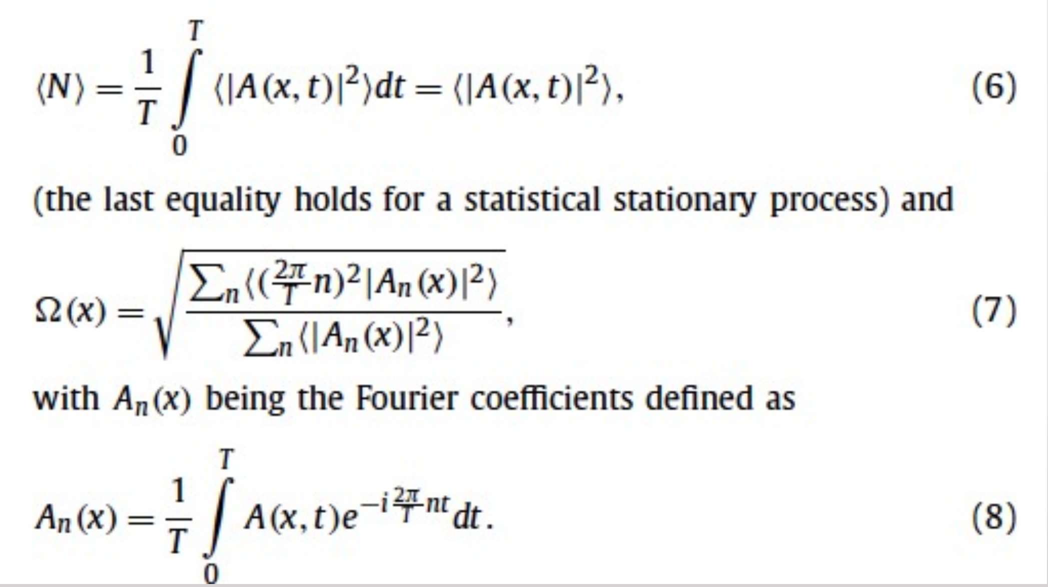
$$i\frac{\partial A}{\partial x} = \beta\frac{\partial^2 A}{\partial t^2} + \alpha|A|^2 A$$

$$H = \frac{1}{T}\int_0^T \beta\left|\frac{\partial A}{\partial t}\right|^2 dt - \frac{1}{T}\int_0^T \frac{\alpha}{2}|A|^4 dt$$

$$\boxed{\kappa(x) = \kappa(x_0) + 2\frac{\beta}{\alpha}\frac{1}{\langle N\rangle}\left[\Omega(x)^2 - \Omega(x_0)^2\right]}$$

kurtosis

$$\kappa = \frac{\langle|A|^4\rangle}{\langle|A|^2\rangle^2} = \frac{\int |A|^4 P(|A|)d|A|}{(\int |A|^2 P(|A|)d|A|)^2}$$



$$\langle N\rangle = \frac{1}{T}\int_0^T \langle|A(x,t)|^2\rangle dt = \langle|A(x,t)|^2\rangle, \qquad (6)$$

(the last equality holds for a statistical stationary process) and

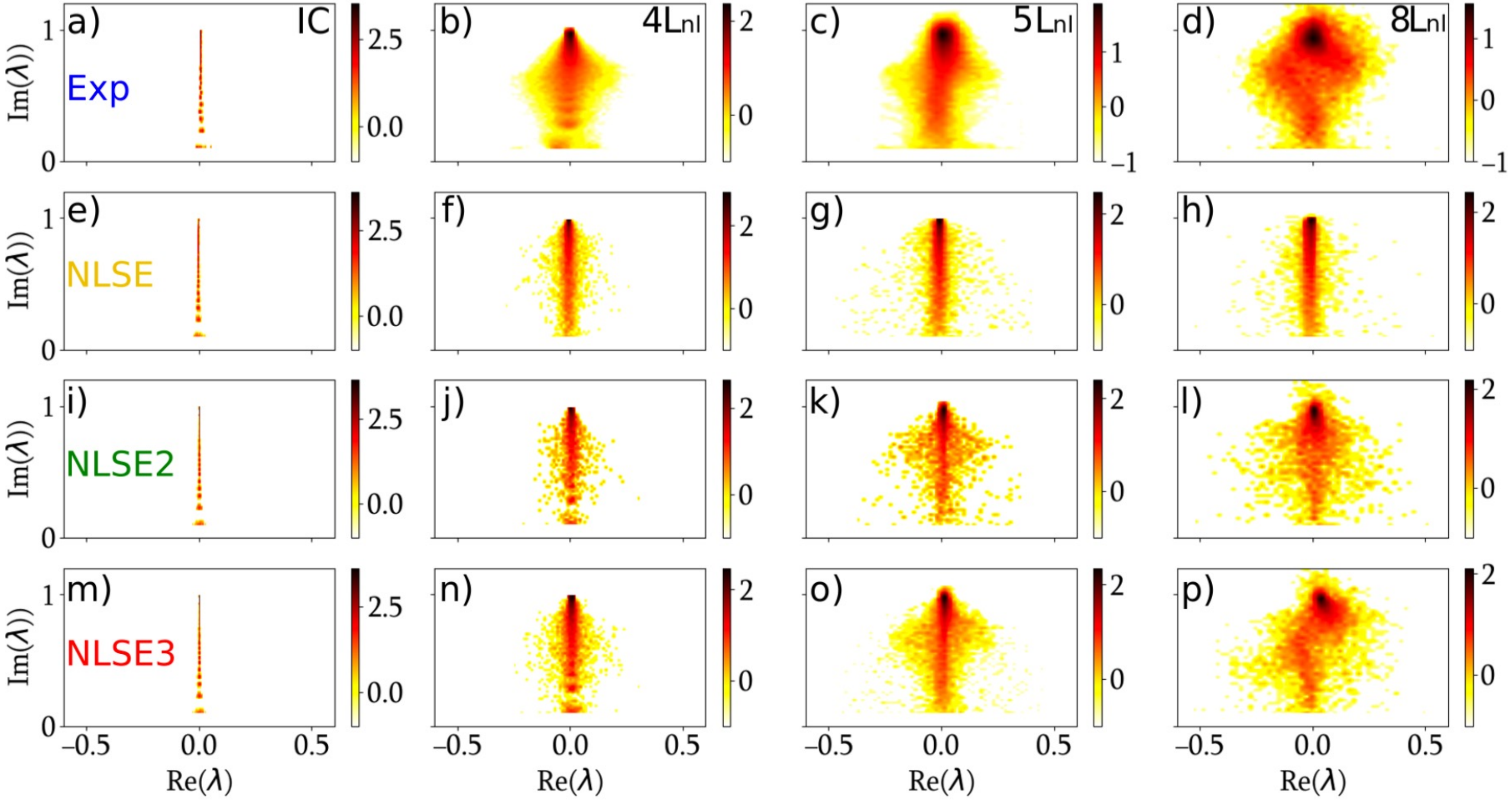
$$\Omega(x) = \sqrt{\frac{\sum_n\langle(\frac{2\pi}{T}n)^2|A_n(x)|^2\rangle}{\sum_n\langle|A_n(x)|^2\rangle}}, \qquad (7)$$

with $A_n(x)$ being the Fourier coefficients defined as

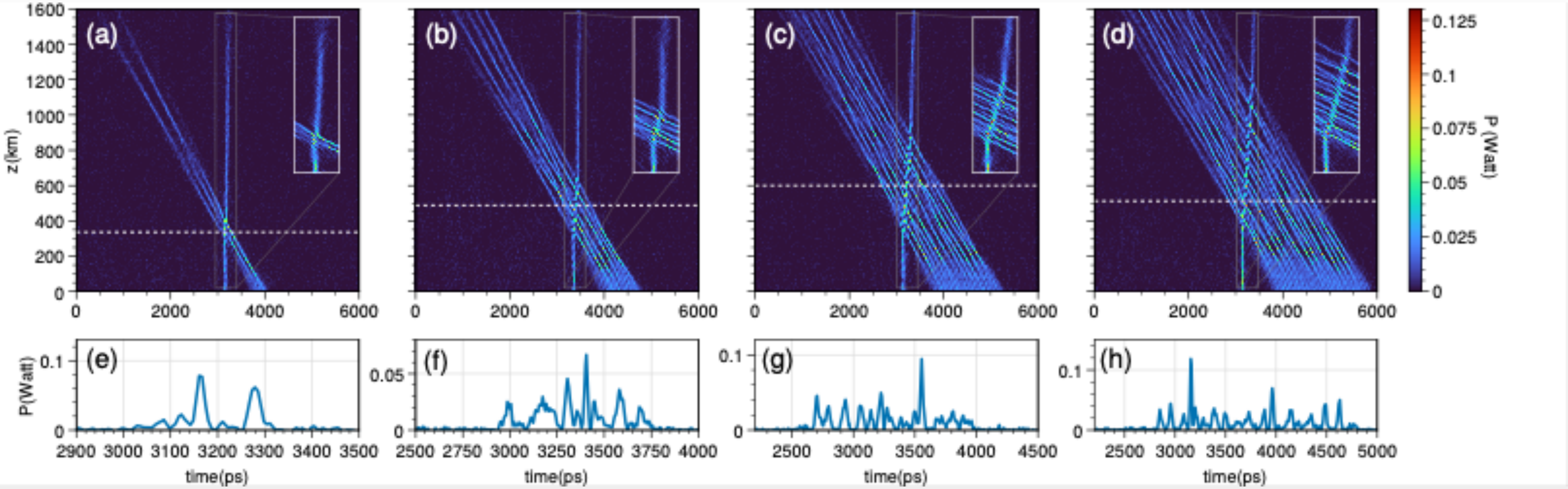
$$A_n(x) = \frac{1}{T}\int_0^T A(x,t)e^{-i\frac{2\pi}{T}nt}dt. \qquad (8)$$

Density of state : Experiments + Numerical simulations

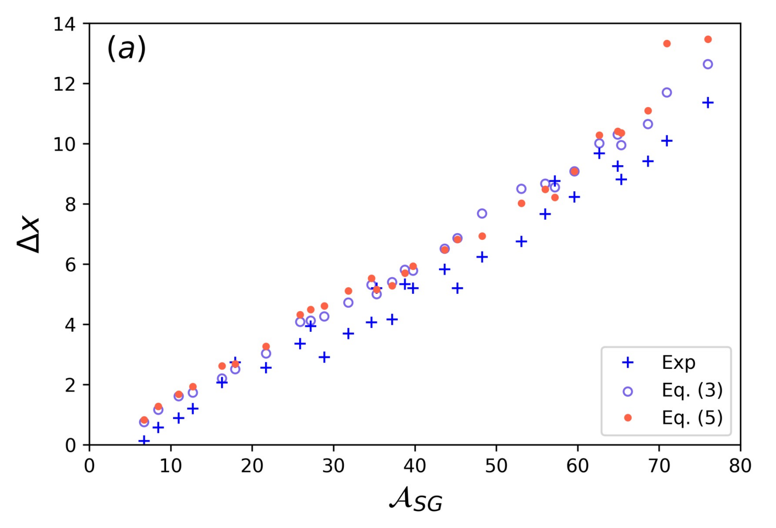
$$\frac{\partial\psi}{\partial z} = -i\frac{\beta_2}{2}\frac{\partial^2\psi}{\partial t^2} + i\gamma|\psi|^2\psi - \alpha\psi + \frac{\beta_3}{6}\frac{\partial^3\psi}{\partial t^3} - i\gamma T_r\frac{\partial|\psi|^2}{\partial t}\psi$$



3. Refraction of a soliton by a soliton gas

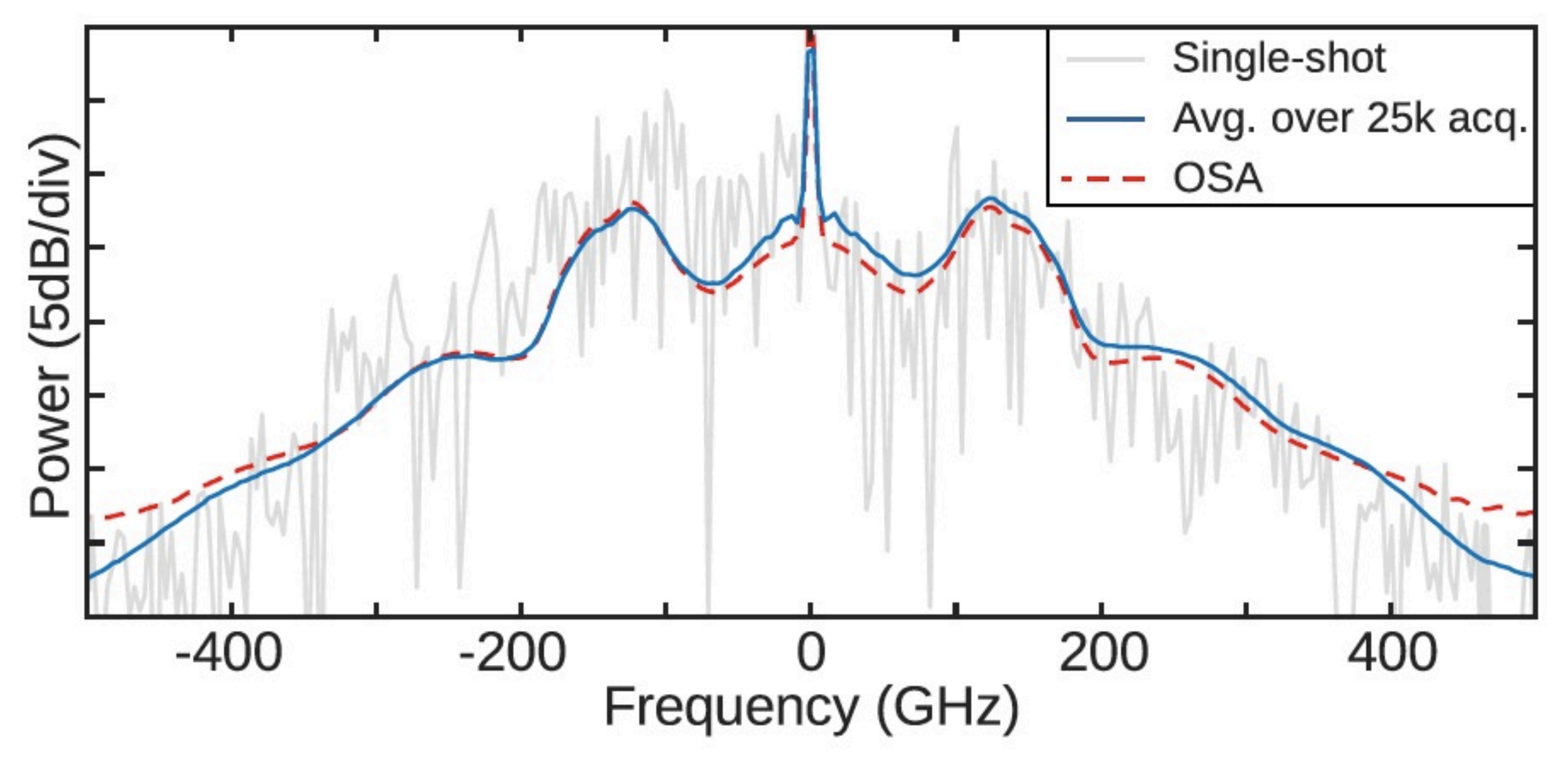


$$\mathcal{A}_{SG} = \int |\psi_{SG}(x)| dx$$



"Soliton refraction through an optical soliton gas", P. Suret et al, arXiv:2303.13421 [nlin.PS] (2023)

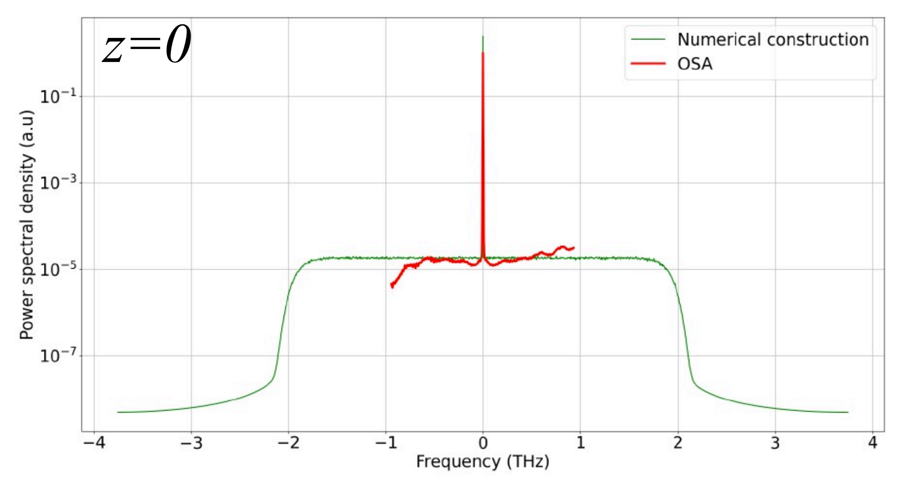
Modulation Instability in optical fiber experiments

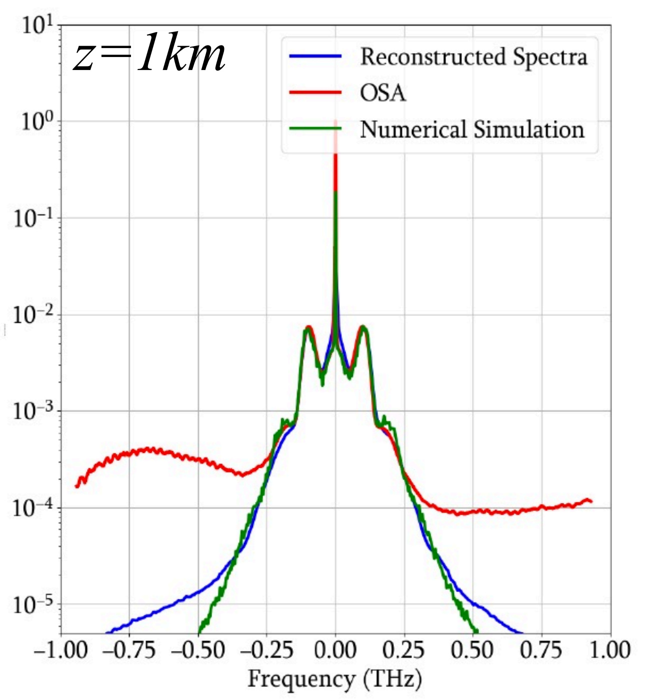


A Lebel, *et al*., Opt. Lett. 46 (2), 298-301 (2021)

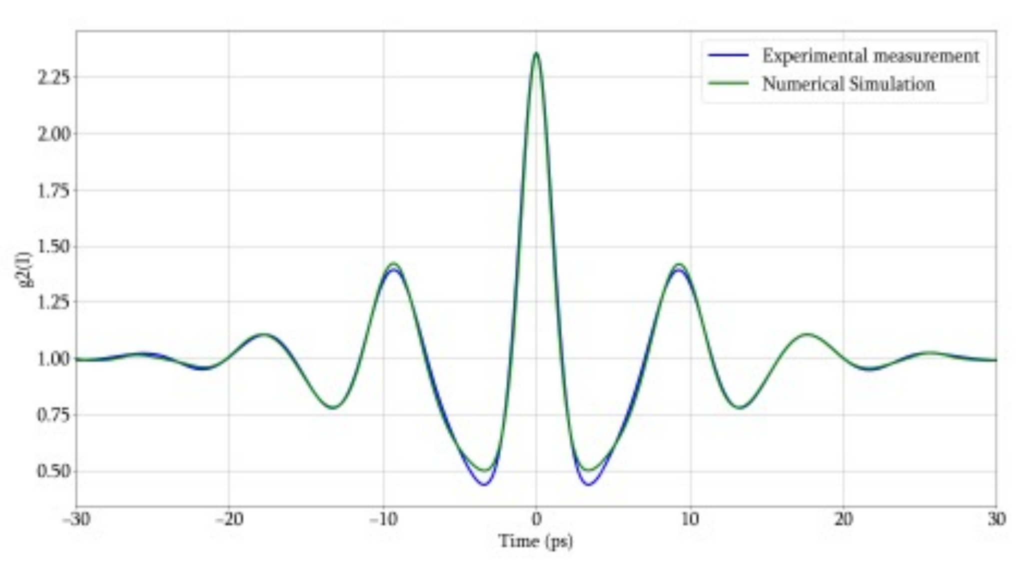
Modulation Instability in optical fiber experiments

- Spectrum





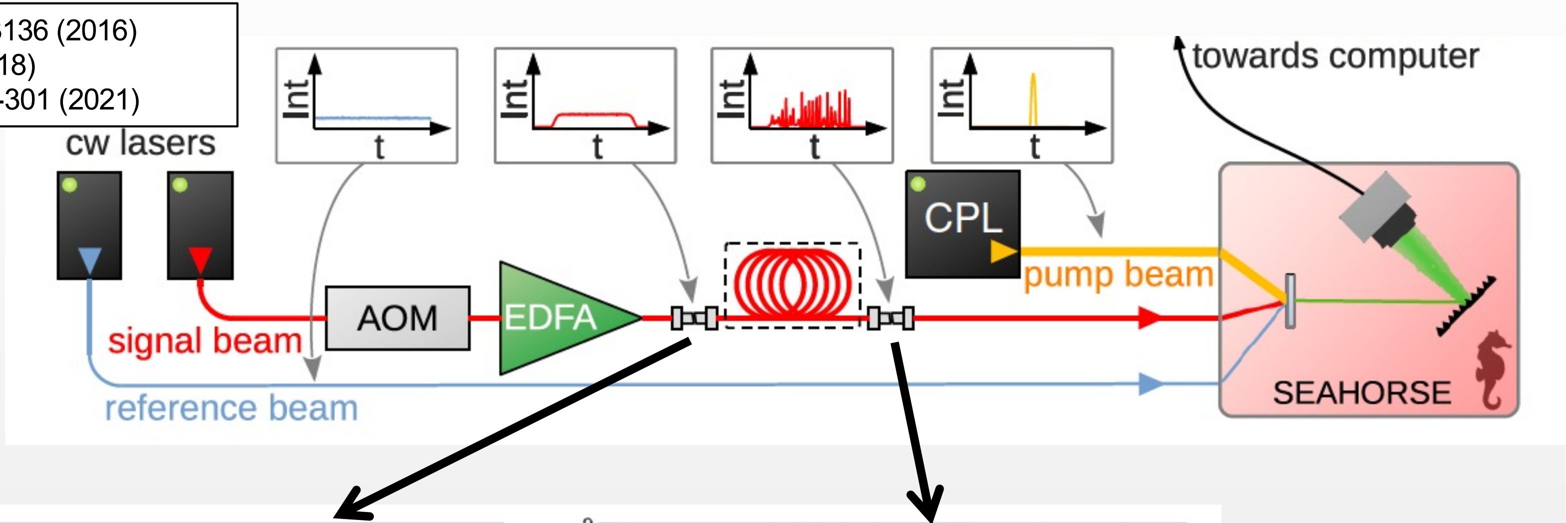
- $g^{(2)}$

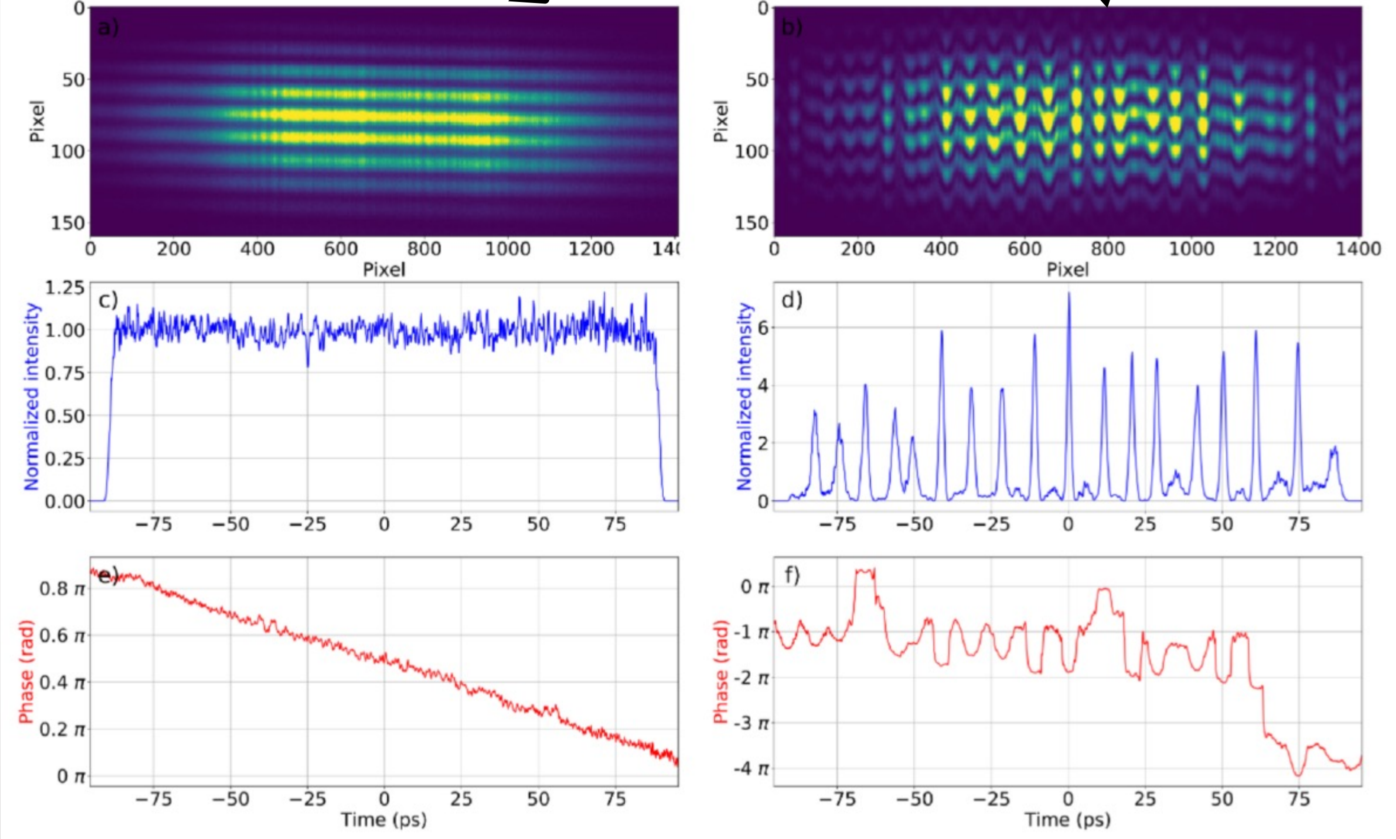


Ultrafast measurement (SEAHORSE) in optical fiber experiments

P. Suret *et al.,* Nat. Commun. **7**, 13136 (2016)
A. Tikan *et al.,* Nat. Photon. **12** (2018)
A Lebel *et al.* Opt. Lett. 46 (2), 298-301 (2021)

SMF28, P = 4W
β_2 = -21.7ps^2/km
$\chi^{(3)}$ = 1.3W^{-1}km^{-1}
L=1km
λ=1.55μm

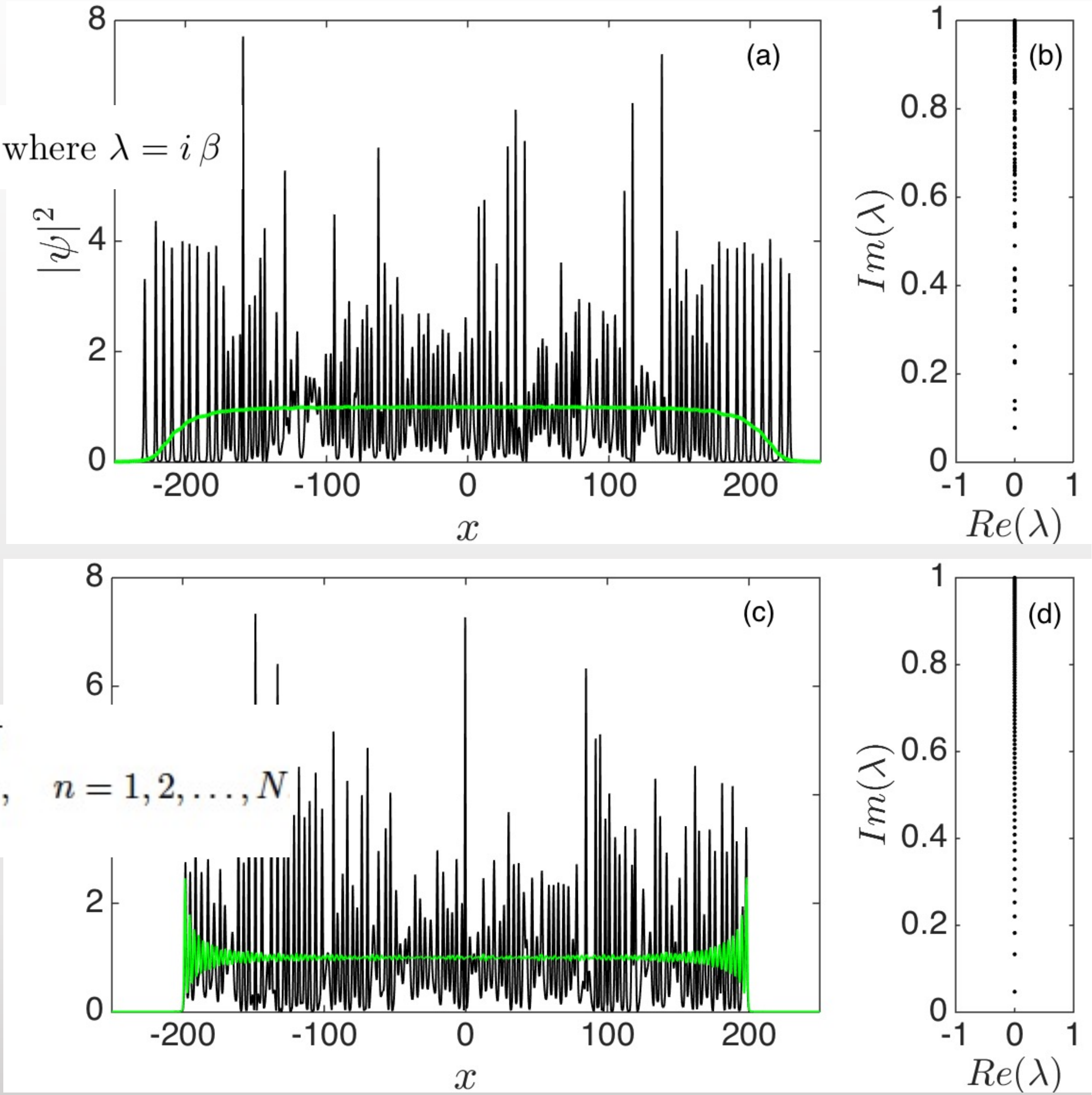




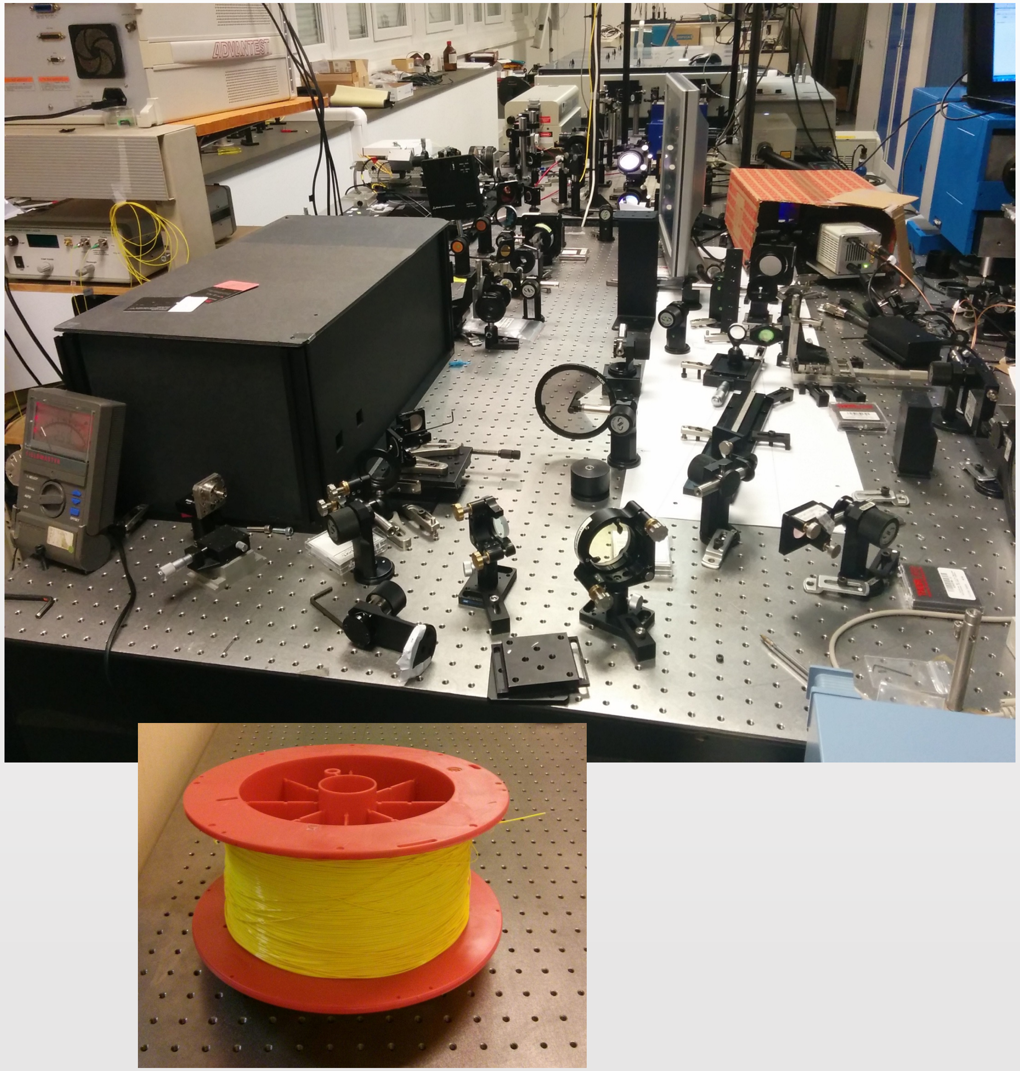
Random eigenvalues vs Borh Sommerfeld

$$f(\lambda) = f(\beta) = \beta/(\sqrt{1-\beta^2}) \text{ where } \lambda = i\,\beta$$

$$\lambda_n = i\,\beta_n = i\sqrt{1-\left[\frac{\pi(n-\frac{1}{2})}{L_0}\right]^2}, \quad n = 1, 2, \ldots, N$$



NLS Experiments in optical fibers and in water tank



- Phlam, University of Lille, France
- Ecole Centrale of Nantes, France

Noise-driven Modulation instability (water tank experiments)

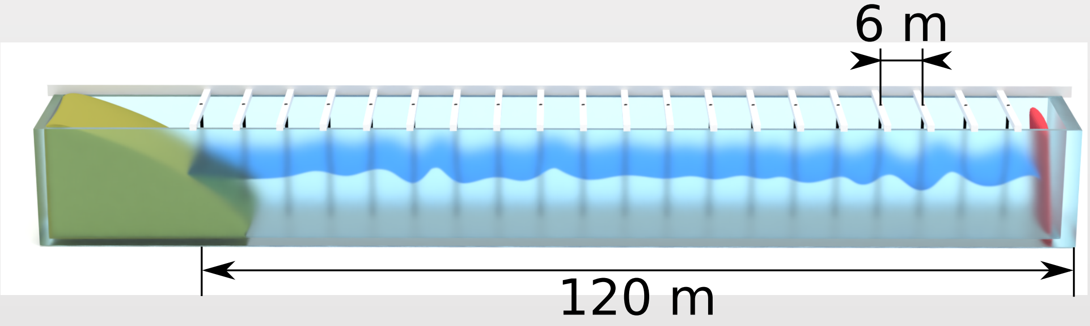
F. Copie, S. Randoux, A. Tikan, P. Suret
Phlam, Univ Lille, France

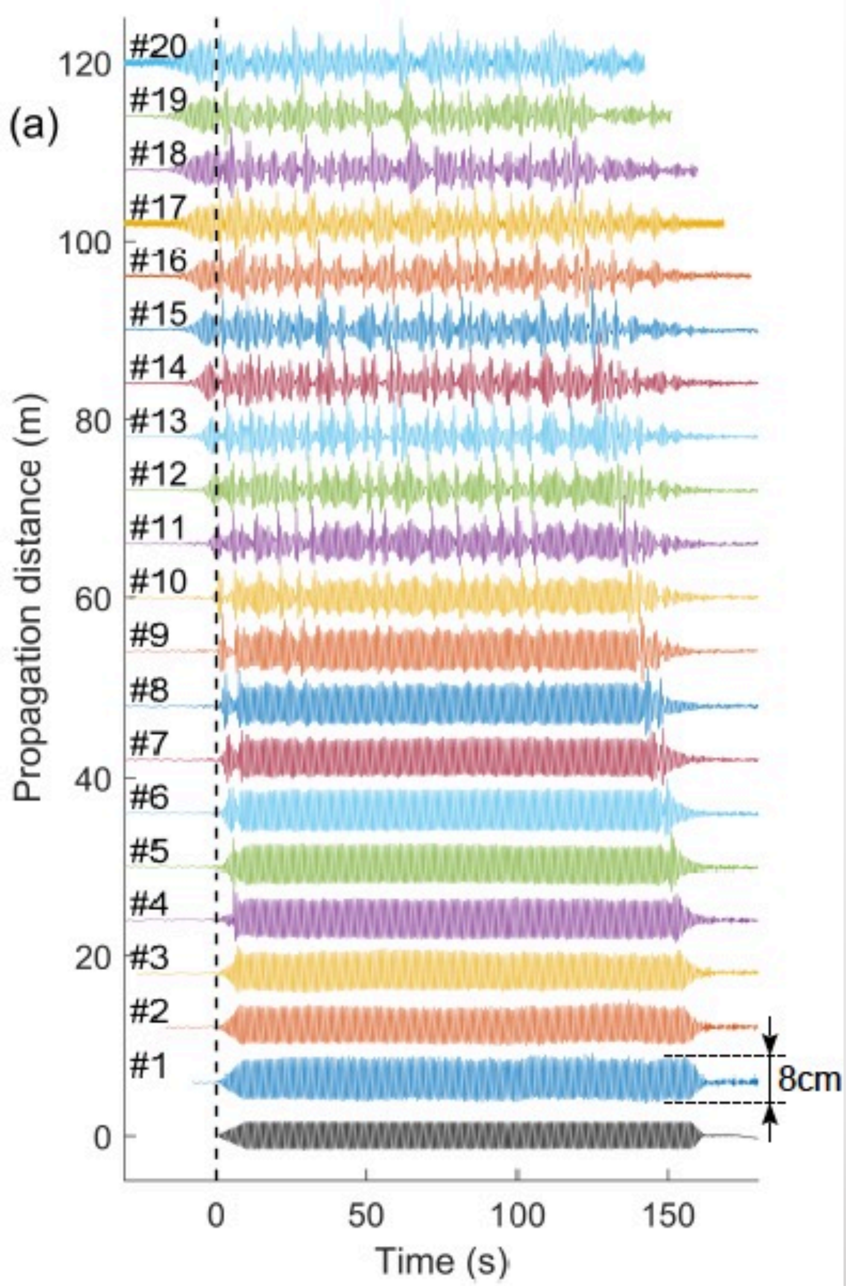
Eric Falcon, Annette Cazaubiel (PhD)
MSC, *Univ. Paris Diderot, France*

Guillaume Michel, Gaurav Prabhudesai (ENS, PhD),
Ecole Norm. Sup., France

Félicien Bonnefoy, Guillaume Ducrozet, *Ecole Centrale de Nantes*

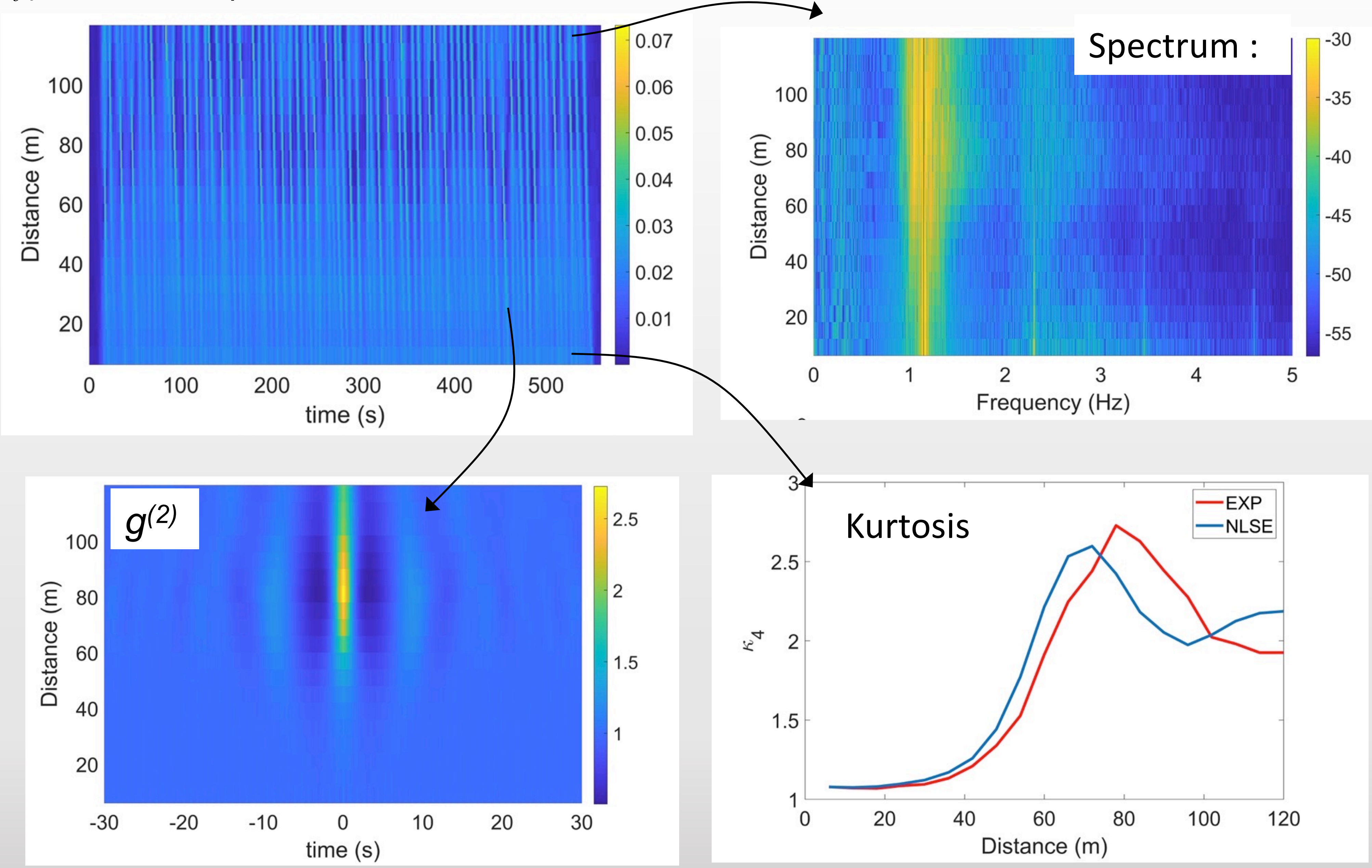
Amin Chabchoub, *Univ. Sydney, Australia*





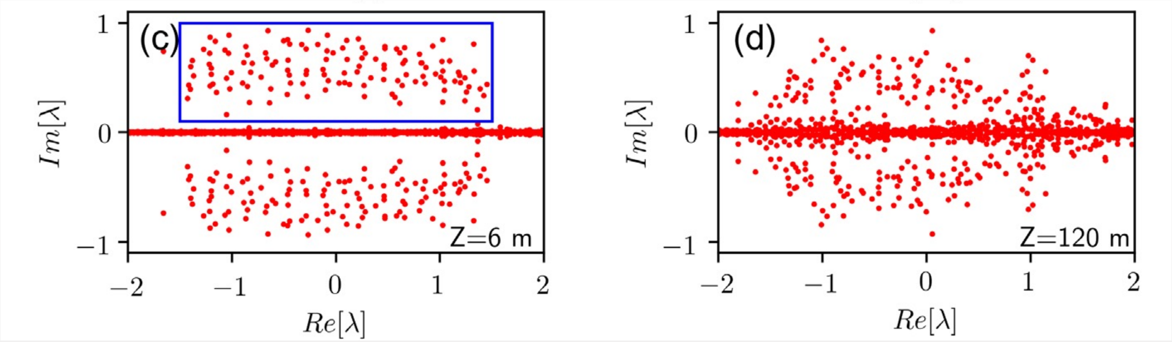
Noise-driven Modulation instability (water tank experiments)

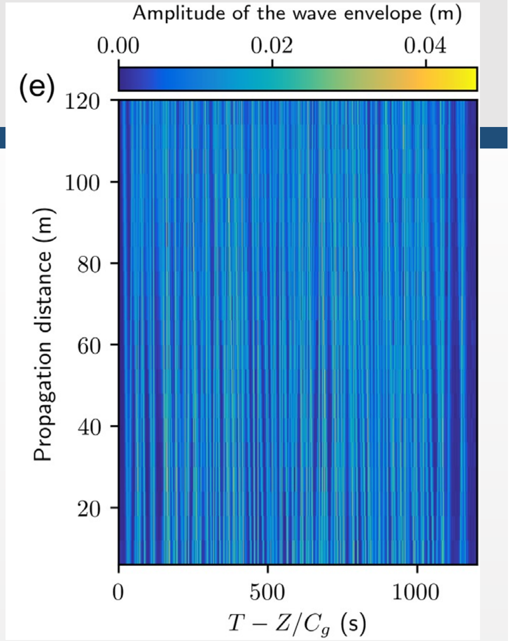
f_0=1.15Hz, steepness=0.1, noise=10%

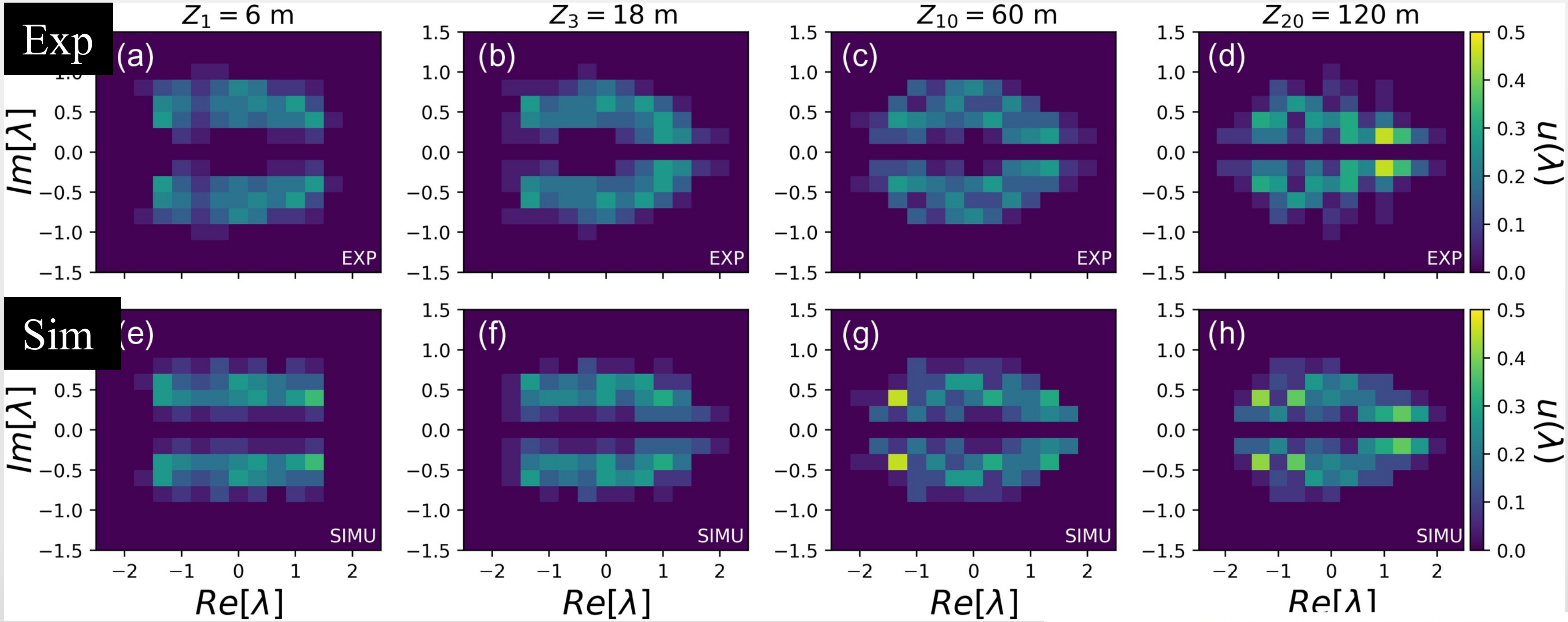


Water Tank experiments

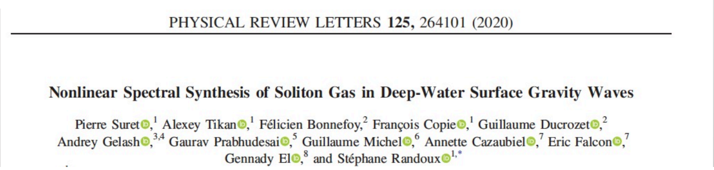
- First measurement of the DOS of SG in experiments (N=128)







Simulations : Euler's equations



PHYSICAL REVIEW LETTERS **125,** 264101 (2020)

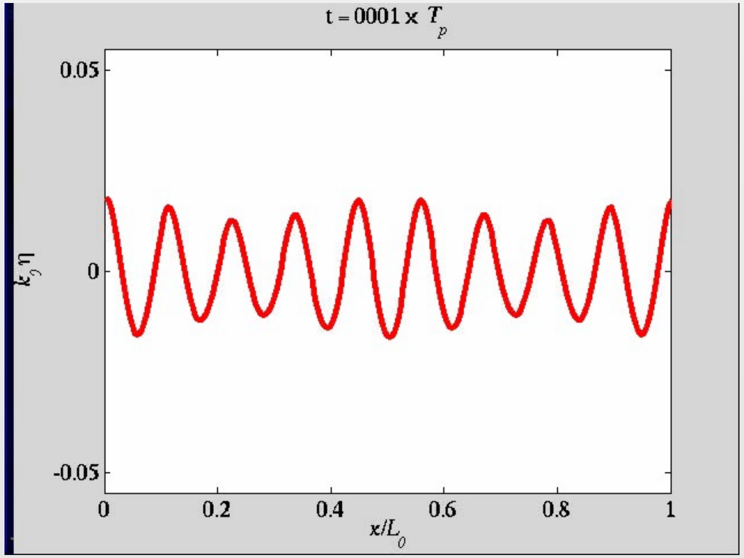
Nonlinear Spectral Synthesis of Soliton Gas in Deep-Water Surface Gravity Waves

Pierre Suret,[1] Alexey Tikan,[1] Félicien Bonnefoy,[2] François Copie,[1] Guillaume Ducrozet,[2]
Andrey Gelash,[3,4] Gaurav Prabhudesai,[5] Guillaume Michel,[6] Annette Cazaubiel,[7] Eric Falcon,[7]
Gennady El,[8] and Stéphane Randoux[1,*]

Modulation Instability

- Benjamin-Feir instability (1967)
- Deep Water waves
- Sideband instability (breathers)

N. Akhmediev *et al.*,, Sov. Phys. JETP 62, 894 (1985).
N. Akhmediev and V. Korneev,, Theor. Math. Phys. 69, 1089 (1986).
N. Akhmediev, *et al.* Phys. Lett. A 373, 675 (2009).



from Nobuhito Mori, reuse with permission

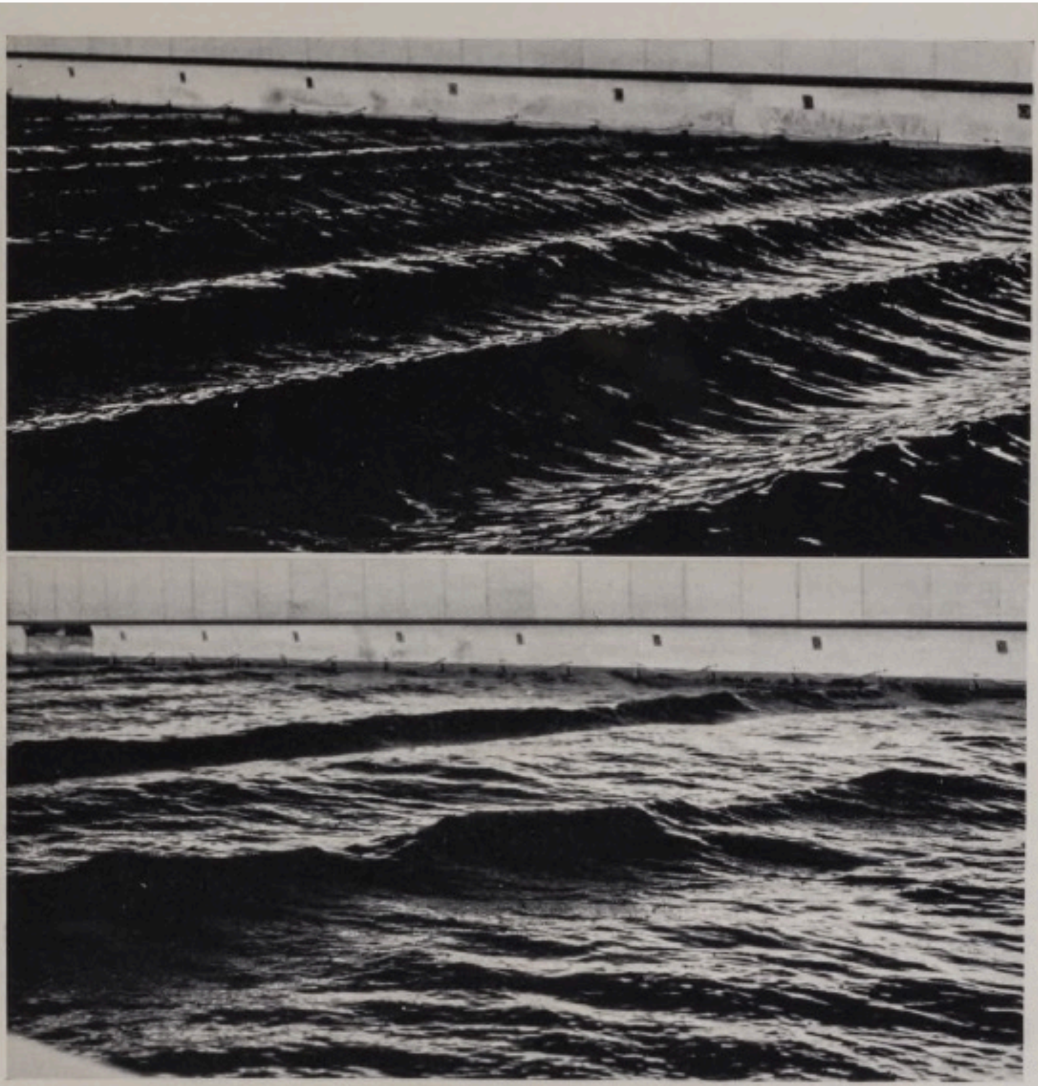
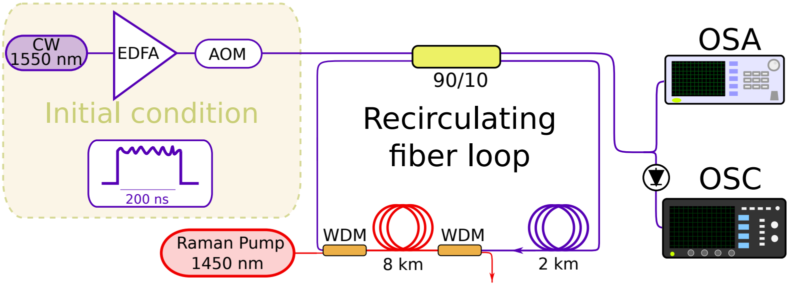


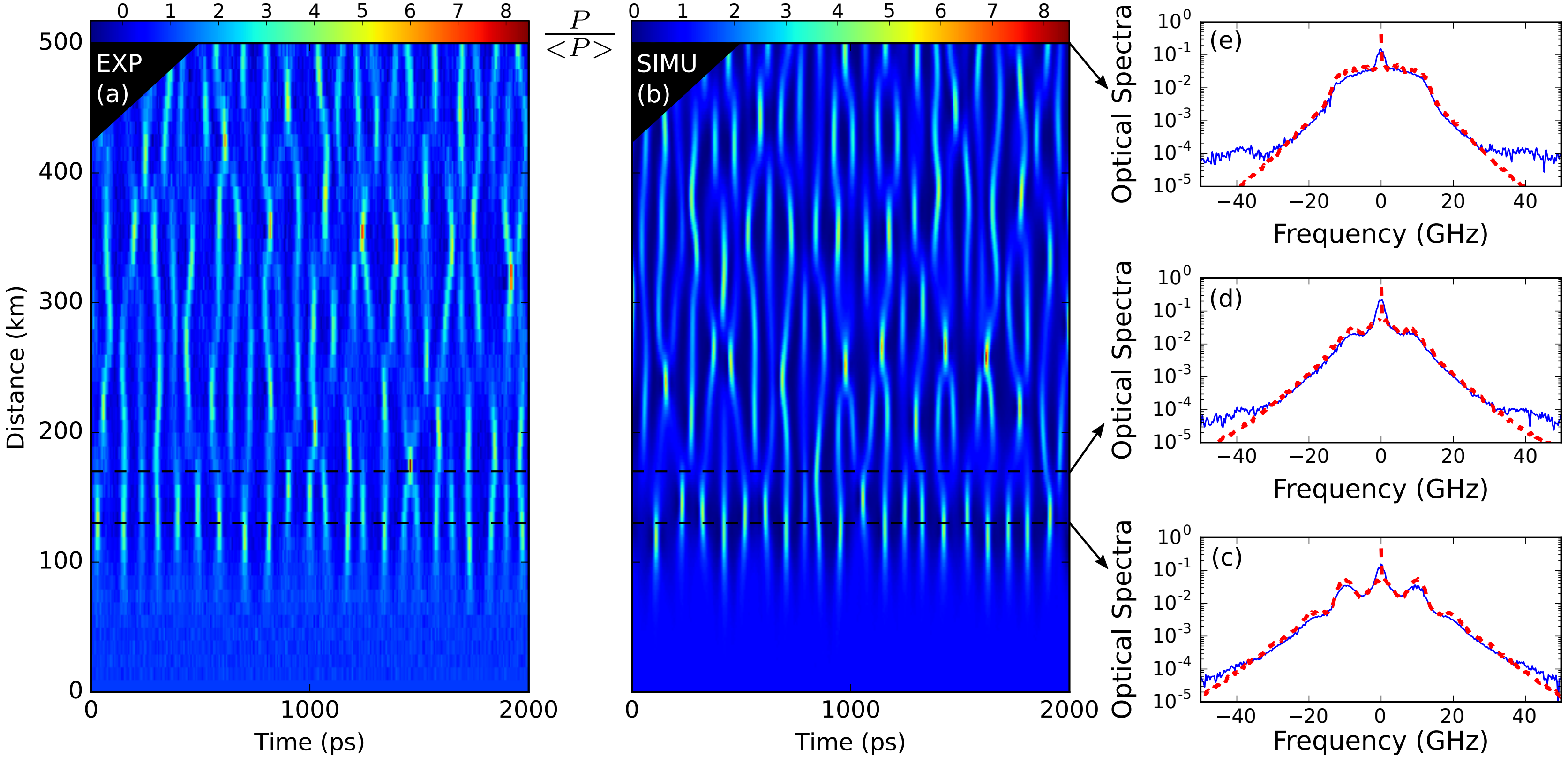
FIGURE 1. Photographs of a progressive wavetrain at two stations, illustrating disintegration due to instability: (*a*) view near to wavemaker; (*b*) view at 200 ft. farther from wavemaker. Fundamental wavelength, 7·2 ft.

Benjamin, T. Brooke; Feir, J.E. (1967). Journal of Fluid Mechanics. 27 (3) p.417–430
Benjamin, T.B. (1967). Proceedings of the Royal Society of London. A. 299 (1456) p.59–76

Modulation Instability in optical fiber experiments

A. E. Kraych, D. Agafontsev, S. Randoux, and P. Suret, Phys. Rev. Lett. **123**, 093902 (2019)





MI in optical fiber

of spectra

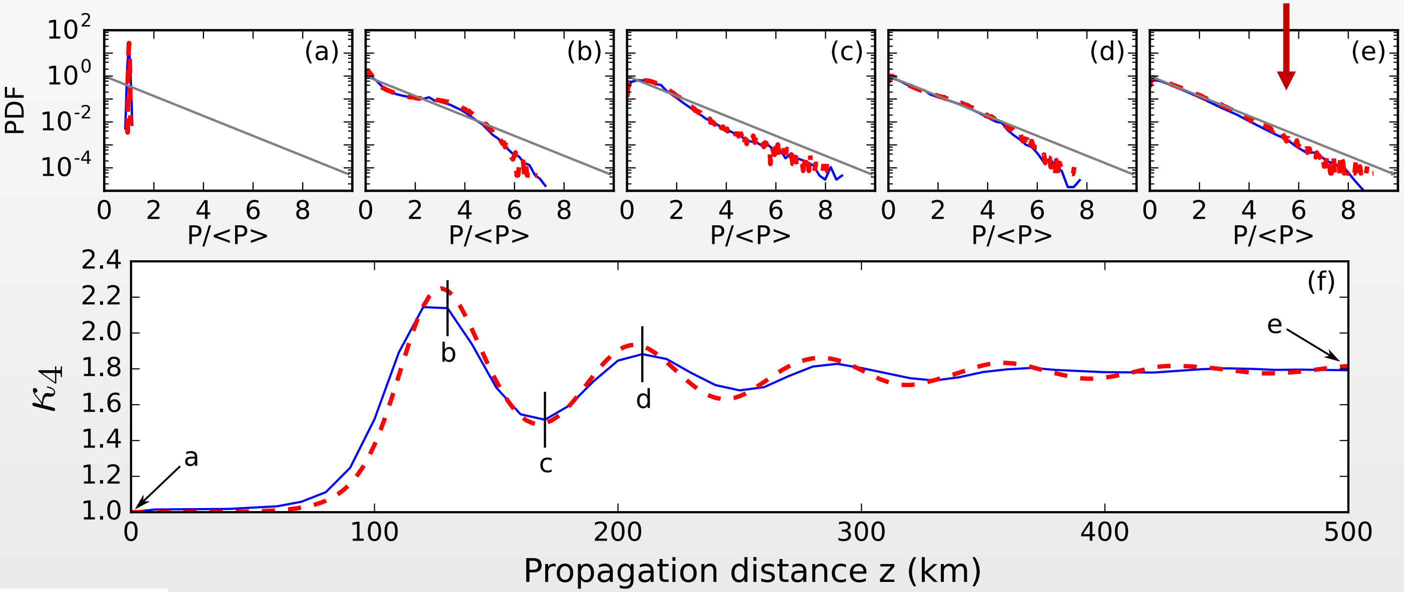
A. E. Kraych *et al.*, Phys. Rev. Lett. **123**, 093902 (2019)

Noise-driven Modulation instability (optical fiber experiments)

A. E. Kraych, D. Agafontsev, S. Randoux, and P. Suret, Phys. Rev. Lett. **123**, 093902 (2019)

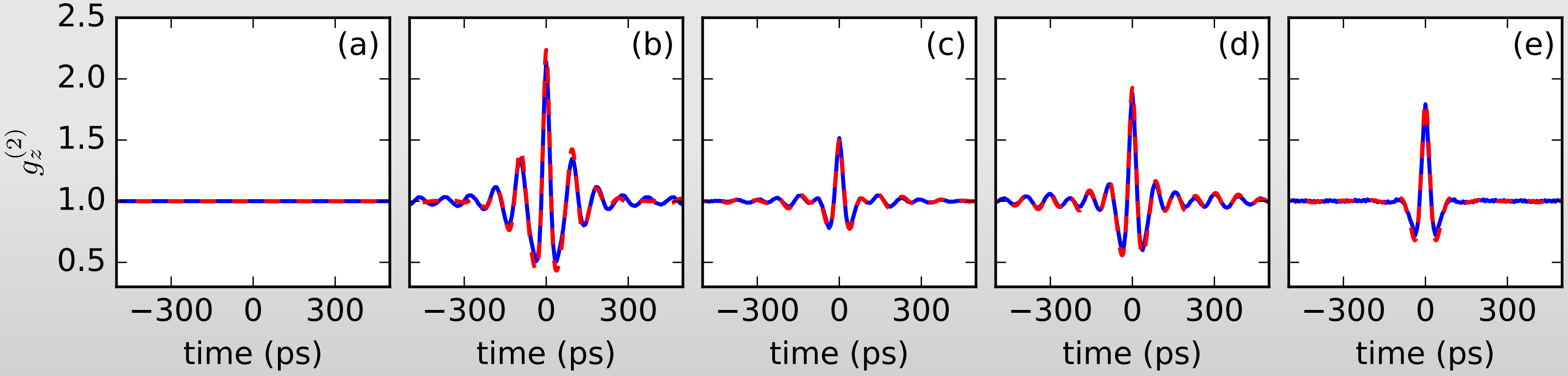
$$P = |\psi|^2$$

Gaussian statistics of ψ



$$\kappa_4(z) = \frac{\langle P(z,t)^2 \rangle}{\langle P(z,t) \rangle^2}$$

$$g_z^{(2)}(\tau) = \frac{\langle P(z,t)P(z,t+\tau) \rangle}{\langle P(z,t) \rangle^2}$$



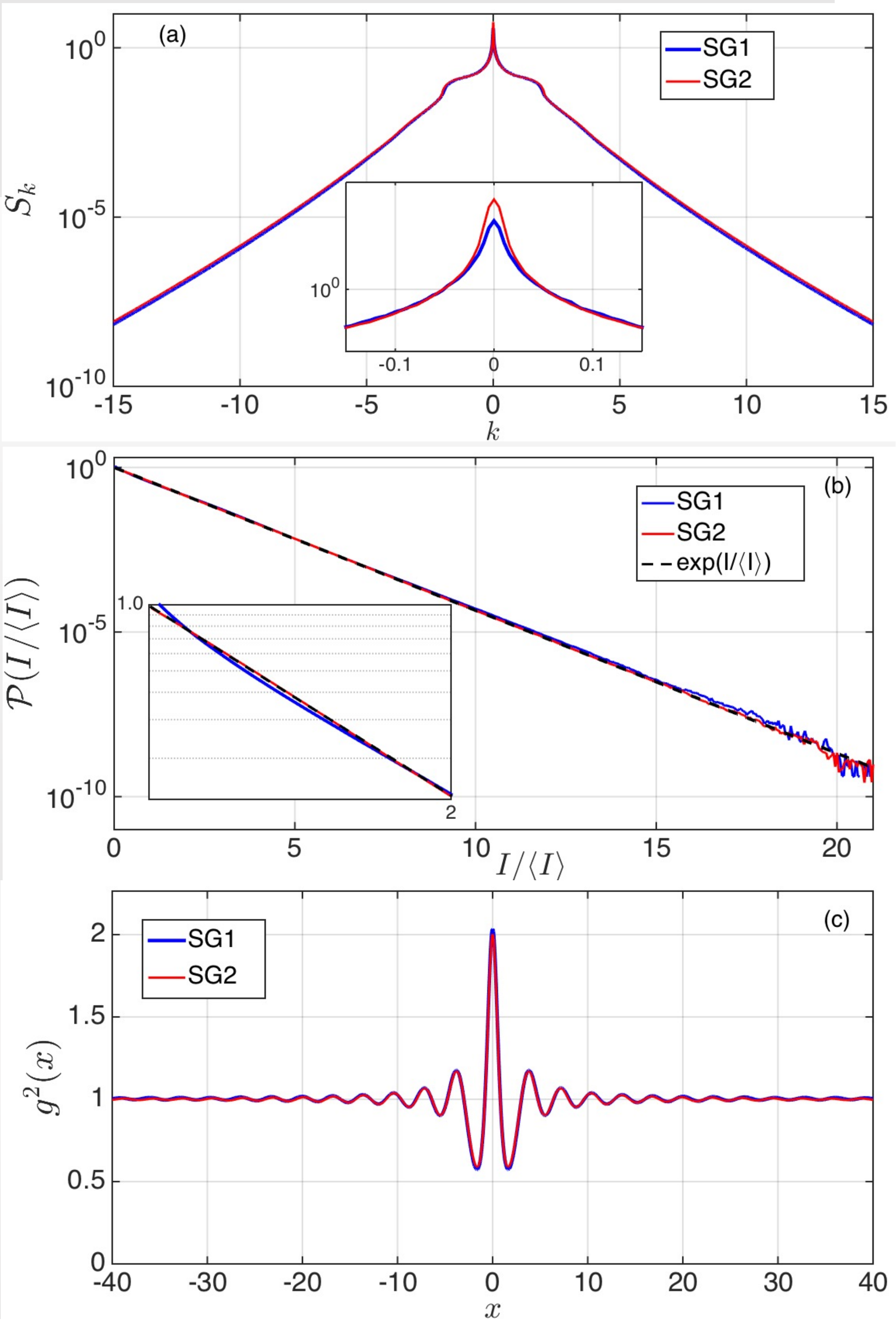
Random eigenvalues vs Borh Sommerfeld

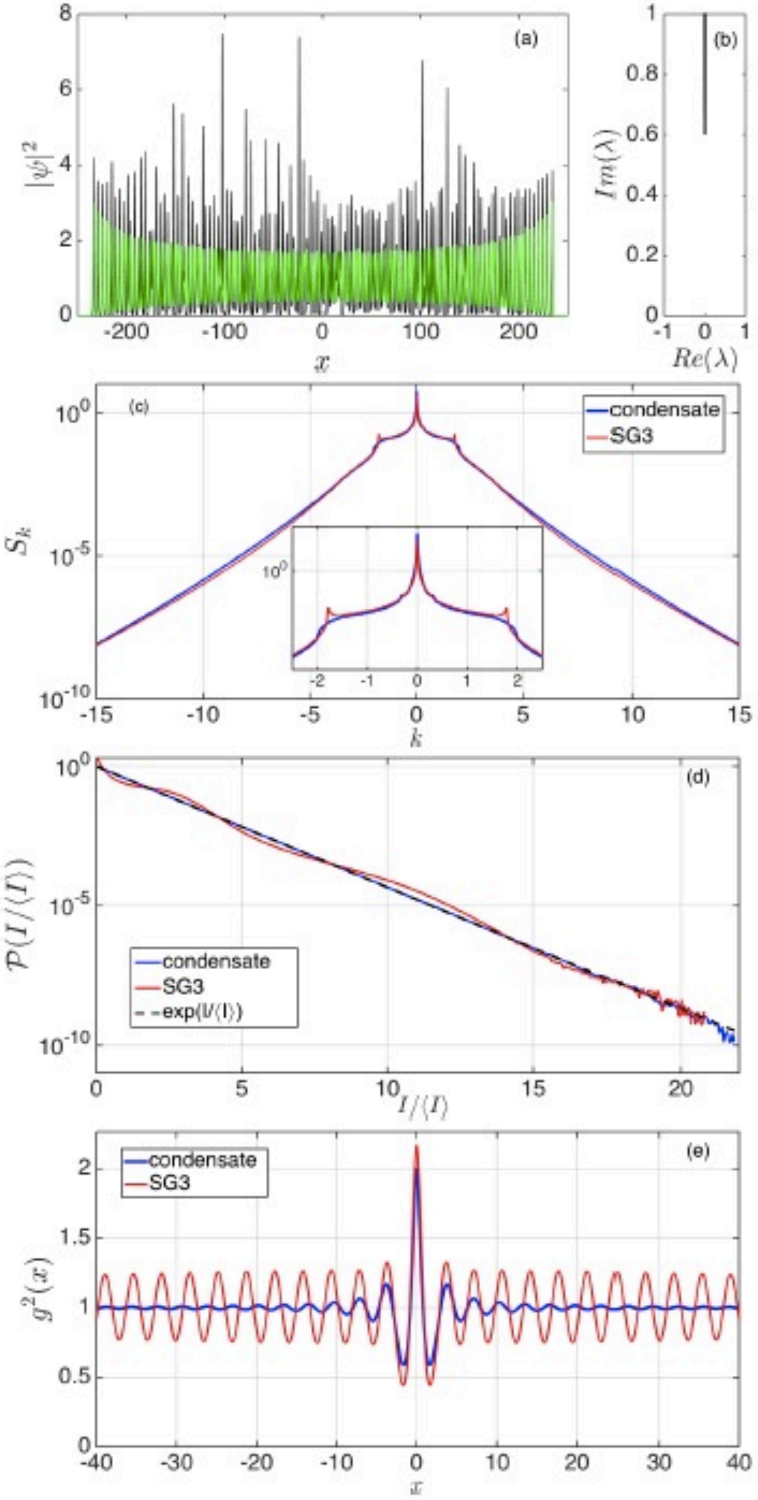
SG1

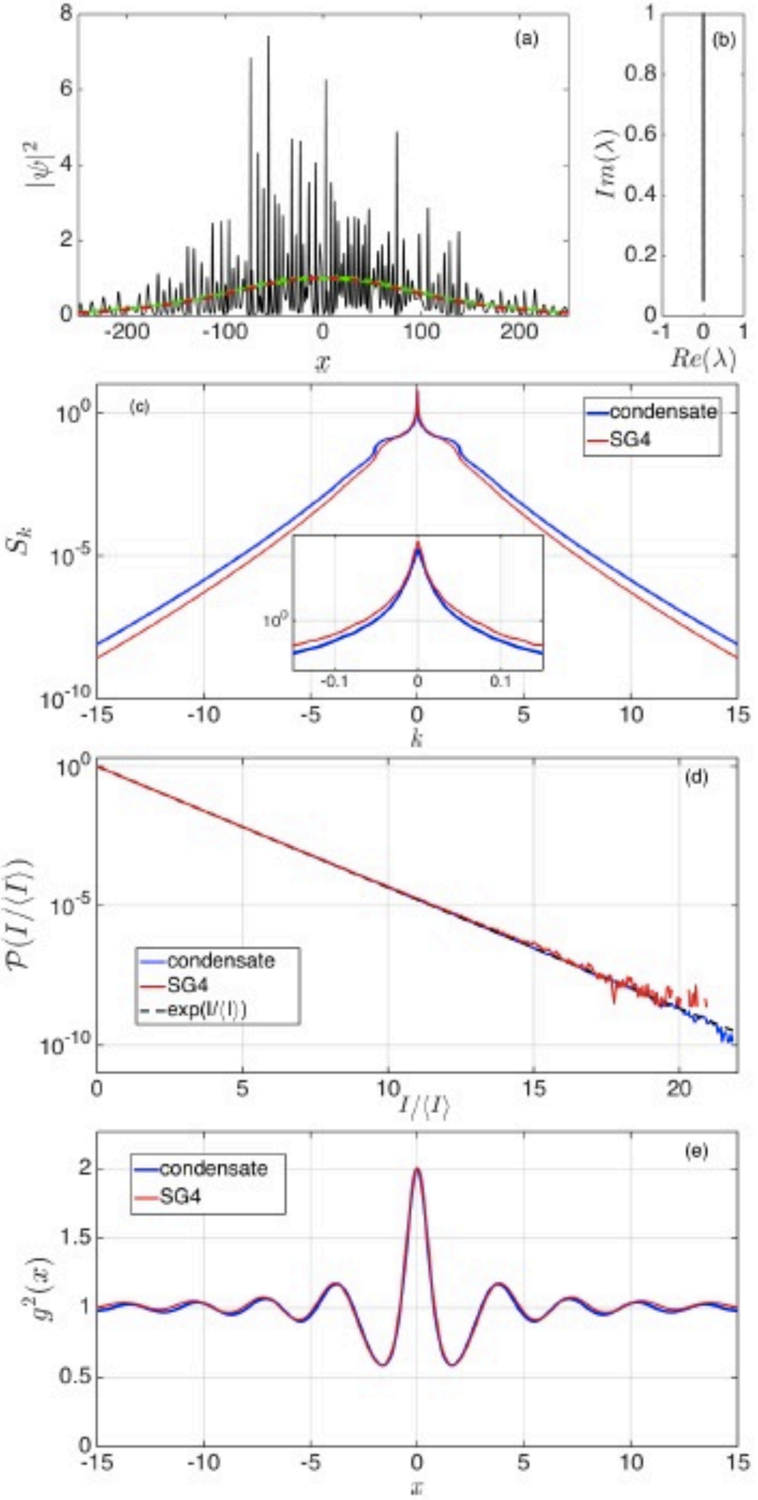
$$f(\lambda) = f(\beta) = \beta/(\sqrt{1-\beta^2}) \text{ where } \lambda = i\,\beta$$

SG2

$$\lambda_n = i\,\beta_n = i\sqrt{1-\left[\frac{\pi(n-\frac{1}{2})}{L_0}\right]^2}$$

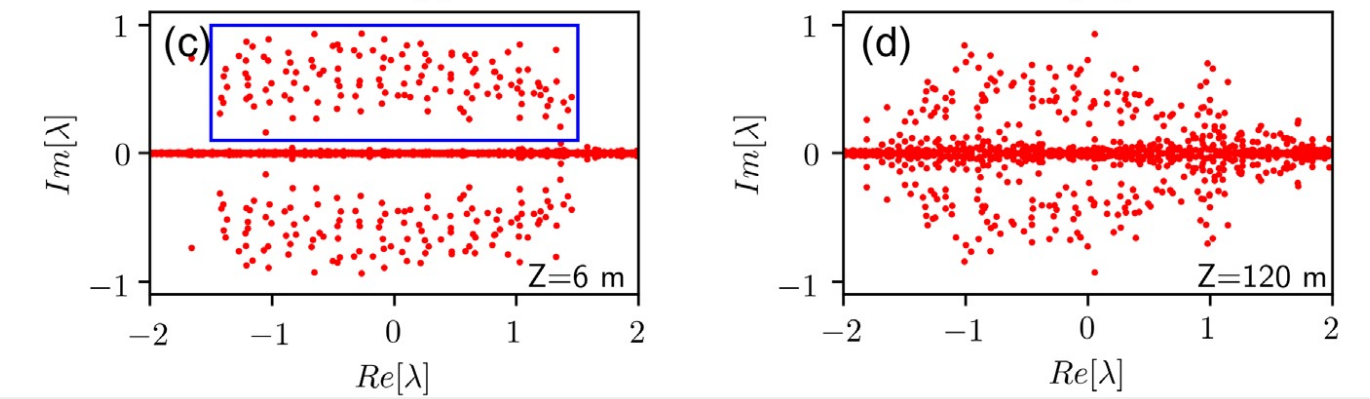


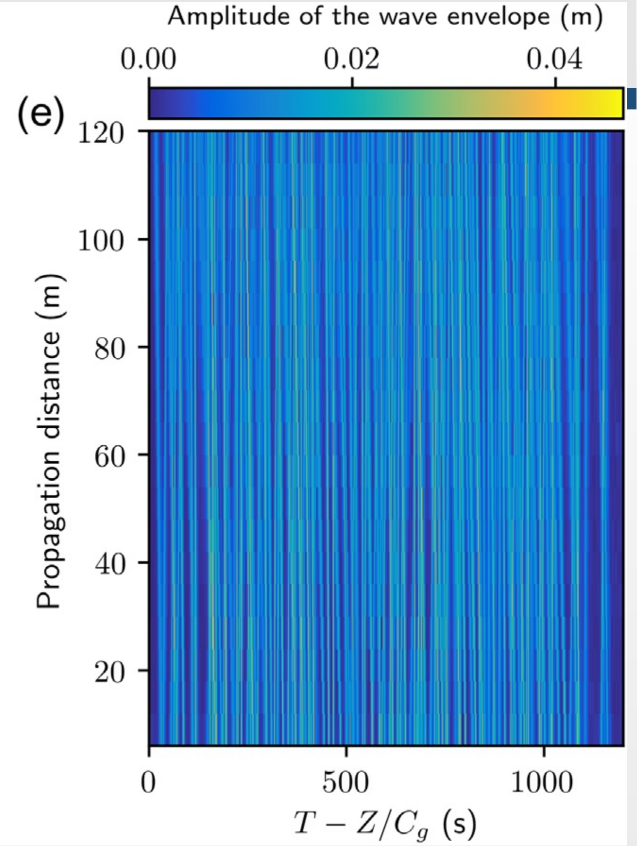

(a)
$|\psi|^2$
x
(b)
$Im(\lambda)$
$Re(\lambda)$
(c)
S_k
k
condensate
SG3
(d)
$\mathcal{P}(I/\langle I\rangle)$
$I/\langle I\rangle$
condensate
SG3
exp(I/⟨I⟩)
(e)
$g^2(x)$
x
condensate
SG3


(a)
$|\psi|^2$
x
(b)
$Im(\lambda)$
$Re(\lambda)$
(c)
S_k
k
condensate
SG4
(d)
$\mathcal{P}(I/\langle I\rangle)$
$I/\langle I\rangle$
condensate
SG4
exp(I/⟨I⟩)
(e)
$g^2(x)$
x
condensate
SG4

Water Tank experiments

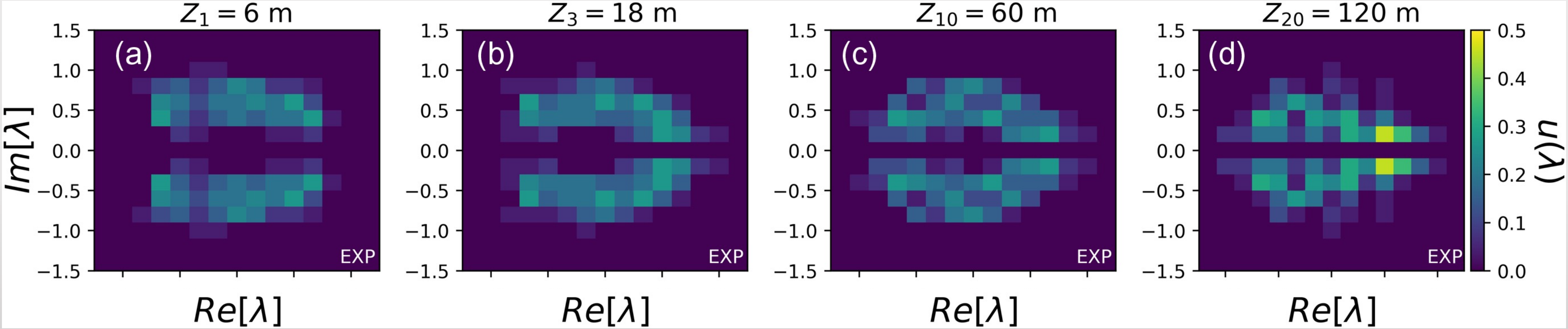
- ➢ N=128



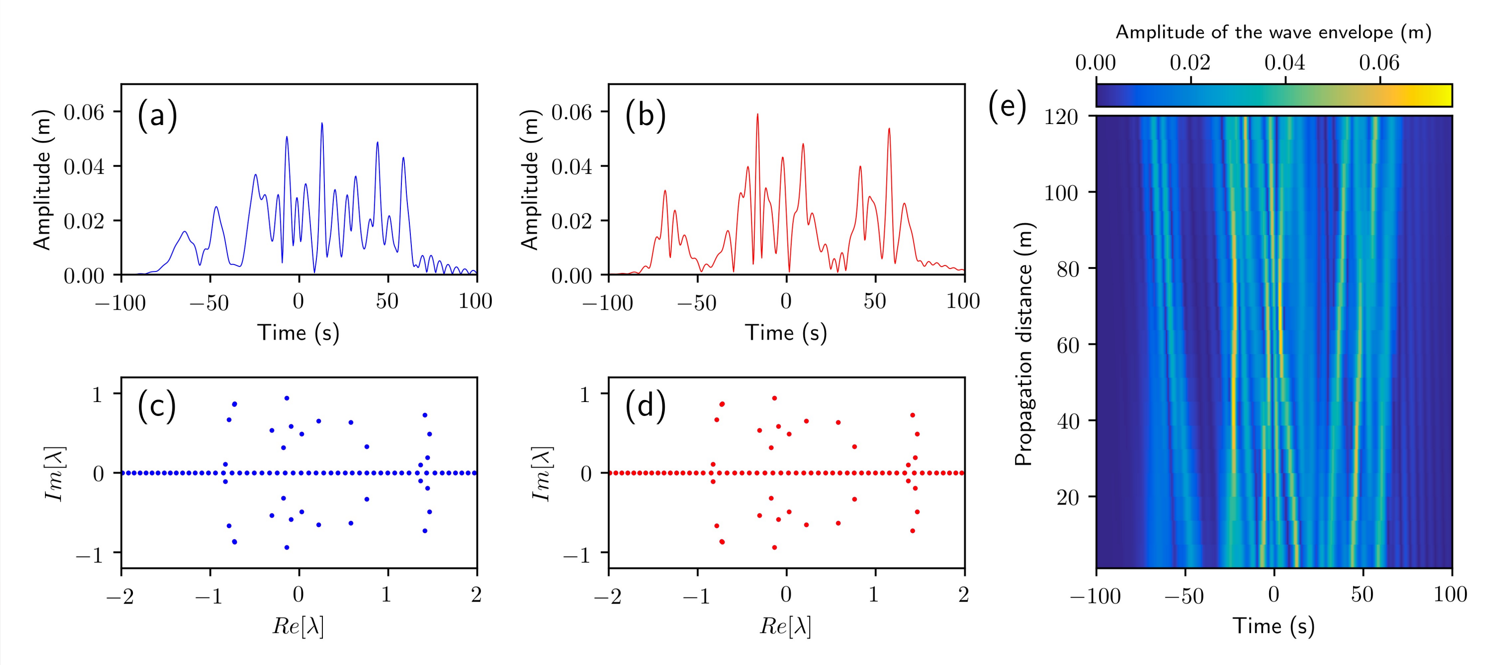


301-231-850

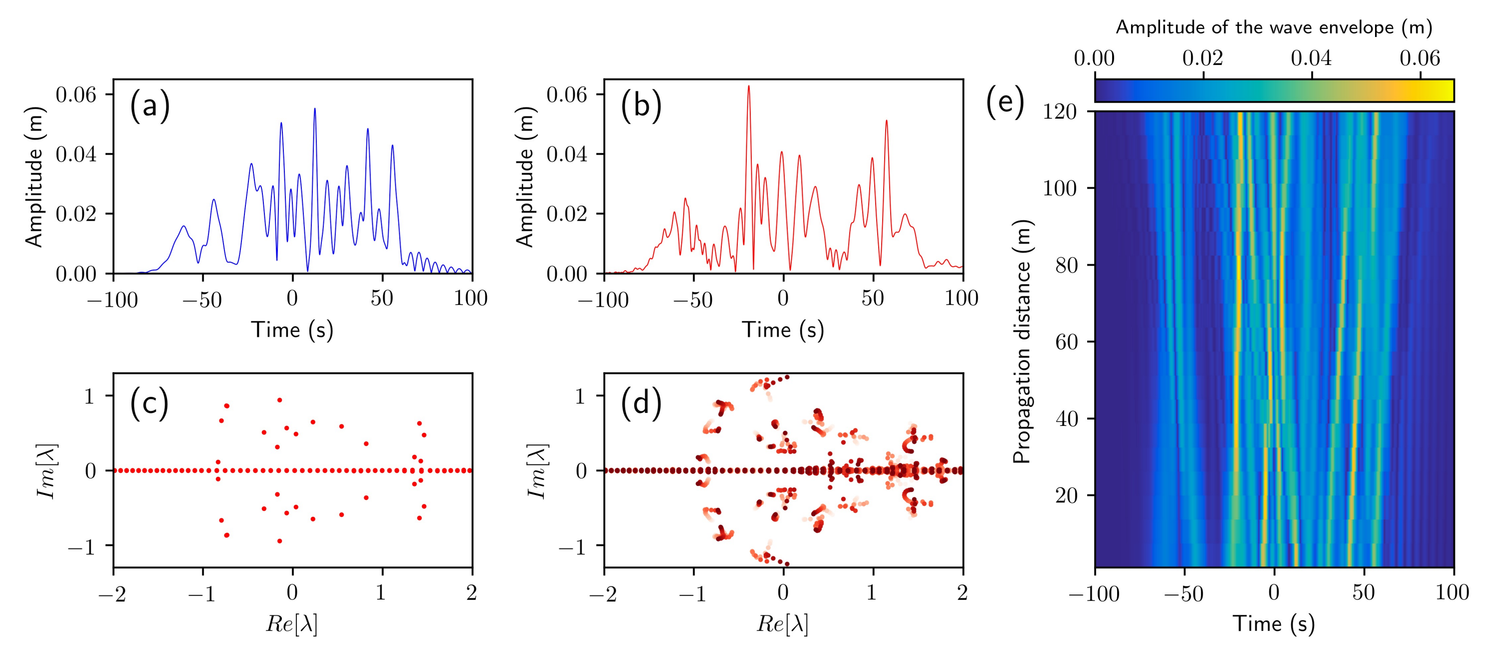
- ➢ First measurement of the DOS of SG in experiments (N=128)



Numerical simulations (high order terms)



NLS eq.



Dysthe eq.

The concept of soliton gas (integrable systems)

➢ 1971 Soliton gas in integrable system (ex: KdV, focusing 1DNLS)

V. E. Zakharov, *Kinetic equation for solitons*, Sov. Phys. JETP 33, 538 (1971)

- ✓ Solitons with elastic collisions
- ✓ Random parameters

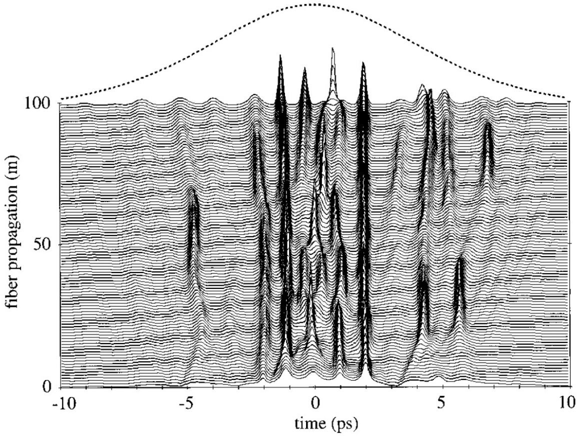
➢ 1973 Solitons in optical fibers

Hasegawa, A., Tappert, F., Mollenauer, L. F., Stolen, R. H., & Gordon, L. P. *Experimental observation of pico second pulse narrowing and solitons in optical fiber.* Appl. Phys. Lett., ***23***, (1973).

F. M. Mitschke and L. F. Mollenauer , *Experimental observation of interaction forces between solitons in optical fibers*, Opt. Lett., **12**, 5, (1987)

➢ 1997 Numerical simulations of solitons gas in optical fibers

A. Schwache and F. Mitschke, “Properties of an optical soliton gas”, Phys. Rev. E, **55**, 6 (1997)



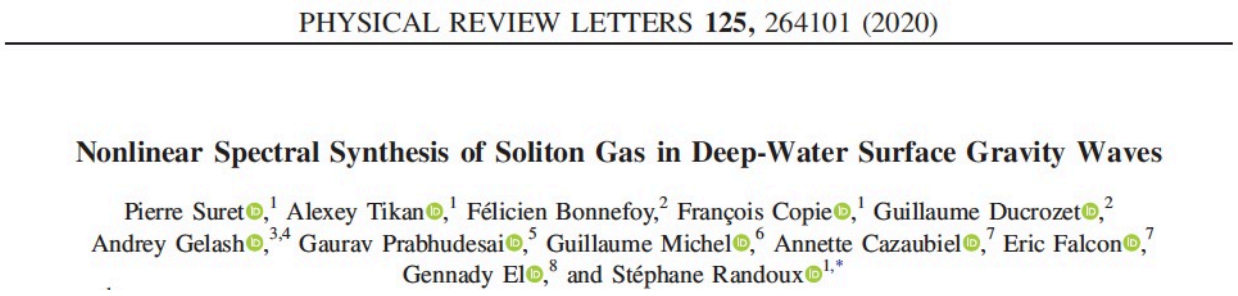
➢ Recent experiments (water waves + optical fibe

Redor I. *et al.*, Experimental evidence of a hydrodynamic soliton gas. *Phys. Rev. Lett.*, **122**, 21, (201
Suret P., et al. "Nonlinear spectral synthesis of soliton gas in deep-water surface gravity waves." *Phys. Rev. Le*
Suret, Pierre, et al. "Soliton refraction through an optical soliton gas." *arXiv preprint arXiv:2303.13421* (2023)

Outline

Theoretical framework: focusing 1D nonlinear Schrodinger Equation (1DNLS)

1. Experimental realization of soliton gas (water waves)



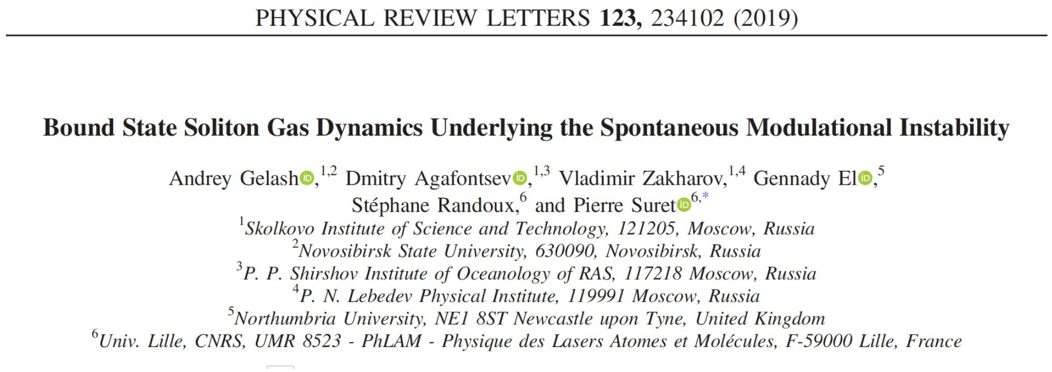
PHYSICAL REVIEW LETTERS **125,** 264101 (2020)

Nonlinear Spectral Synthesis of Soliton Gas in Deep-Water Surface Gravity Waves

Pierre Suret,[1] Alexey Tikan,[1] Félicien Bonnefoy,[2] François Copie,[1] Guillaume Ducrozet,[2]
Andrey Gelash,[3,4] Gaurav Prabhudesai,[5] Guillaume Michel,[6] Annette Cazaubiel,[7] Eric Falcon,[7]
Gennady El,[8] and Stéphane Randoux[1,*]

2. Spontaneous modulation instability

- Modelling statistical properties with soliton gas



PHYSICAL REVIEW LETTERS **123,** 234102 (2019)

Bound State Soliton Gas Dynamics Underlying the Spontaneous Modulational Instability

Andrey Gelash,[1,2] Dmitry Agafontsev,[1,3] Vladimir Zakharov,[1,4] Gennady El,[5]
Stéphane Randoux,[6] and Pierre Suret[6,*]
[1]*Skolkovo Institute of Science and Technology, 121205, Moscow, Russia*
[2]*Novosibirsk State University, 630090, Novosibirsk, Russia*
[3]*P. P. Shirshov Institute of Oceanology of RAS, 117218 Moscow, Russia*
[4]*P. N. Lebedev Physical Institute, 119991 Moscow, Russia*
[5]*Northumbria University, NE1 8ST Newcastle upon Tyne, United Kingdom*
[6]*Univ. Lille, CNRS, UMR 8523 - PhLAM - Physique des Lasers Atomes et Molécules, F-59000 Lille, France*

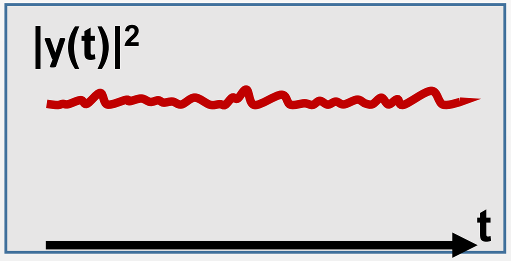
3. Refraction of a soliton by a soliton gas (optical fibers)

"Soliton refraction through an optical soliton gas", P. Suret et al, arXiv:2303.13421 [nlin.PS] (2023)

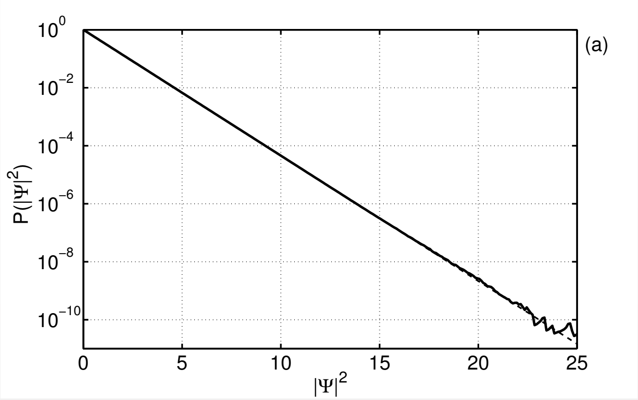
Influence of the initial condition

✓ **Noise driven modulational instability**

D.S. Agafontsev and V.E. Zakharov, Nonlinearity, 2015

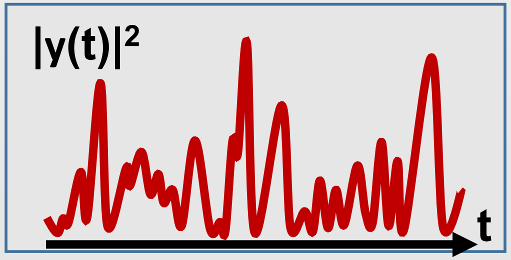


Stationnary state :
Gaussian statistics

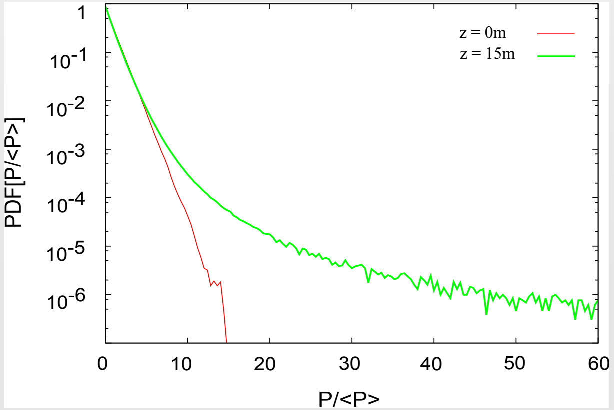


✓ **Strongly fluctuating initial condition**

P. Walczak *et al.*, Phys. Rev. Lett., 2015



Stationnary state :
strongly non Gaussian
statistics

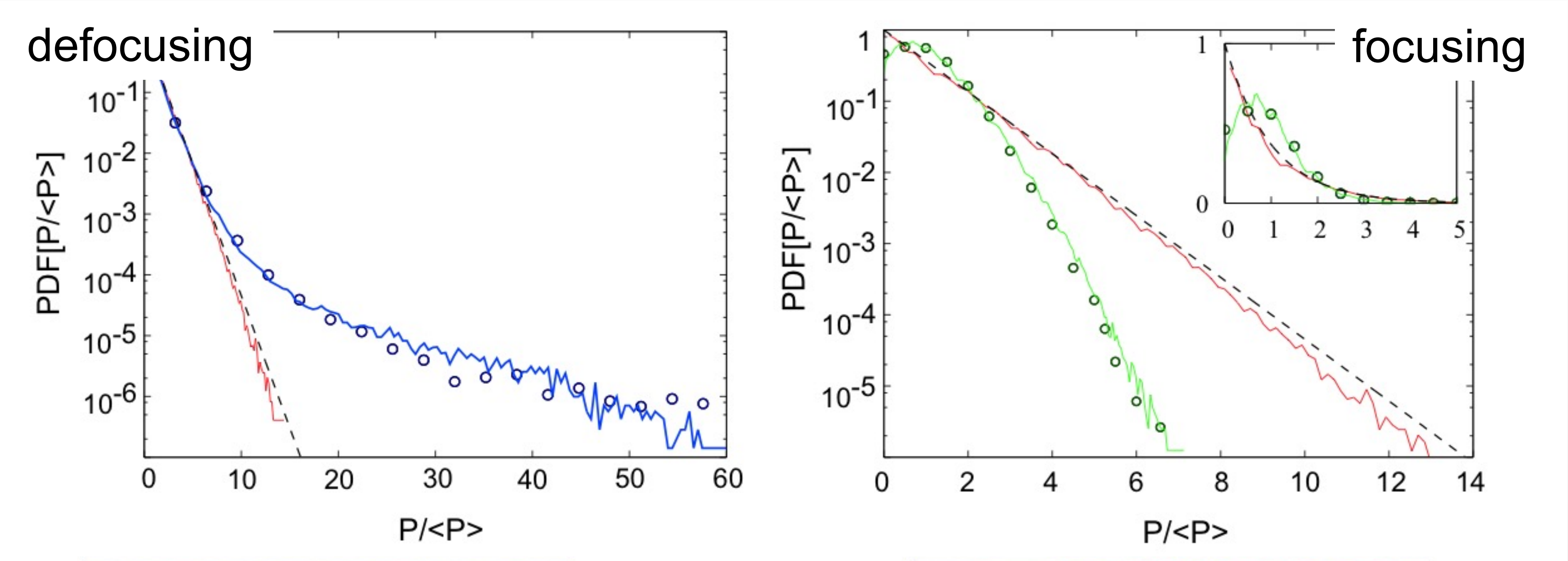


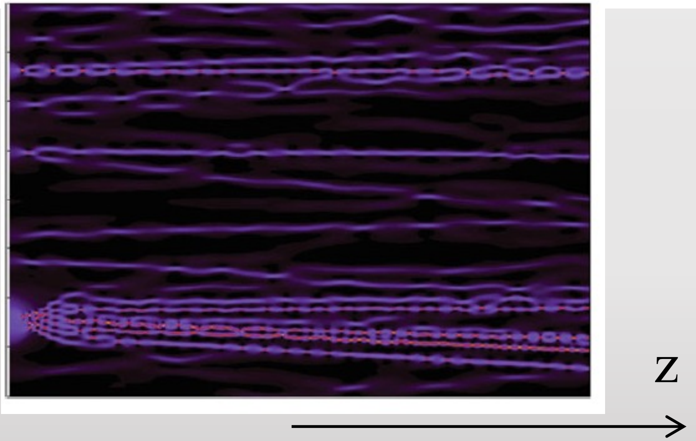
✓ **Transition between the two cases** J. Soto-Crespo et al., Phys. Rev. Lett., 2016

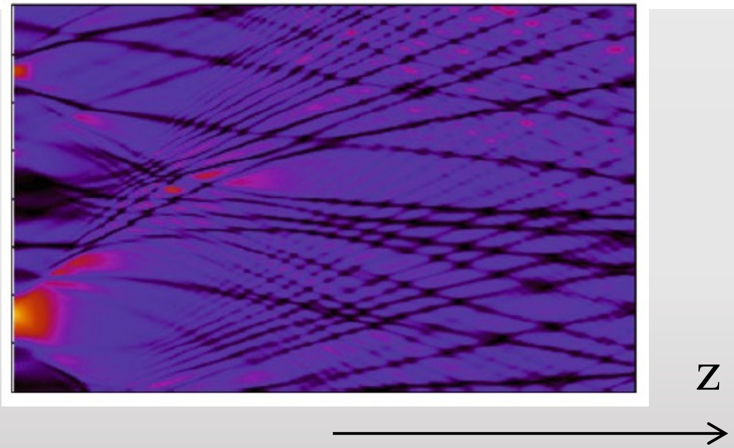
✓ **Theoretical description ? Soliton Gas ?**

Example of integrable turbulence

- ✓ **Existence of a stationary state**
- ✓ Focusing / defocusing NLS : heavy/ low tail (PDF)



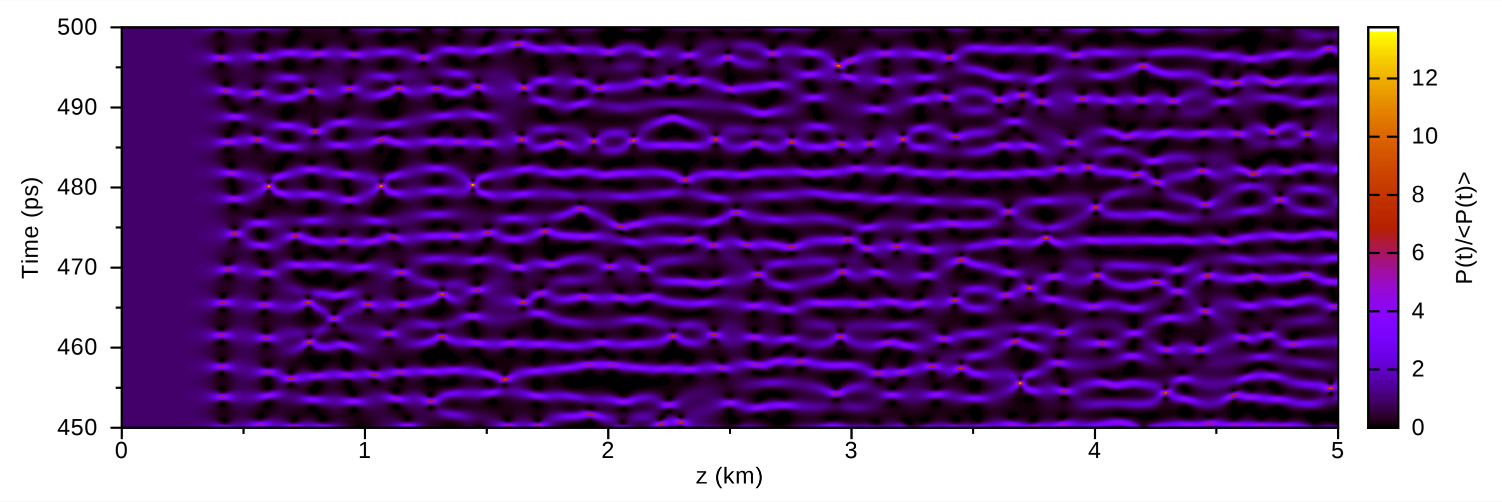




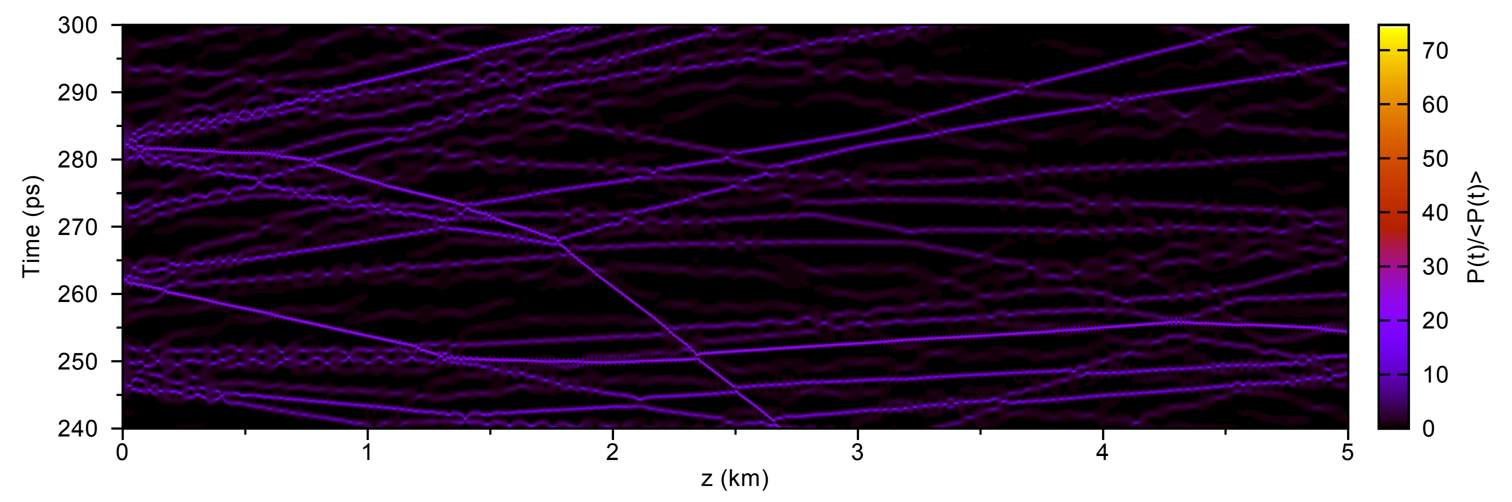
Randoux and Suret (2016), in “Rogue and shock waves in nonlinear dispersive media”, Springer

Influence of the initial condition

- ✓ **Noise driven modulational instability** Dudley et al. Nat. Photon. 8, 75 (2014)

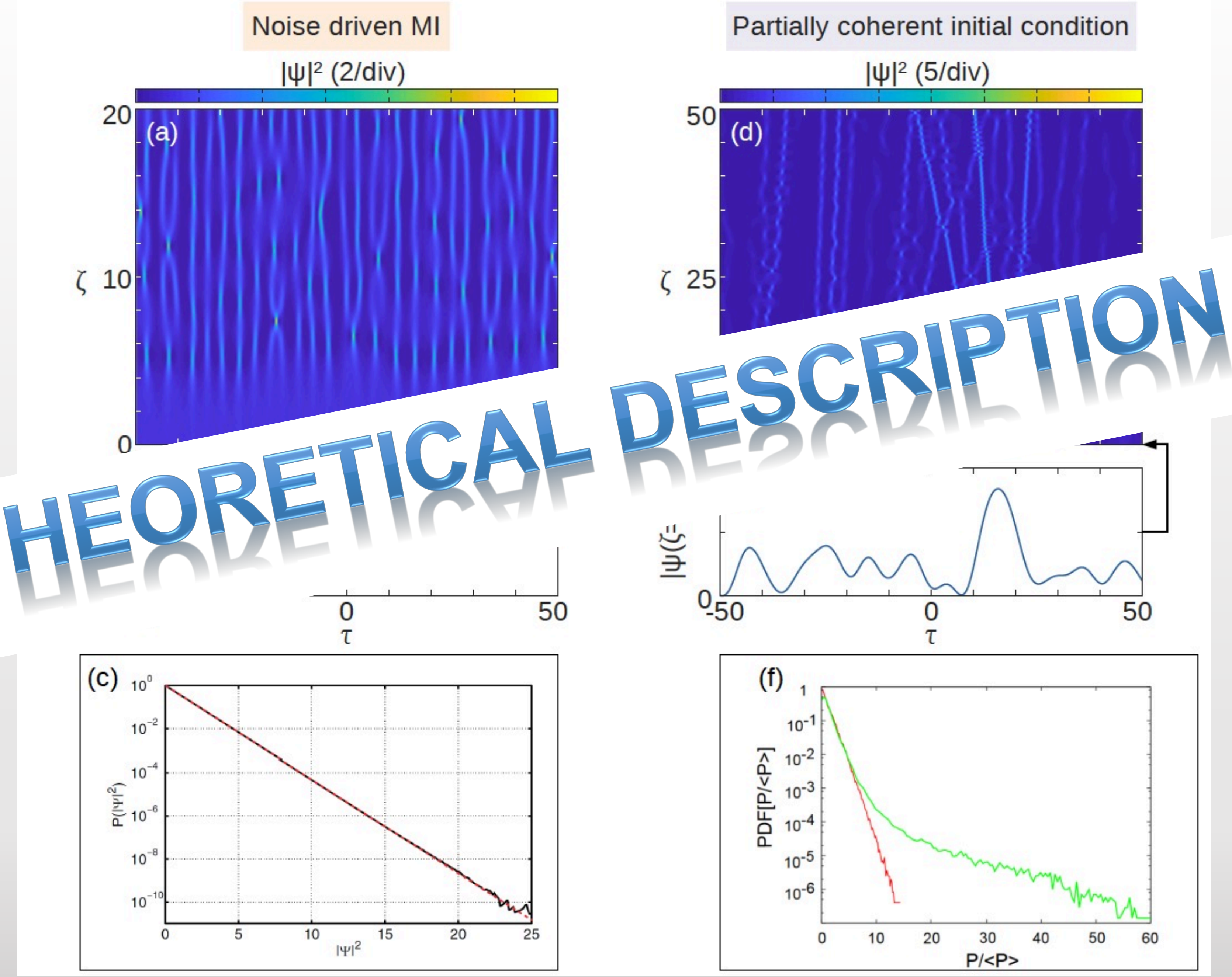


- ✓ **Strongly fluctuating initial condition**



- ➢ movie

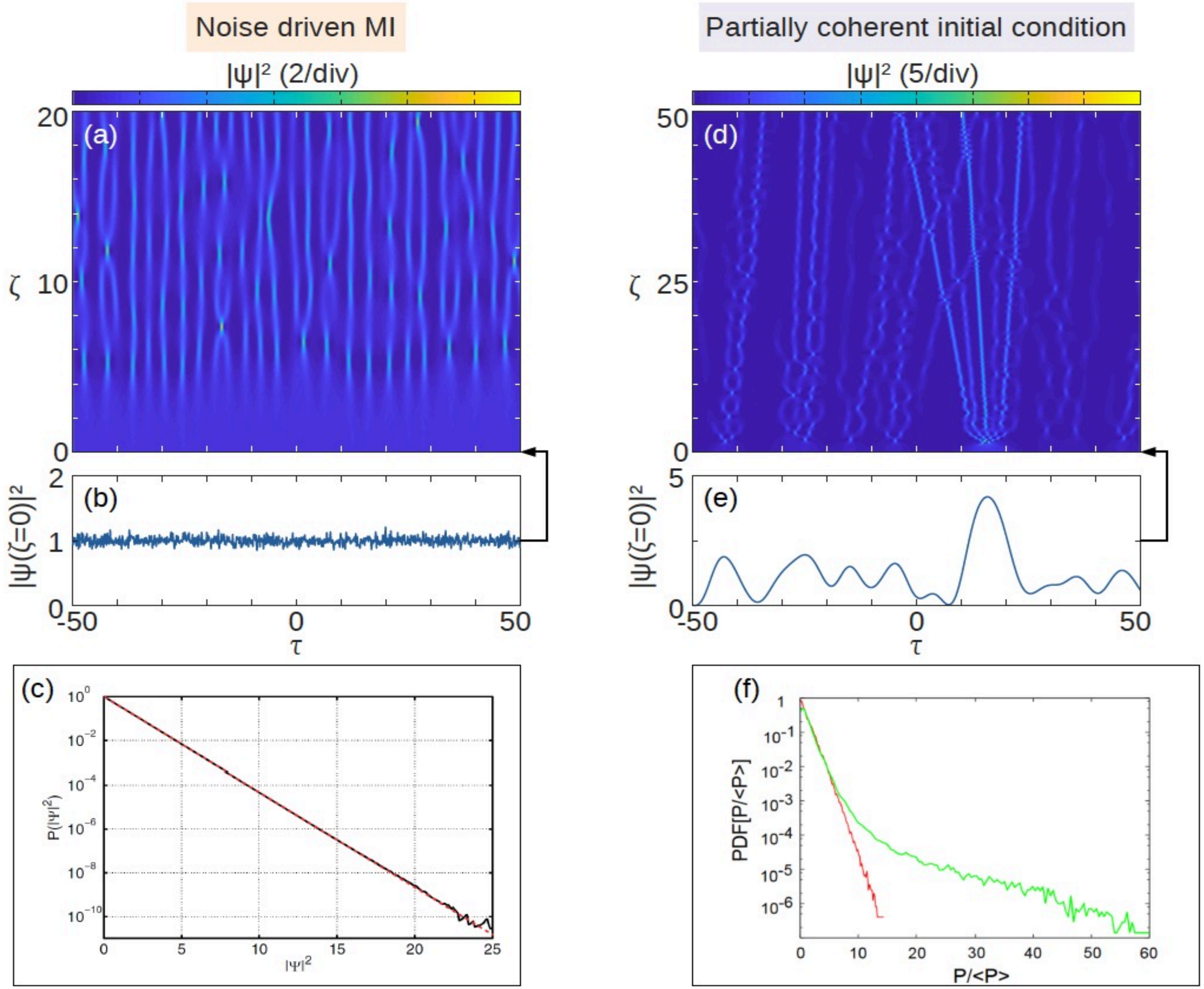
Influence of the initial condition (focusing NLS)



THEORETICAL DESCRIPTION ?

F Copie, S Randoux, P Suret, Reviews in Physics 5, 100037 (2020)

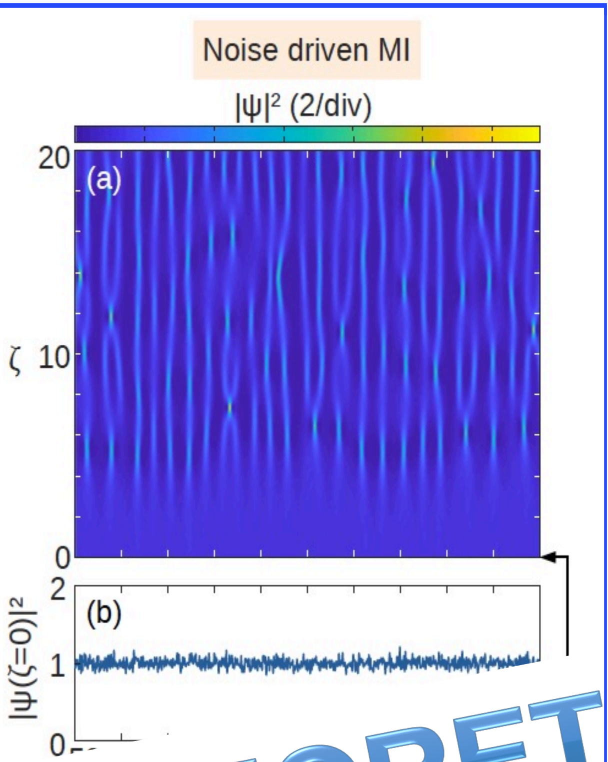
Influence of the initial condition (focusing NLS)

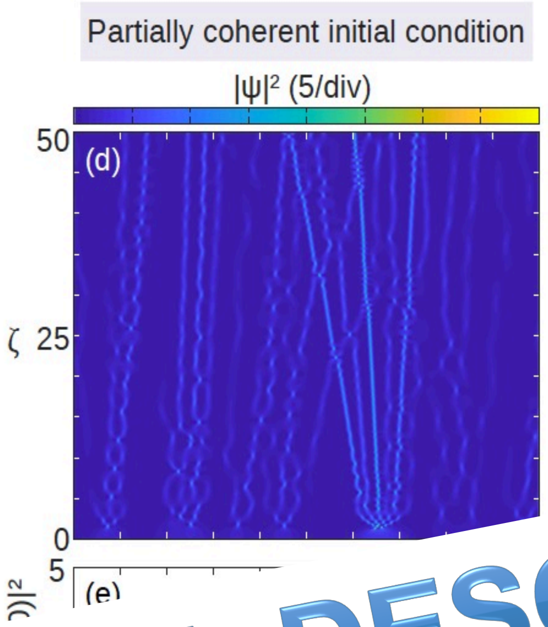


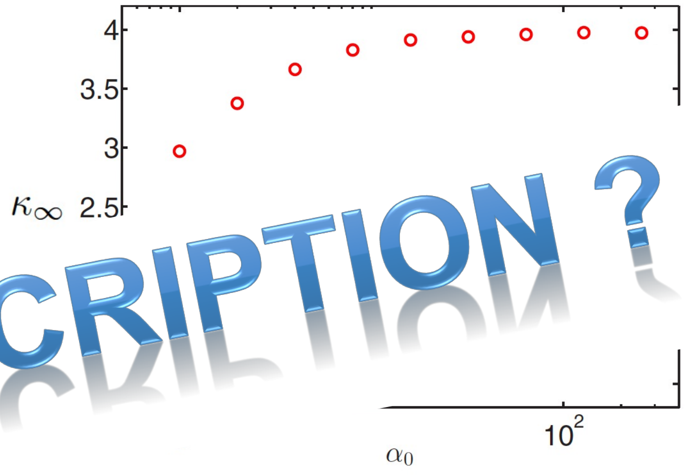
F Copie, S Randoux, P Suret, Reviews in Physics 5, 100037 (2020)

Kurtosis (focusing NLS)

$$\kappa = \frac{\langle |\psi|^4 \rangle}{\langle |\psi|^2 \rangle^2}$$







THEORETICAL DESCRIPTION ?

$$\kappa_\infty = 2$$

$$\max(\kappa_\infty) = 4$$

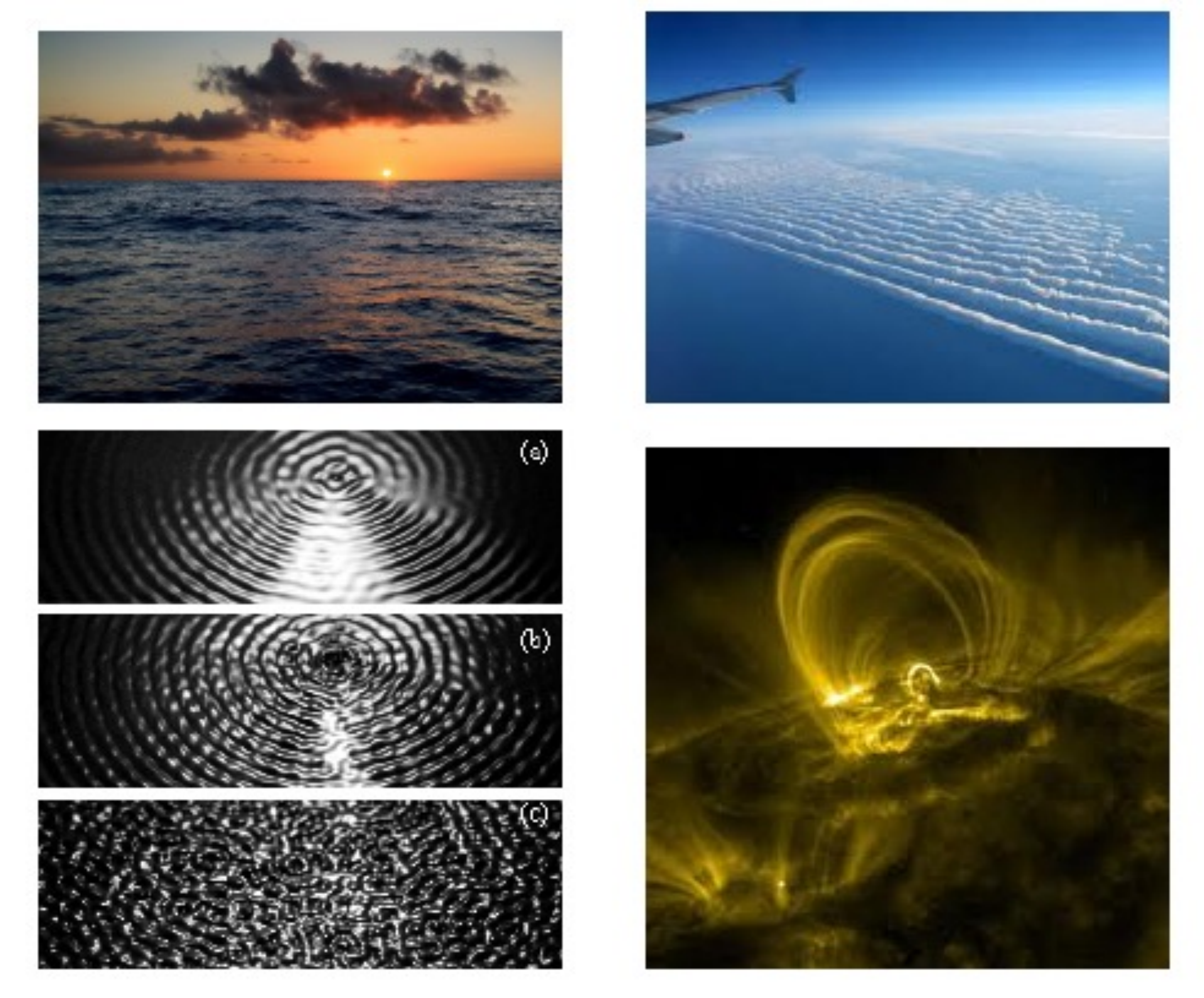
$$\alpha_0 = \frac{|\langle H_{nl} \rangle|}{H_l}\bigg|_{t=0} = \frac{\langle |\psi|^2 \rangle}{\delta k^2}\bigg|_{t=0}$$

$$\delta k^2 = \frac{\sum_k k^2 |\psi_k|^2}{\sum_k |\psi_k|^2}$$

D. Agafontsev, S. Randoux and P. Suret, Phys. Rev. E, 103, 032209 (2021)

Wave Turbulence

- ✓ **Waves**
- ✓ **Dispersion**
- ✓ **Nonlinearity**
- ✓ **randomness**



Wave turbulence = nonlinear dispersive random waves

hydrodynamics, oceanography, optics, mechanics, plasma physics…