From N-solitons to a gas.

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Encoding solitons in a Riemann-Hilbert problem

- Solitons are eigenvalues of scattering problem: $\mathscr{L}\mathbf{w} = z_k\mathbf{w}, \qquad \mathbf{w}_1^-(z_k) = \gamma_k \mathbf{w}_2^+(z_k)$
- Get encoded into RH problem as (simple) poles with prescribed residues

$$\mathbf{M}(z) := \left[a(z)^{-1}\mathbf{W}_1^-(z), \mathbf{W}_2^+(z)\right] e^{izx\sigma_3} \qquad \Rightarrow \qquad \operatorname*{Res}_{z=z_k} \mathbf{M}(z) = \lim_{z\to z_k} \mathbf{M}(z) \begin{bmatrix} 0 & 0 \\ c_k e^{2i\theta(z_k;x,t)} & 0 \end{bmatrix}$$

$$c_k := \frac{\gamma_k}{a'(z_k)}$$

- Can be encoded differently by changing the normalization of $\mathbf{M}(z)$

$$\widetilde{\mathbf{M}}(z) := \left[\mathbf{W}_1^-(z), a(z)^{-1}\mathbf{W}_2^+(z)\right] e^{izx\sigma_3} \qquad \Rightarrow \qquad \operatorname*{Res}_{z=z_k} \mathbf{M}(z) = \lim_{z\to z_k} \mathbf{M}(z) \begin{bmatrix} 0 & \widetilde{c}_k e^{-2i\theta(z_k;x,t)} \\ 0 & 0 \end{bmatrix}$$

$$\widetilde{c}_k = \frac{1}{a'(z_k)^2}\frac{1}{c_k} = \frac{1}{\gamma_k}\frac{1}{a'(z_k)}$$

Encoding solitons in a Riemann-Hilbert problem

- Solitons are eigenvalues of scattering problem: $\mathscr{L}\mathbf{w} = z_k\mathbf{w}$, $\quad \mathbf{w}_1^-(z_k) = \gamma_k w_2^+(z_k)$

- More generally, you can factor the scattering coefficient: $a(z) = a_R(z)a_L(z)$

$$\widehat{\mathbf{M}}(z) := \left[\frac{\mathbf{w}_1^-(z)}{a_R(z)}, \frac{\mathbf{w}_2^+(z)}{a_L(z)}\right] e^{izx\sigma_3}$$

  - If $a_R(z_k) = 0,\ a_L(z_k) \neq 0$:

$$\operatorname*{Res}_{z=z_k} \widehat{\mathbf{M}}(z) = \lim_{z\to z_k} \widehat{\mathbf{M}}(z) \begin{bmatrix} 0 & 0 \\ \widehat{c}_{R,k}\, e^{2i\theta(z_k;x,t)} & 0 \end{bmatrix}, \quad \widehat{c}_{R,k} = \gamma_k \frac{a_L(z_k)}{a_R'(z_k)} = c_k a_L(z_k)^2$$

  - If $a_R(z_k) \neq 0,\ a_L(z_k) = 0$:

$$\operatorname*{Res}_{z=z_k} \widehat{\mathbf{M}}(z) = \lim_{z\to z_k} \widehat{\mathbf{M}}(z) \begin{bmatrix} 0 & \widehat{c}_{L,k}\, e^{-2i\theta(z_k;x,t)} \\ 0 & 0 \end{bmatrix}, \quad \widehat{c}_{L,k} = \frac{1}{\gamma_k}\frac{a_R(z_k)}{a_L'(z_k)} = \frac{1}{c_k}\frac{1}{a_L'(z_k)^2}$$

Encoding solitons in a Riemann-Hilbert problem

- At the level of the RHP we “flip the triangularity” of residues

$$\mathbf{M}(z_k) \text{ has simple poles at each } z_k \in \mathcal{Z}: \qquad \operatorname*{Res}_{z=z_k} \mathbf{M}(z) = \lim_{z\to z_k} \mathbf{M}(z) \begin{bmatrix} 0 & 0 \\ c_k e^{2i\theta(z_k;x,t)} & 0 \end{bmatrix}$$

- Make a change of variable: $\mathcal{Z} = \mathcal{Z}_L \cup \mathcal{Z}_R, \quad \mathcal{Z}_L \cap \mathcal{Z}_R = \emptyset$

$$\widehat{\mathbf{M}}(z) = \mathbf{M}(z) \left[\prod_{z_k \in \mathcal{Z}_L} \left(\frac{z - z_k}{z - z_k^*} \right) \right]^{\sigma_3}$$

▶ For $z_k \in \mathcal{Z}_R$: $\displaystyle \operatorname*{Res}_{z=z_k} \widehat{\mathbf{M}}(z) = \lim_{z\to z_k} \widehat{\mathbf{M}}(z) \begin{bmatrix} 0 & 0 \\ \widehat{c}_{R,k}\, e^{2i\theta(z_k;x,t)} & 0 \end{bmatrix}, \quad \widehat{c}_{R,k} = c_k \left(\prod_{z_\ell \in \mathcal{Z}_L} \frac{z_k - z_\ell}{z_k - z_\ell^*} \right)^2$

▶ For $z_k \in \mathcal{Z}_L$: $\displaystyle \operatorname*{Res}_{z=z_k} \widehat{\mathbf{M}}(z) = \lim_{z\to z_k} \widehat{\mathbf{M}}(z) \begin{bmatrix} 0 & \widehat{c}_{L,k}\, e^{-2i\theta(z_k;x,t)} \\ 0 & 0 \end{bmatrix}, \quad \widehat{c}_{L,k} = \frac{1}{c_k} \left(\frac{1}{2\,\mathrm{Im}(z_k)} {\prod_{z_\ell \in \mathcal{Z}_L}}' \frac{z_k - z_\ell}{z_k - z_\ell^*} \right)^{-2}$

Using soliton flipping to recover phase shift

- One soliton solution with data: $z_0 = \xi_0 + i\eta_0, \quad c_0 = -2i\eta_0 e^{2\eta_0 x_0} e^{i\phi_0}$

$$\psi_{\text{sol}}(x,t) = 2\eta_0 \operatorname{sech}\left(2\eta_0(x + 2\xi_0 t - x_0)\right) e^{-2i(\xi_0 x + (\xi_0^2 - \eta_0^2)t)} e^{-i\phi_0}$$

- Two soliton with data: $\{(z_1, c_1), (z_2, c_2)\}$ $\quad z_k = \xi_k + i\eta_k \quad c_k = -2i\eta_k e^{2\eta_k x_k} e^{i\phi_k}$

1. $\mathbf{M}(z) = I + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.
2. $\mathbf{M}(z) = \sigma_2 M(z^*)^* \sigma_2$ for all $z \in \mathbb{C}$.
3. $\mathbf{M}(z)$ has simple poles at each z_k and z_k^* with

Student exercise:
Compute the exact formula
for the two-soliton.

$$\operatorname*{Res}_{z=z_k} \mathbf{M}(z) = \lim_{z \to z_k} \mathbf{M}(z) \begin{bmatrix} 0 & 0 \\ c_k e^{2i\theta(z_k;x,t)} & 0 \end{bmatrix} \qquad \operatorname{Re}(2i\theta(z_k;x,t)) = -2\eta_k(x + 2\xi_k t)$$

✦ ***Key idea:*** For finitely many solitons, exponentially small residues can be ignored at the cost of exponentially small errors.

Using soliton flipping to recover phase shift

$$\operatorname*{Res}_{z=z_k} \mathbf{M}(z) = \lim_{z \to z_k} \mathbf{M}(z) \begin{bmatrix} 0 & 0 \\ c_k e^{2i\theta(z_k;x,t)} & 0 \end{bmatrix}$$

$$\mathrm{Re}(2i\theta(z_k;x,t)) = -2\eta_k(x+2\xi_k t)$$

As $t \to +\infty$, we consider five cases (assume w.l.o.g. $\xi_1 < \xi_2$):

1. $x \geq -2(\xi_1 - \epsilon)t$: Both exponentials small

$$\mathrm{Re}(2i\theta(z_1;x,t)) \leq -4t\epsilon\eta_1$$
$$\mathrm{Re}(2i\theta(z_2;x,t)) \leq -4\eta_2 t(\xi_2 - \xi_1 + \epsilon)$$

$$\psi(x,t) = \mathcal{O}\left(e^{-ct}\right)$$

Using soliton flipping to recover phase shift

$$\operatorname*{Res}_{z=z_k} \mathbf{M}(z) = \lim_{z\to z_k} \mathbf{M}(z) \begin{bmatrix} 0 & 0 \\ c_k e^{2i\theta(z_k;x,t)} & 0 \end{bmatrix} \qquad \mathrm{Re}(2i\theta(z_k;x,t)) = -2\eta_k(x+2\xi_k t)$$

As $t \to +\infty$, we consider five cases (assume w.l.o.g. $\xi_1 < \xi_2$):

2. $x = -2\xi_1 t + o(t)$: Residue at z_2 negligible: $\mathrm{Re}(2i\theta(z_2;x,t)) \leq -4\eta_2 t(\xi_2 - \xi_1) + o(t)$

- Dropping the residue at z_2 gives a one soliton problem to solve with data (z_1, c_1)

$$|\psi(x,t)| = 2\eta_1 \operatorname{sech}\left(2\eta_1(x+2\xi_1 t - x_1)\right) \qquad x = -2\xi_1 t + o(t),\ t\to\infty$$

Using soliton flipping to recover phase shift

$$\operatorname*{Res}_{z=z_k} \mathbf{M}(z) = \lim_{z \to z_k} \mathbf{M}(z) \begin{bmatrix} 0 & 0 \\ c_k e^{2i\theta(z_k;x,t)} & 0 \end{bmatrix}$$

$$\mathrm{Re}(2i\theta(z_k;x,t)) = -2\eta_k(x + 2\xi_k t)$$

As $t \to +\infty$, we consider five cases (assume w.l.o.g. $\xi_1 < \xi_2$):

3. $-2(\xi_2 - \epsilon)t < x < -2(\xi_1 + \epsilon)t$: Residue at z_1 now large!!

$$\mathrm{Re}(2i\theta(z_1;x,t)) \geq 4\eta_1 \epsilon t$$
$$\mathrm{Re}(2i\theta(z_2;x,t)) \leq -4\eta_2 \epsilon t$$

- To deal with the exponentially large residue at z_1 we flip the triangularity:

$$\widehat{\mathbf{M}}(z) = \mathbf{M}(z) \left(\frac{z - z_1}{z - z_1^*} \right)^{\sigma_3}$$

$$\operatorname*{Res}_{z=z_1} \widehat{\mathbf{M}}(z) = \lim_{z \to z_1} \widehat{\mathbf{M}}(z) \begin{bmatrix} 0 & \widehat{c}_1 \, e^{-2i\theta(z_1;x,t)} \\ 0 & 0 \end{bmatrix}, \quad \widehat{c}_1 = \frac{1}{c_1} \left(\frac{1}{2\,\mathrm{Im}(z_1)} \right)^{-2}$$

$$\operatorname*{Res}_{z=z_2} \widehat{\mathbf{M}}(z) = \lim_{z \to z_2} \widehat{\mathbf{M}}(z) \begin{bmatrix} 0 & 0 \\ \widehat{c}_2 \, e^{2i\theta(z_2;x,t)} & 0 \end{bmatrix}, \quad \widehat{c}_2 = c_2 \left(\frac{z_2 - z_1}{z_2 - z_1^*} \right)^2$$

- All residues are exponentially small again: $\psi(x,t) = \mathcal{O}\left(e^{-ct}\right)$

Using soliton flipping to recover phase shift

$$\operatorname*{Res}_{z=z_k} \mathbf{M}(z) = \lim_{z \to z_k} \mathbf{M}(z) \begin{bmatrix} 0 & 0 \\ c_k e^{2i\theta(z_k;x,t)} & 0 \end{bmatrix} \qquad \mathrm{Re}(2i\theta(z_k;x,t)) = -2\eta_k(x + 2\xi_k t)$$

As $t \to +\infty$, we consider five cases (assume w.l.o.g. $\xi_1 < \xi_2$):

4. $x = -2\xi_2 t + o(t)$:

$$\mathrm{Re}(2i\theta(z_1;x,t)) \geq 4\eta_1 t(\xi_2 - \xi_1) + o(t)$$
$$\mathrm{Re}(2i\theta(z_2;x,t)) = o(t)$$

- Upper triangular residue at z_1 is negligible:

$$\widehat{\mathbf{M}}(z) = \mathbf{M}(z) \left(\frac{z - z_1}{z - z_1^*} \right)^{\sigma_3} \qquad \operatorname*{Res}_{z=z_2} \widehat{\mathbf{M}}(z) = \lim_{z \to z_2} \widehat{\mathbf{M}}(z) \begin{bmatrix} 0 & 0 \\ \widehat{c}_2\, e^{2i\theta(z_2;x,t)} & 0 \end{bmatrix}, \quad \widehat{c}_2 = c_2 \left(\frac{z_2 - z_1}{z_2 - z_1^*} \right)^2$$

- See one soliton with a shifted position:

$$|\psi(x,t)| = 2\eta_2 \operatorname{sech}\left(2\eta_2(x + 2\xi_2 t - x_2 - \Delta_2)\right) \quad x = -2\xi_2 t + o(t), \qquad \Delta_2 = \frac{1}{\eta_2} \log \left| \frac{z_2 - z_1}{z_2 - z_1^*} \right|$$

Using soliton flipping to recover phase shift

$$\operatorname*{Res}_{z=z_k} \mathbf{M}(z) = \lim_{z\to z_k} \mathbf{M}(z) \begin{bmatrix} 0 & 0 \\ c_k e^{2i\theta(z_k;x,t)} & 0 \end{bmatrix} \qquad \mathrm{Re}(2i\theta(z_k;x,t)) = -2\eta_k(x+2\xi_k t)$$

As $t \to +\infty$, we consider five cases (assume w.l.o.g. $\xi_1 < \xi_2$):

5. $x \le -2(\xi_2+\epsilon)t$:

$$\mathrm{Re}(2i\theta(z_1;x,t)) \ge 4\eta_1 t(\xi_2 - \xi_1 + \epsilon)$$
$$\mathrm{Re}(2i\theta(z_2;x,t)) \ge 4\eta_2 \epsilon t$$

- Now we flip both triangularities to get to a problem with small residues:

$$\widehat{\mathbf{M}}(z) = \mathbf{M}(z)\left(\prod_{k=1}^{2} \frac{z-z_k}{z-z_k^*}\right)^{\sigma_3} \qquad \operatorname*{Res}_{z=z_k} \widehat{\mathbf{M}}(z) = \lim_{z\to z_k} \widehat{\mathbf{M}}(z) \begin{bmatrix} 0 & \widehat{c}_k\, e^{-2i\theta(z_k;x,t)} \\ 0 & 0 \end{bmatrix},$$

- Again get uniform decay estimate:

$$\psi(x,t) = \mathcal{O}\left(e^{-ct}\right)$$

Using soliton flipping to recover phase shift

- The argument we just gave shows that as $t \to \infty$, 2-soliton can be expressed as

$$\psi(x,t) = \psi_{\text{sol}}(x - x_1^+, t; z_1) + \psi_{\text{sol}}(x - x_2^+, z_2) + \mathcal{O}\left(e^{-ct}\right), \qquad t \to +\infty$$

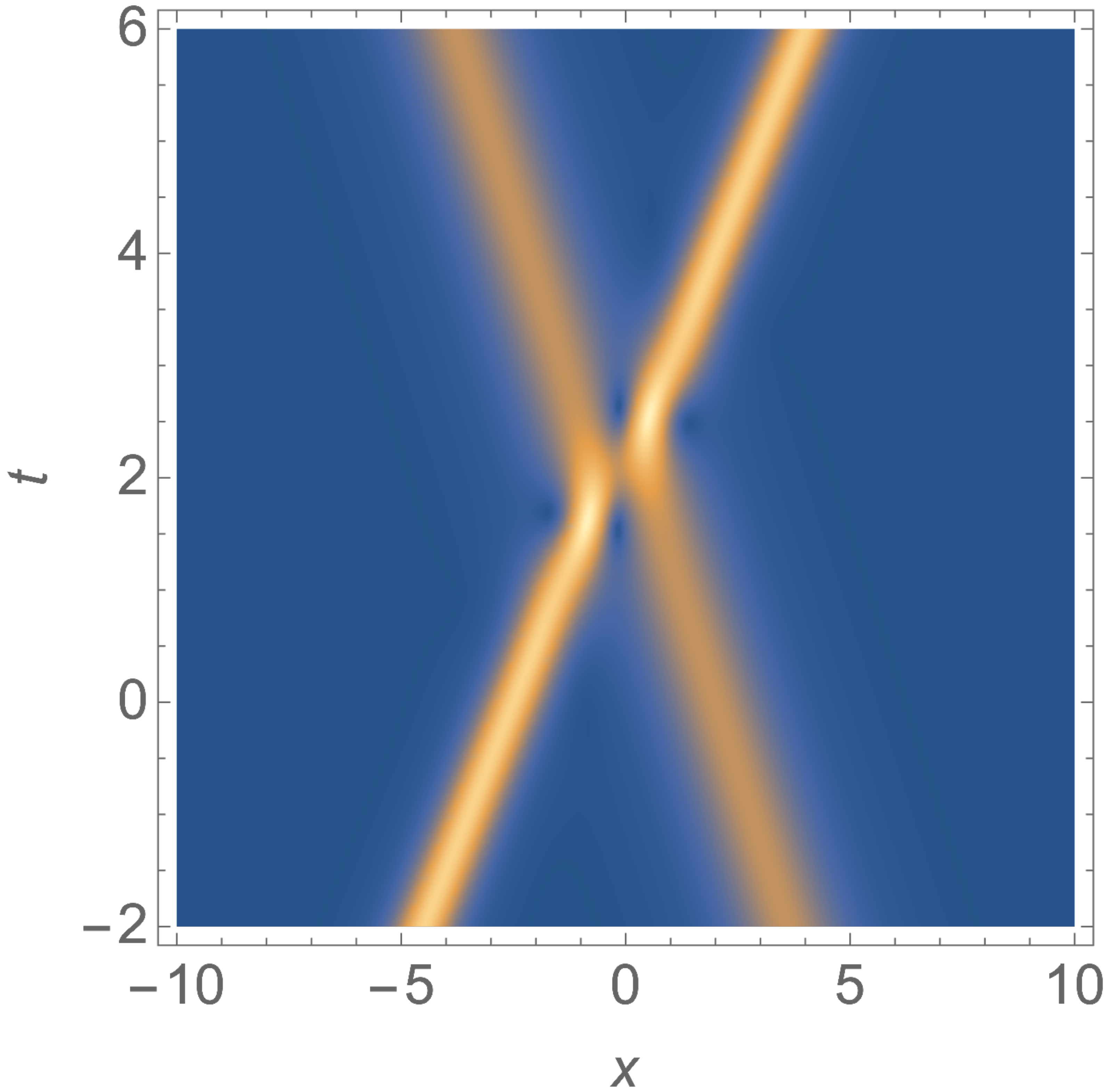
- The argument can be repeated for $t \to -\infty$, with minor alterations

$$\psi(x,t) = \psi_{\text{sol}}(x - x_1^-, t; z_1) + \psi_{\text{sol}}(x - x_2^-, z_2) + \mathcal{O}\left(e^{-c|t|}\right), \qquad t \to -\infty$$

- The phase shifts can be explicitly calculated:

$$\Delta_1 := x_1^+ - x_1^- = \frac{1}{\operatorname{Im} z_1} \log\left|\frac{z_1 - z_2^*}{z_1 - z_2}\right|$$

$$\Delta_2 := x_2^+ - x_2^- = -\frac{1}{\operatorname{Im} z_2} \log\left|\frac{z_2 - z_1^*}{z_2 - z_1}\right|$$



Analyzing many soliton problems

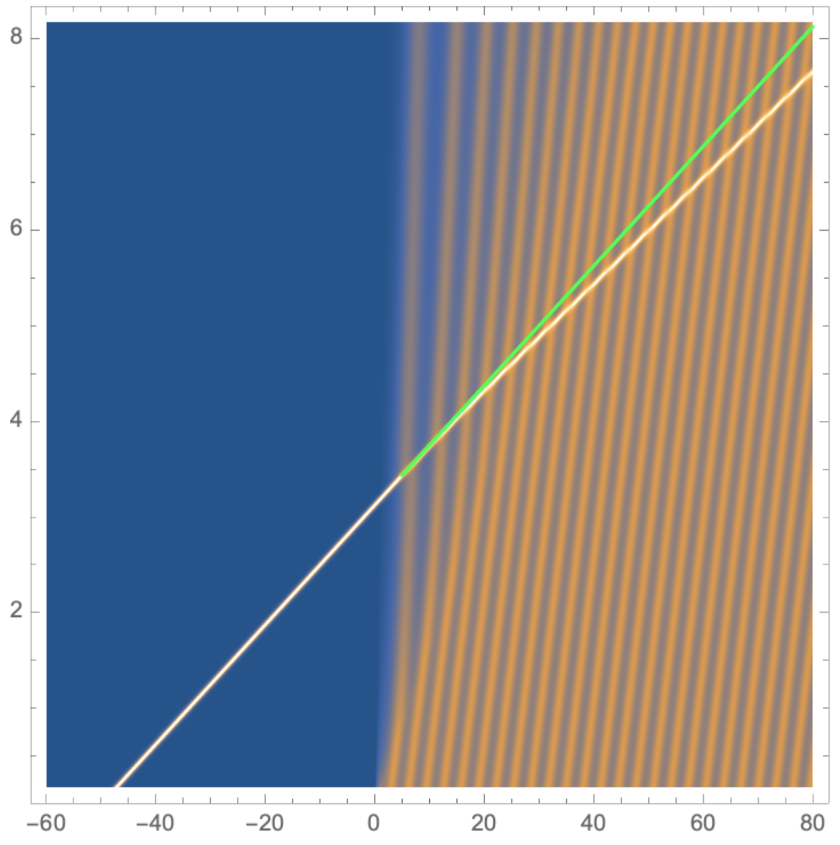
- For ***finite and fixed*** number N, the 2-soliton analysis extends to N-soliton problem

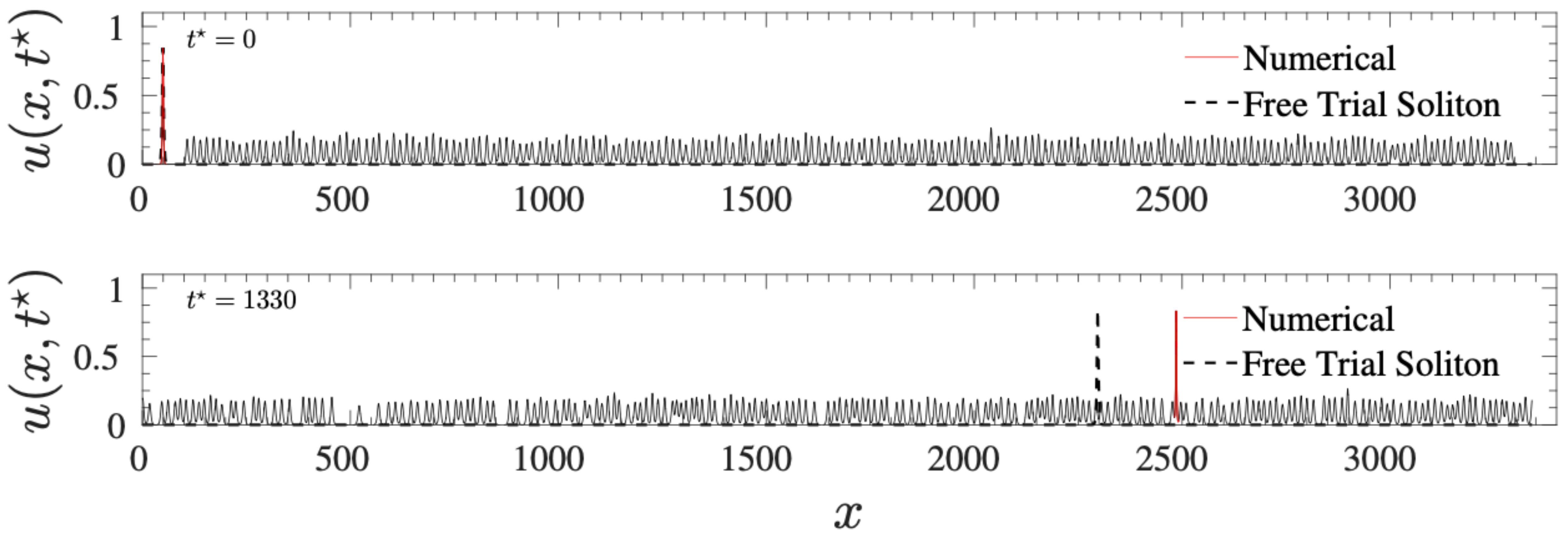
$$\Delta_k := \frac{1}{\operatorname{Im} z_k} \sum_{j \neq k} \operatorname{sgn}(\operatorname{Re}(z_j - z_k)) \log \left| \frac{z_k - z_j^*}{z_k - z_j} \right|, \quad k = 1, \ldots, N$$

- If N grows at a rate proportional to (x, t) analysis will become much more delicate:

$$\widehat{c}_k := c_k e^{2i\theta(z_k; x, t)} \left(\prod_{z_j \in \mathcal{Z}_L} \frac{z_k - z_j}{z_k - z_j^*} \right)^2 \qquad \left(\prod_{z_j \in \mathcal{Z}_L} \frac{z_k - z_j}{z_k - z_j^*} \right) \sim e^{N\Phi(z_k)} \qquad \text{if } |\mathcal{Z}_L| = cN \text{ and } N \gg 1$$

- The accumulation of many solitons interactions may begin to effect the macroscopic soliton velocity





Analyzing many soliton problems

- This idea was originally observed by Zakharov (JETP 1971) in context of KdV equation:

$$(k = i\kappa, i\chi) \qquad \Longrightarrow \qquad u_{\mathrm{sol}}(x,t) = 2\kappa\,\mathrm{sech}^2(2\kappa(x - 4\kappa^2 t - x_0))$$

- Assume solitons have joint distribution function $f(\kappa; x_0, t)$ which is sufficiently "dilute"

$$s(\kappa) = 4\kappa^2 + \int_0^\infty \frac{1}{\kappa} \log\left|\frac{\kappa+\eta}{\kappa-\eta}\right| (4\kappa^2 - 4\eta^2) f(\eta)\,\mathrm{d}\eta, \qquad f_t + (s(\kappa) f)_x = 0$$

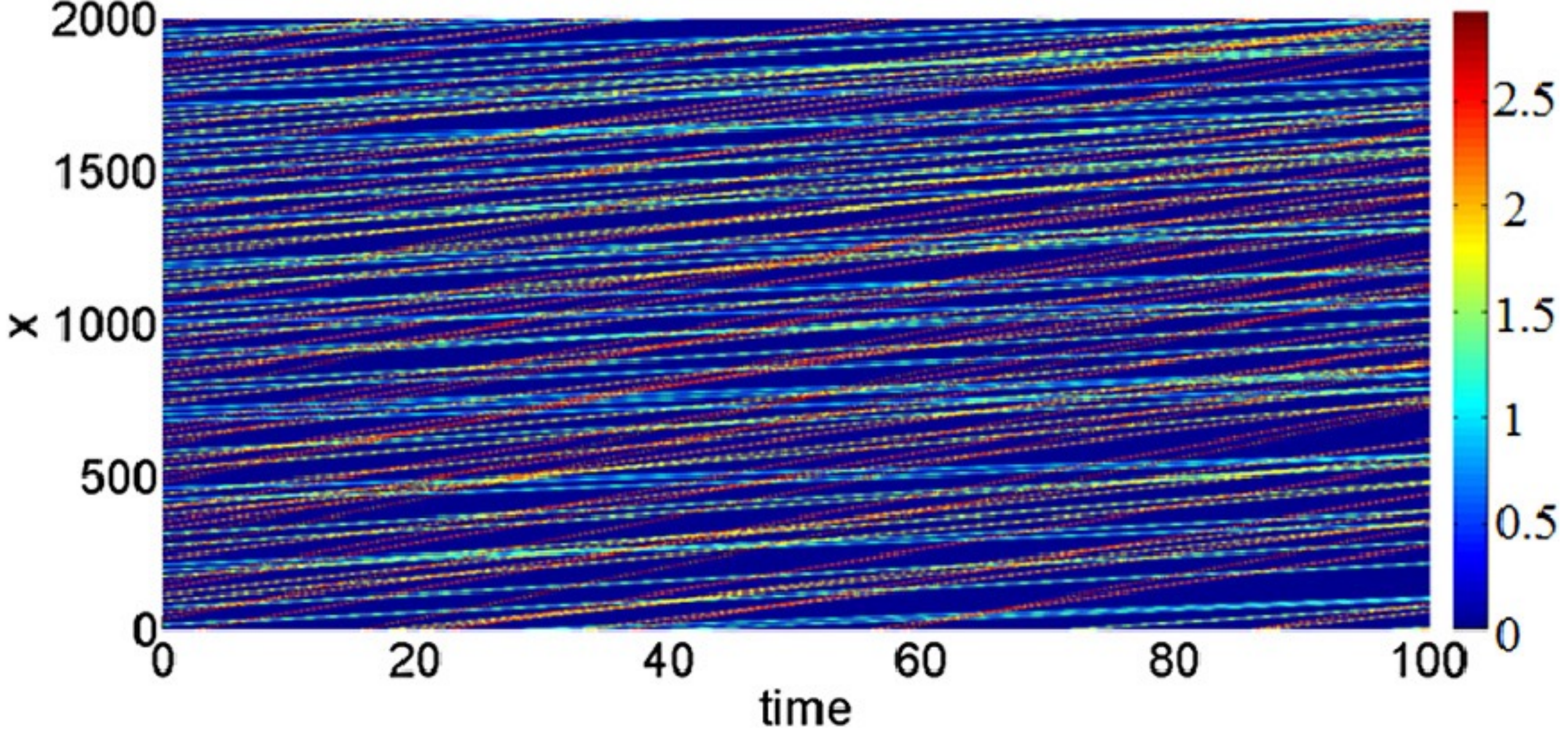
- Derivations for dense "gas" of solitons and extended to other equation (NLS, mKdV, …) by El and collaborators.

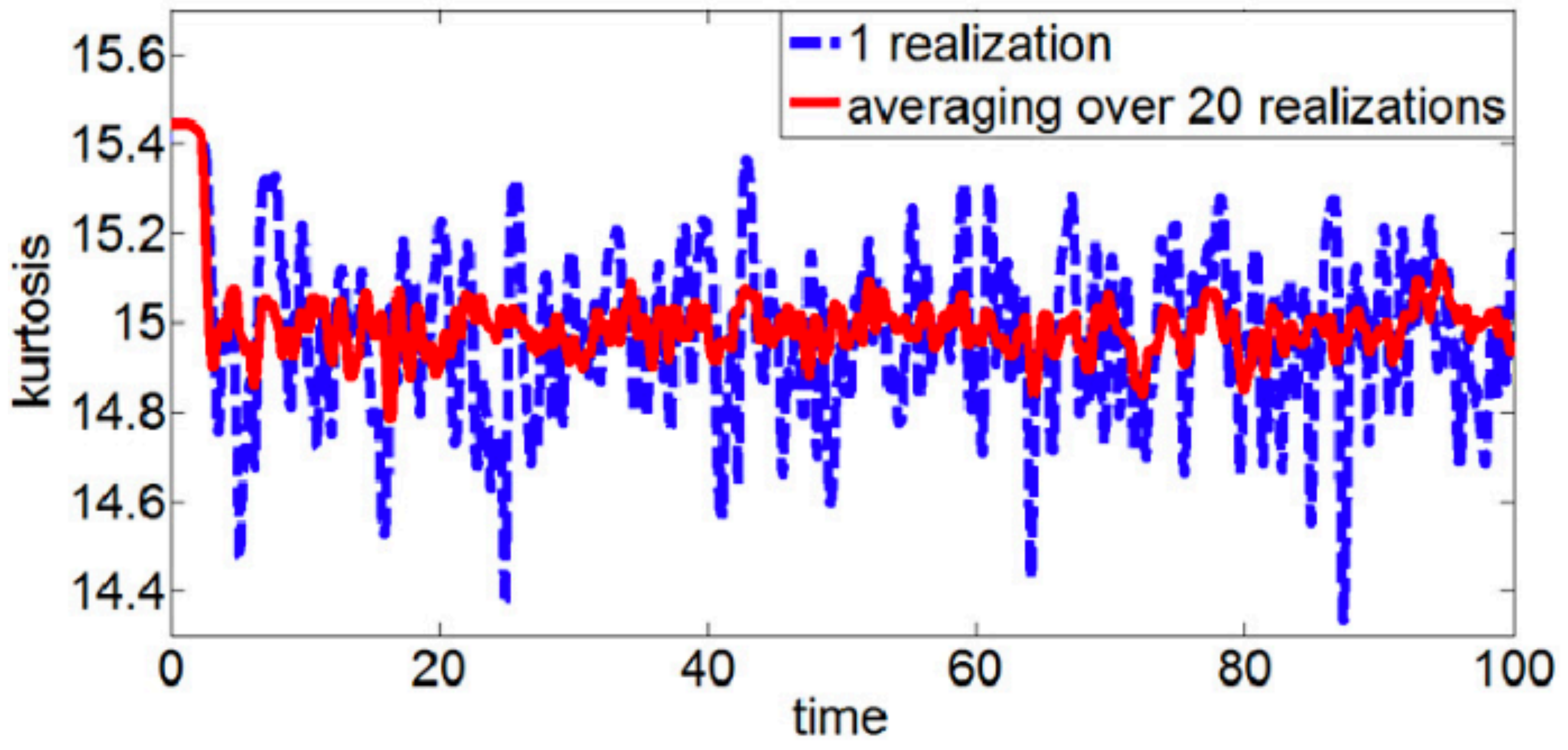
  - Modeled as "thermodynamic limit" of high genus quasi-periodic (finite gap) solutions of the PDE

$$s(\kappa) = 4\kappa^2 + \int_0^\infty \frac{1}{\kappa} \log\left|\frac{\kappa+\eta}{\kappa-\eta}\right| (s(\kappa) - s(\eta)) f(\eta)\,\mathrm{d}\eta \qquad f_t + (s(\kappa) f)_x = 0$$

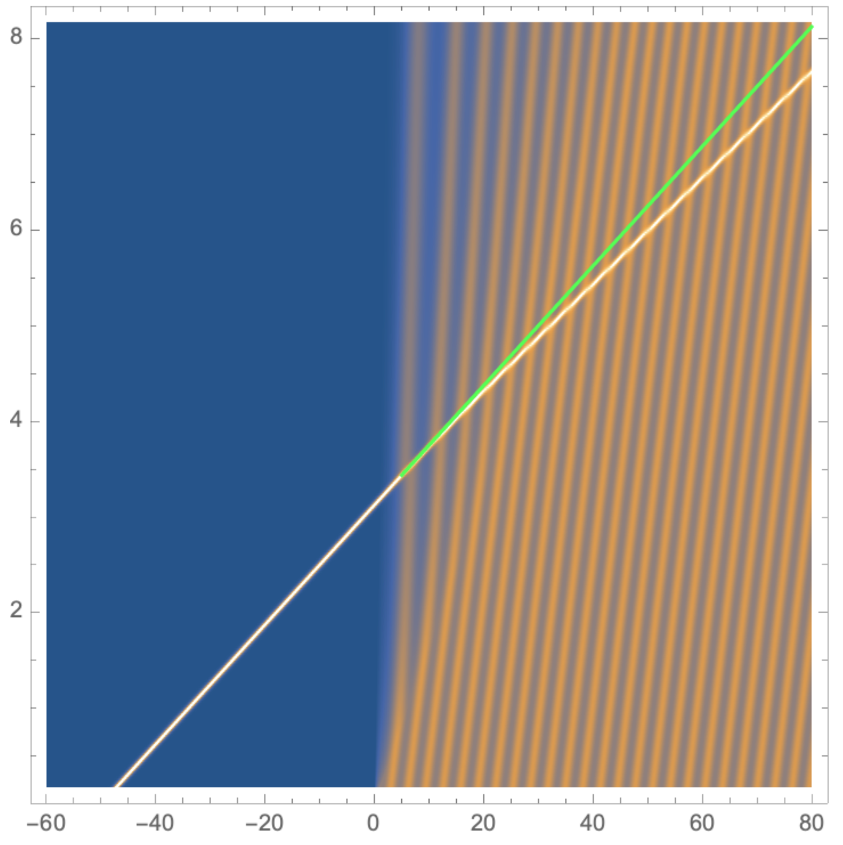
What is a soliton gas?   What's interesting about them?

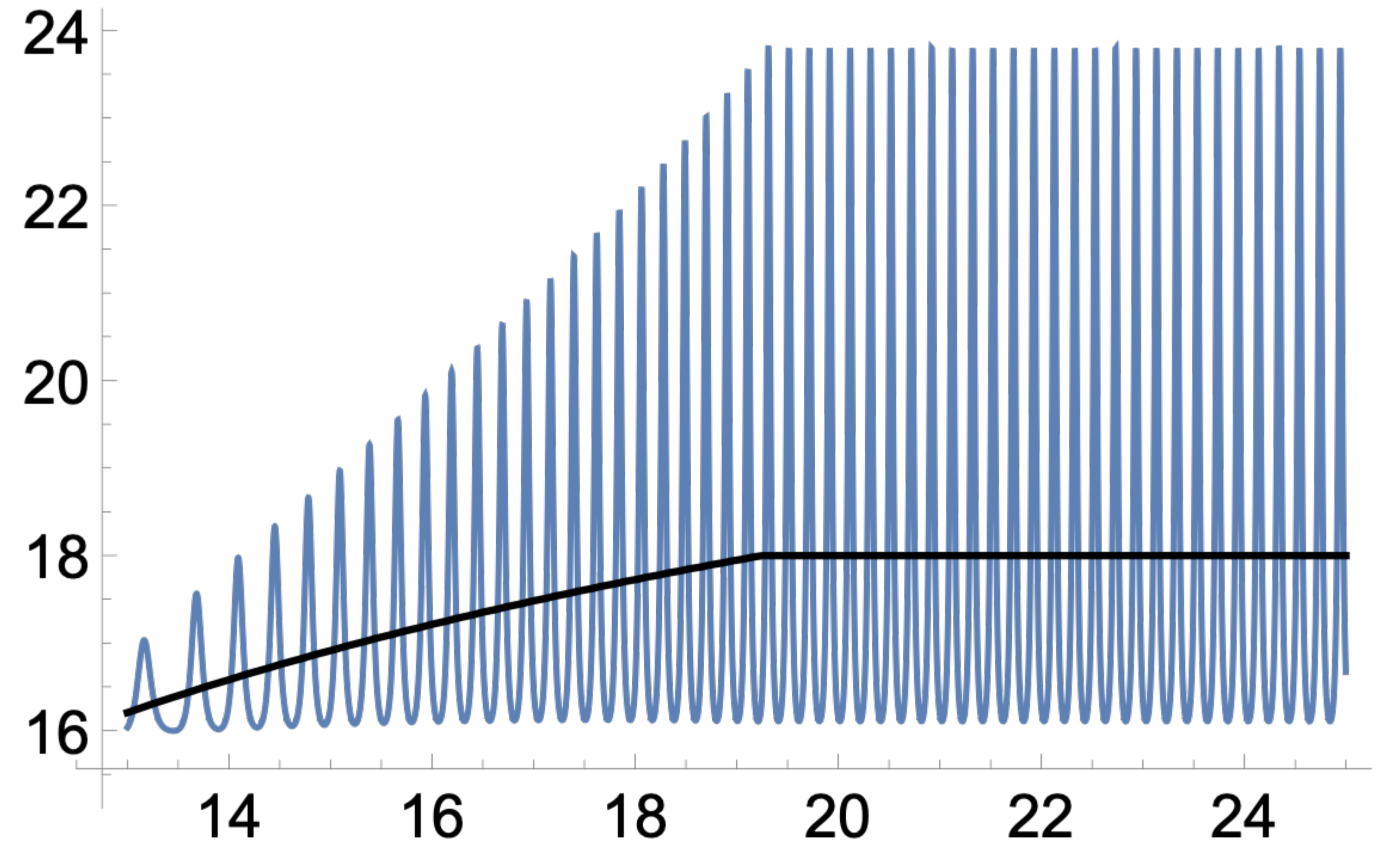
[Shurgaline, Pelinovki, '16]





- Study potentials consisting of many, many solitons
- How do the many soliton-soliton interactions effect the resulting dynamics?
- Can consider the statistics of randomly sampled solitons
- Can study deterministic soliton gases (realizations) to get finer details of interactions

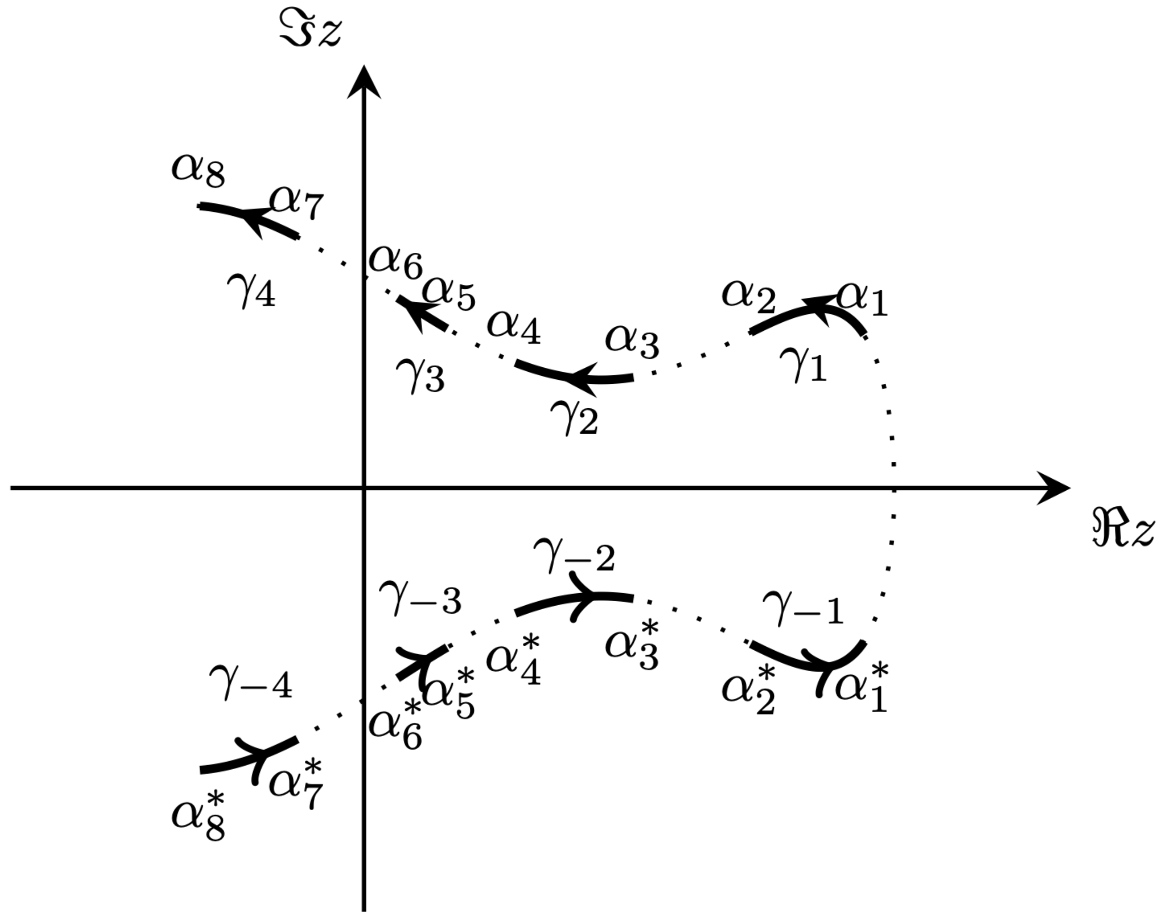




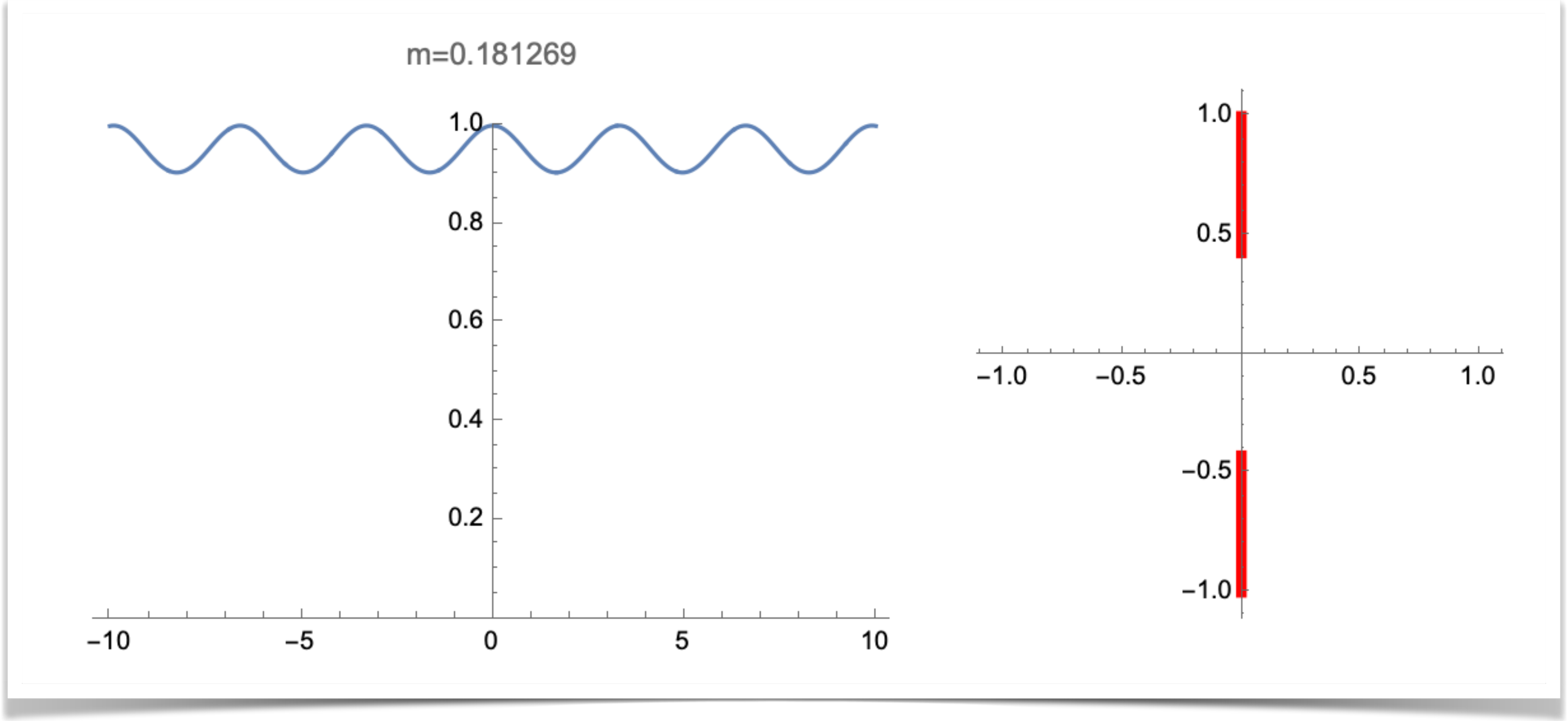
Modeling a soliton gas

- As the number of solitons $N \to \infty$, analysis of exact solutions is analytically intractable

Finite-Gap Models: model soliton gas with exact quasi-period solutions: $\psi = \psi(\theta_1, \ldots, \theta_N)$, $\theta_j = k_j x - \omega_j t + \theta_j^0$



- Solutions modeled by spectral data supported on N bands in spectral z-plane
- As $N \to \infty$, bands shrink to points accumulating on some finite set $\mathscr{A}$



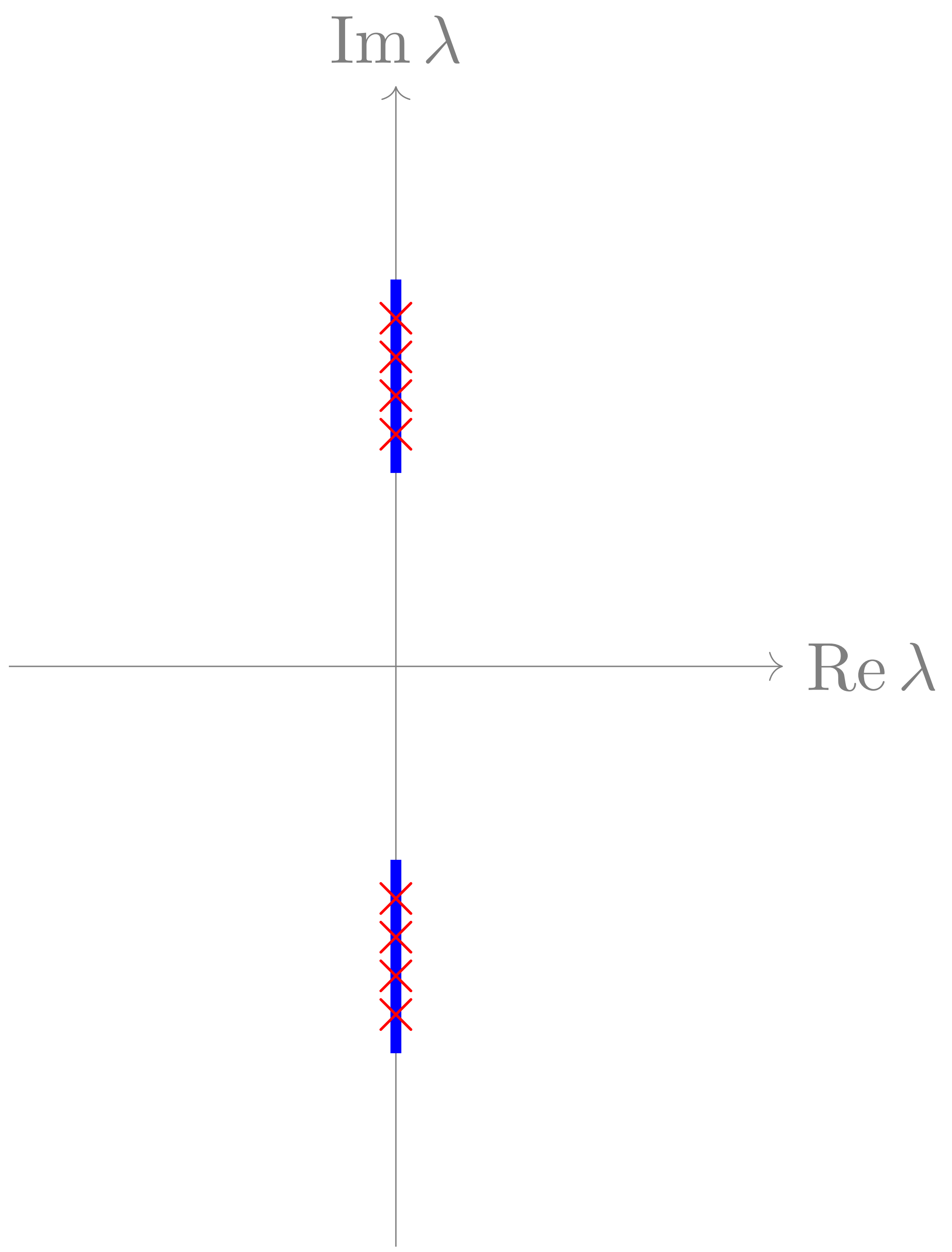
$$\psi(x,t) = \mathrm{dn}(x|m)e^{i(1-m/2)t}$$

Modeling a soliton gas

- As the number of solitons $N \to \infty$, analysis of exact solutions is analytically intractable

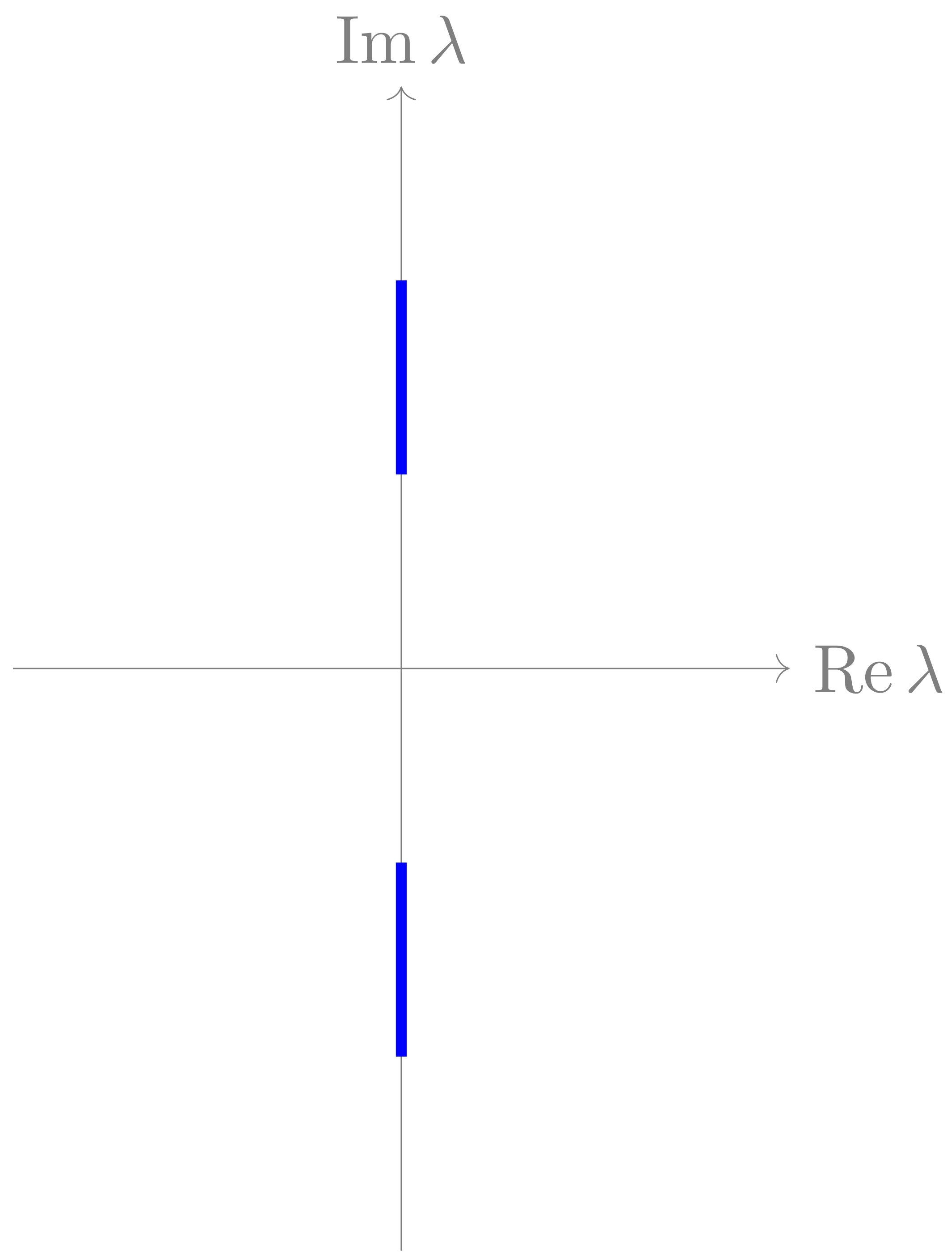
Primitive Potentials: Zakharov *et. al.* (Physica D 2016) formal limit of N-solitons accumulating on a curve

N-soliton problem



$N \to \infty$

Primitive Potential Problem



Questions:
How can me make this limit precise?
In what sense does it converge?
What does a primitive potential look like?

$$\operatorname*{Res}_{z=z_j} \mathbf{M}(z) = \lim_{z\to z_j} \mathbf{M}(z) \begin{bmatrix} 0 & 0 \\ c_j e^{2i\theta(z;x,t)} & 0 \end{bmatrix}$$

$$\mathbf{M}_+(z) = \mathbf{M}_-(z) \begin{bmatrix} \frac{1-R_1R_2}{1+R_1R_2} & \frac{2iR_2}{1+R_1R_2} e^{-2i\theta(z;x,t)} \\ \frac{2iR_2}{1+R_1R_2} e^{2i\theta(z;x,t)} & \frac{1-R_1R_2}{1+R_1R_2} \end{bmatrix}$$

Intermezzo: Interpolating poles in a RH problem

Express $\mathbf{M}(z)$ in terms of its columns: $\mathbf{M}(z) = \left[\mathbf{M}_1(z)\ ,\ \mathbf{M}_2(z)\right]$

The residue condition relates the poles of each column:

$$\operatorname*{Res}_{z=z_j} \mathbf{M}(z) = \lim_{z\to z_j} \mathbf{M}(z) \begin{bmatrix} 0 & 0 \\ c_j e^{2i\theta(z;x,t)} & 0 \end{bmatrix}$$

$$\mathbf{M}_1(z) = \frac{c_j e^{2i\theta(z_j;x,t)}}{z-z_j}\mathbf{M}_2(z_j) + \text{ holomorphic}, \quad z \to z_j$$

$$\mathbf{M}_2(z) = \text{ holomorphic}, \quad z \to z_j$$

The following linear combination is clearly locally holomorphic: $\mathbf{M}_1(z) - \dfrac{c_j}{z-z_j} e^{2i\theta(z;x,t)}\mathbf{M}_2(z)$

Define: $$\mathbf{H}(z) = \begin{cases} \mathbf{M}(z) \begin{bmatrix} 1 & 0 \\ -\frac{c_j}{z-z_j} e^{2i\theta(z;x,t)} & 1 \end{bmatrix} & |z-z_j| < r_0 \\ \mathbf{M}(z) & |z-z_j| > r_0 \end{cases}$$

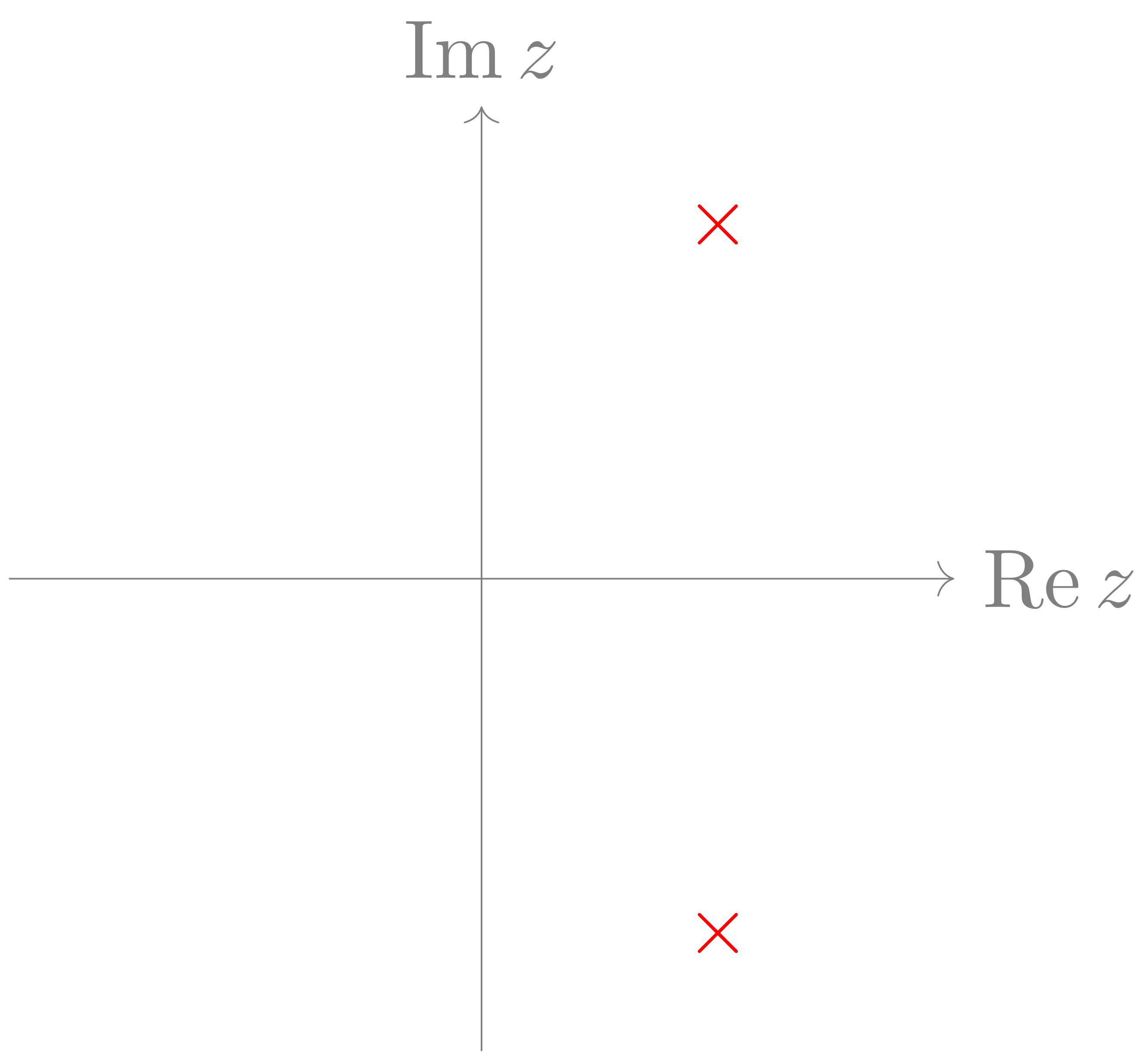
New function $\mathbf{H}(z)$ is now piecewise analytic with a jump across $|z - z_j| = r_0$

Intermezzo: Interpolating poles in a RH problem

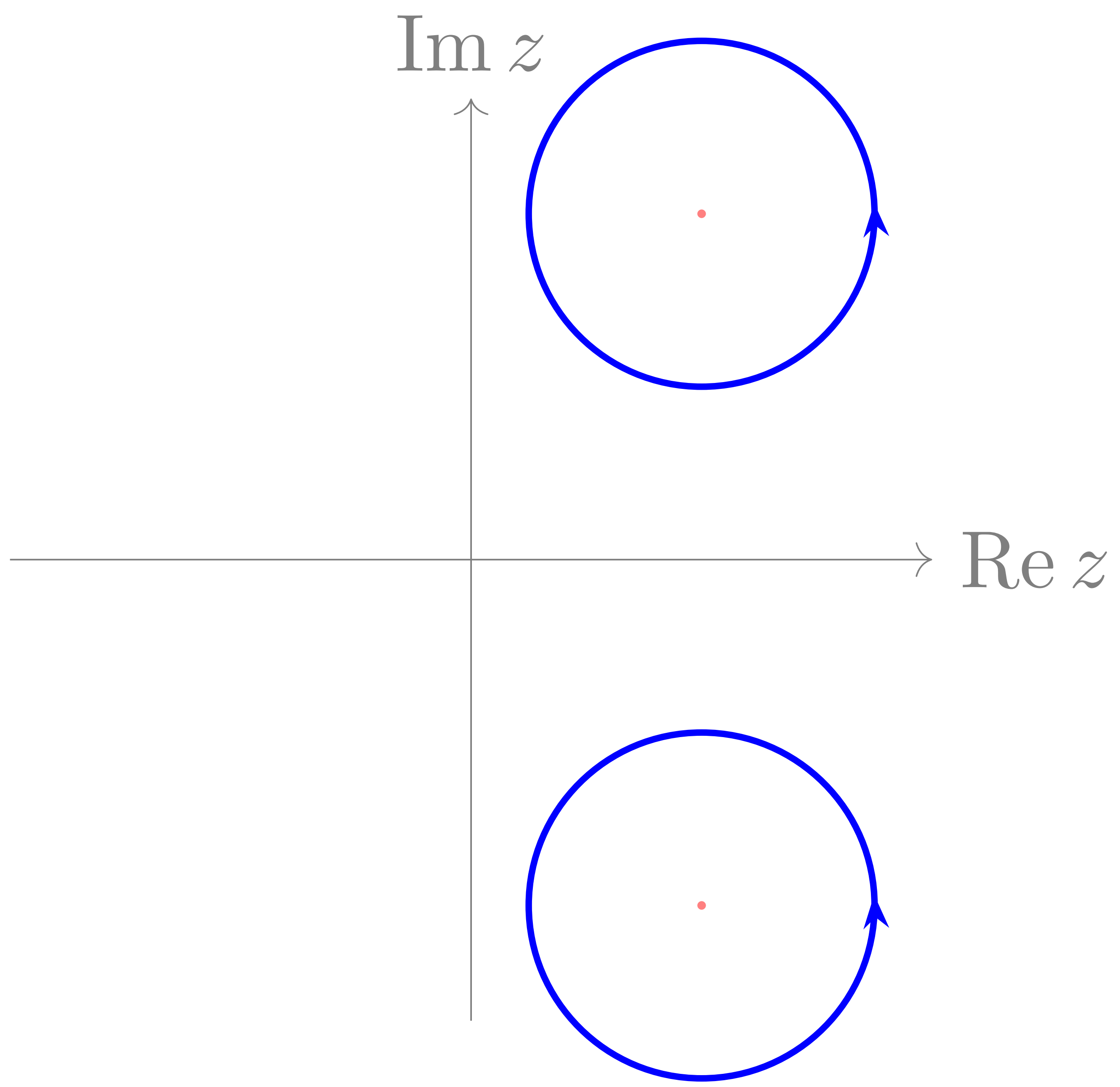
$$\mathbf{H}(z) = \begin{cases} \mathbf{M}(z) \begin{bmatrix} 1 & 0 \\ -\frac{c_j}{z-z_j} e^{2i\theta(z;x,t)} & 1 \end{bmatrix} & |z - z_j| < r_0 \\ \mathbf{M}(z) & |z - z_j| > r_0 \end{cases}$$

$$\mathbf{H}_+(z) = \mathbf{H}_-(z) \begin{bmatrix} 1 & 0 \\ \frac{-c_j}{z - z_j} e^{2i\theta(z;x,t)} & 1 \end{bmatrix}, \qquad |z - z_j| = r$$

Meromorphic problem for $\mathbf{M(z)}$



Holomorphic problem for $\mathbf{H(z)}$



- Purely local: can be repeated for any number of poles
- Algebraic construction: Doesn't depend on the PDE or the scattering background.

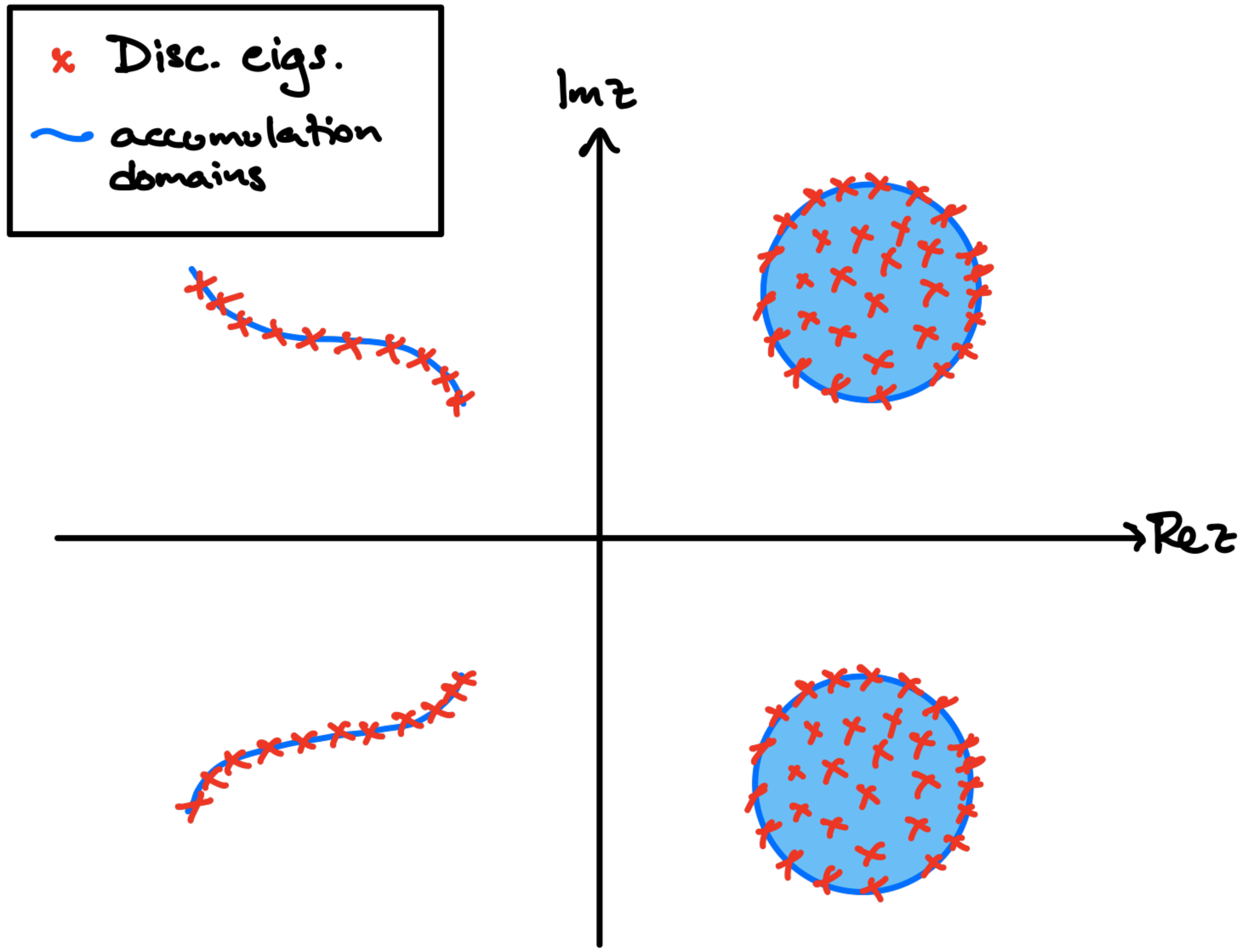
Modeling a soliton gas starting from the n-soliton

- Consider a sequence of n-soliton potentials $\psi_n(x,t)$ for $n = 1,2,\ldots$
- Assume that the poles $z_k^{(n)}$ accumulate on some fixed domain $\mathscr{A}$ (1-D and/or 2-D) .

Given data $\{(z_k^{(n)}, c_k^{(n)})\}_{k=1}^n$, find a 2×2 matrix $\mathbf{m}^{(n)}(z;x,t)$ such that

1. $\mathbf{m}^{(n)}(z)$ is analytic for $z \in \mathbb{C} \setminus \{z_k^{(n)}\}_{k=1}^n$
2. $\mathbf{m}^{(n)}(z) = \mathbf{I} + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$
3. For all $z \in \mathbb{C} \setminus \{z_k^{(n)}\}_{k=1}^n$, $\mathbf{m}^{(n)}(z^*)^* = \boldsymbol{\sigma}_2 \mathbf{m}(z) \boldsymbol{\sigma}_2$
4. $\mathbf{m}^{(n)}(z)$ has simple poles at each $z_k^{(n)}$ (and at $z_k^{(n)*}$) satisfying

$$\operatorname*{Res}_{z=z_k^{(n)}} \mathbf{m}^{(n)}(z) = \lim_{z\to z_k^{(n)}} \mathbf{m}^{(n)}(z) \begin{bmatrix} 0 & 0 \\ c_k^{(n)} \mathrm{e}^{2\mathrm{i}\varphi(z;x,t)} & 0 \end{bmatrix}, \quad k = 1,\ldots n$$

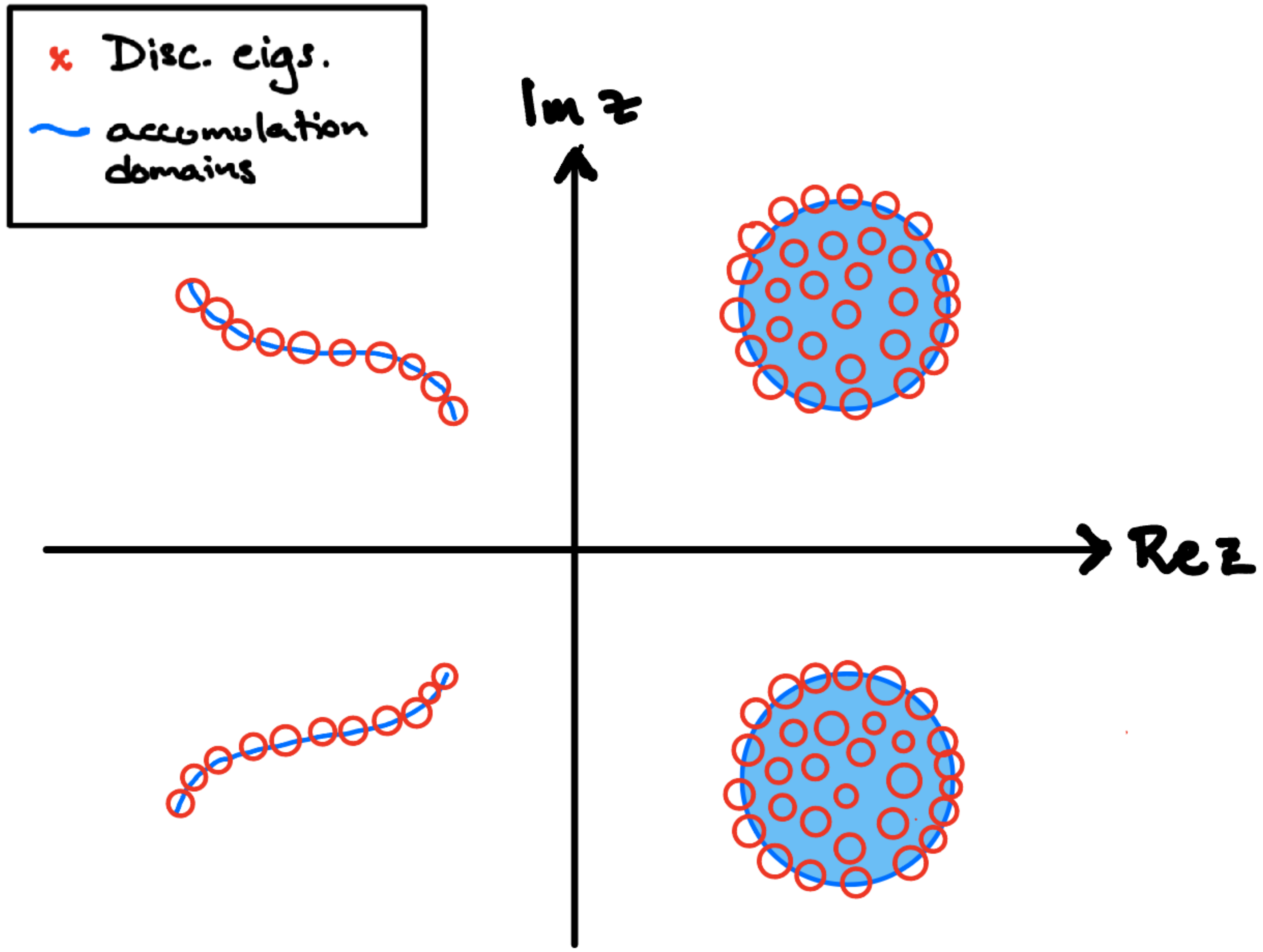


Modeling a soliton gas starting from the n-soliton

- Consider a sequence of n-soliton potentials $\psi_n(x,t)$ for $n = 1,2,\ldots$
- Assume that the poles $z_k^{(n)}$ accumulate on some fixed domain $\mathscr{A}$ (1-D and/or 2-D) .
- Individually interpolate each pole.

1. $\mathbf{H}^{(n)}(z)$ analytic away from $2n$ circles of radius r_0.
2. $\mathbf{H}^{(n)}(z) = \mathbb{I} + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.
3. $\mathbf{H}^{(n)}(z) = \sigma_2 \mathbf{H}^{(n)}(z^*)^* \sigma_2$ for all $z \in \mathbb{C}$.
4. $\mathbf{H}^{(n)}(z)$ has jump on each disk $|z - z_k| = r_0$ given by

$$\mathbf{H}_+^{(n)}(z) = \mathbf{H}_-^{(n)}(z) \begin{bmatrix} 1 & 0 \\ -\frac{c_k}{z-z_k} e^{2i\theta(z;x,t)} & 1 \end{bmatrix}, \quad k = 1, \ldots, n$$

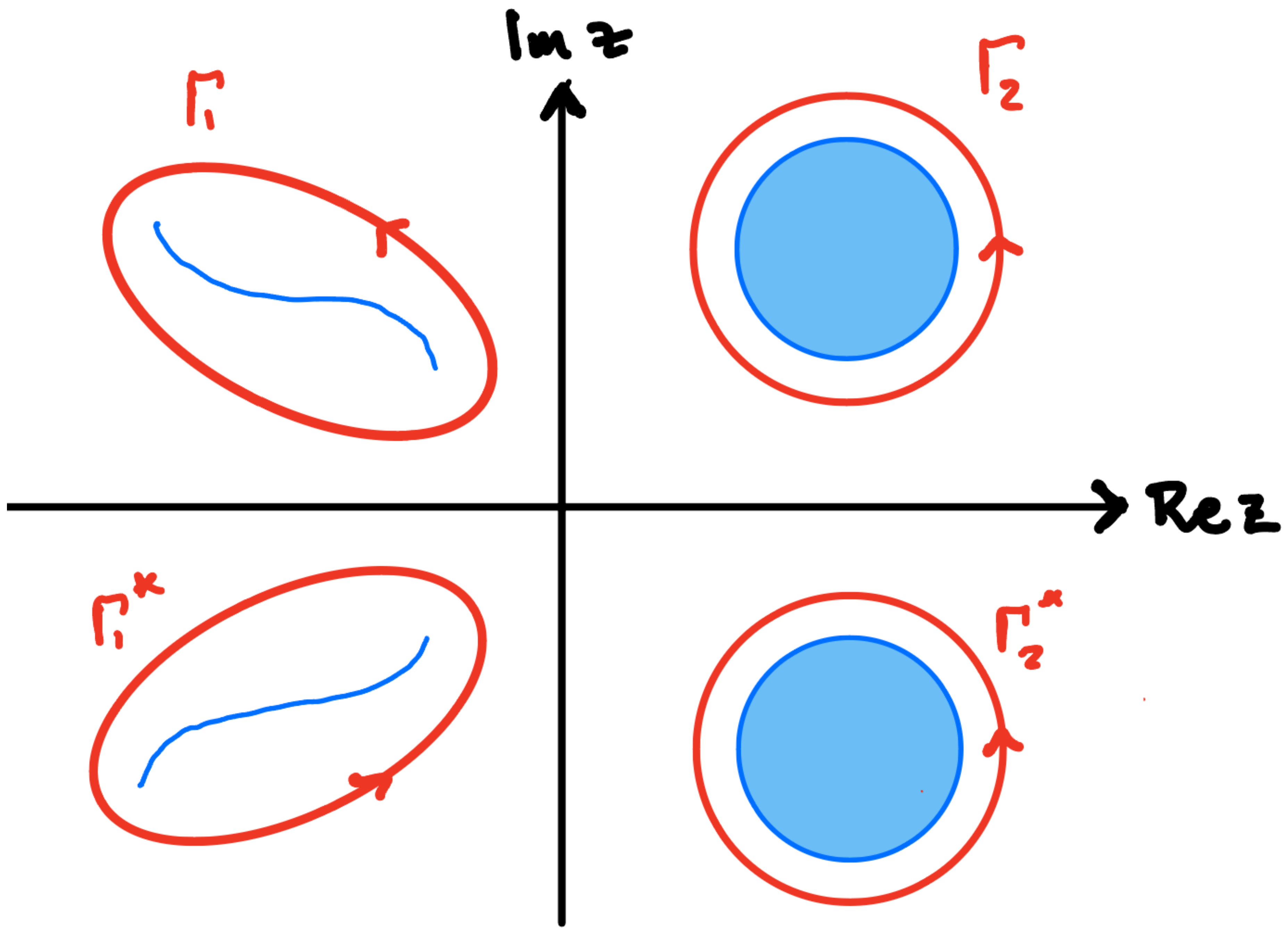


Modeling a soliton gas starting from the n-soliton

- Triangular matrices with unit diagonal commute: $\begin{bmatrix}1 & 0\\ v_1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ v_2 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ v_1+v_2 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ v_2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ v_1 & 1\end{bmatrix}$
- Technical restriction: Residue conditions on each connected component of $\mathscr{A}$ have same triangularity.
- Can expand interpolates to remove all poles in a component simultaneously.

1. $\widetilde{\mathbf{H}}^{(n)}(z)$ analytic away for $z \in \mathbb{C} \setminus \Gamma$
2. $\widetilde{\mathbf{H}}^{(n)}(z) = \mathbb{I} + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.
3. $\widetilde{\mathbf{H}}^{(n)}(z) = \sigma_2 \mathbf{H}^{(n)}(z^*)^* \sigma_2$ for all $z \in \mathbb{C}$.
4. $\widetilde{\mathbf{H}}^{(n)}(z)$ has jump on each component of Γ

$$\widetilde{\mathbf{H}}_+^{(n)}(z) = \widetilde{\mathbf{H}}_-^{(n)}(z) \begin{bmatrix} 1 & 0 \\ -e^{2i\theta(z;x,t)} \displaystyle\sum_{k\in\mathcal{K}_j} \frac{c_k}{z-z_k} & 1 \end{bmatrix}, \quad z \in \Gamma_j$$



Preparing to pass to a limit: soliton gas assumptions

1. Eigenvalues accumulate with some limiting density: $$\frac{1}{n}\sum_{k=1}^{n}\delta(z-z_k) \to \int_{\mathscr{A}} \rho_1(z)\frac{dz}{2\pi i}\chi_1(z) + \rho_2(z,\bar{z})\frac{dz\wedge d\bar{z}}{2\pi i}\chi_2(z)$$
2. Norming constants sampled from a some smooth function: $$c_n = \frac{1}{n}f(z_k)$$

- Assumptions can be relaxed/altered; what one really needs is that the interpolate converges to an integral in the many soliton limit, i.e., $n \to \infty$.

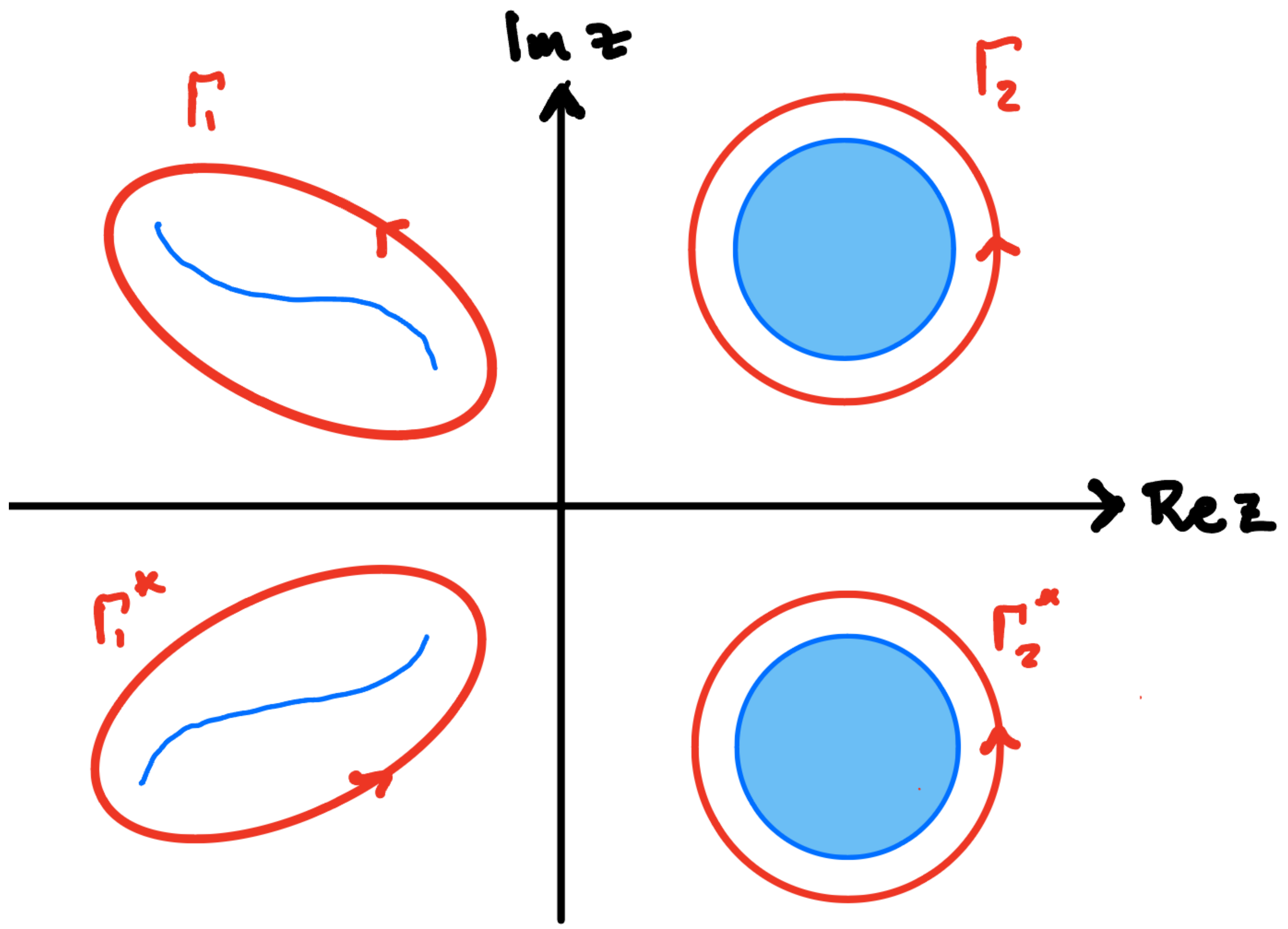
$$\sum_{k=1}^{n}\frac{c_k}{z-z_k} = \frac{1}{n}\sum_{k=1}^{n}\frac{f(z_k)}{z-z_k} = \int_{\mathcal{A}_1}\frac{f(w)\rho(w)}{w-z}\frac{dz}{2\pi i} + \int_{\mathcal{A}_2}\frac{f(w)\rho(w,\bar{w})}{w-z}\frac{dz\wedge d\bar{z}}{2\pi i} + \mathcal{O}\left(n^{-p/d}\right)$$

Preparing to pass to a limit: soliton gas assumptions

$$\sum_{k=1}^{n} \frac{c_k}{z - z_k} = \frac{1}{n} \sum_{k=1}^{n} \frac{f(z_k)}{z - z_k} = \int_{\mathcal{A}_1} \frac{f(w)\rho(w)}{w - z} \frac{dz}{2\pi i} + \int_{\mathcal{A}_2} \frac{f(w)\rho(w,\bar{w})}{w - z} \frac{dz \wedge d\bar{z}}{2\pi i} + \mathcal{O}\left(n^{-p/d}\right)$$

1. $\widetilde{\mathbf{H}}^{(n)}(z)$ analytic away for $z \in \mathbb{C} \setminus \Gamma$
2. $\widetilde{\mathbf{H}}^{(n)}(z) = \mathbb{I} + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.
3. $\widetilde{\mathbf{H}}^{(n)}(z) = \sigma_2 \mathbf{H}^{(n)}(z^*)^* \sigma_2$ for all $z \in \mathbb{C}$.
4. $\widetilde{\mathbf{H}}^{(n)}(z)$ has jump on each component of Γ

$$\widetilde{\mathbf{H}}_+^{(n)}(z) = \widetilde{\mathbf{H}}_-^{(n)}(z) \begin{bmatrix} 1 & 0 \\ -e^{2i\theta(z;x,t)} \displaystyle\sum_{k \in \mathcal{K}_j} \frac{c_k}{z - z_k} & 1 \end{bmatrix}, \quad z \in \Gamma_j$$

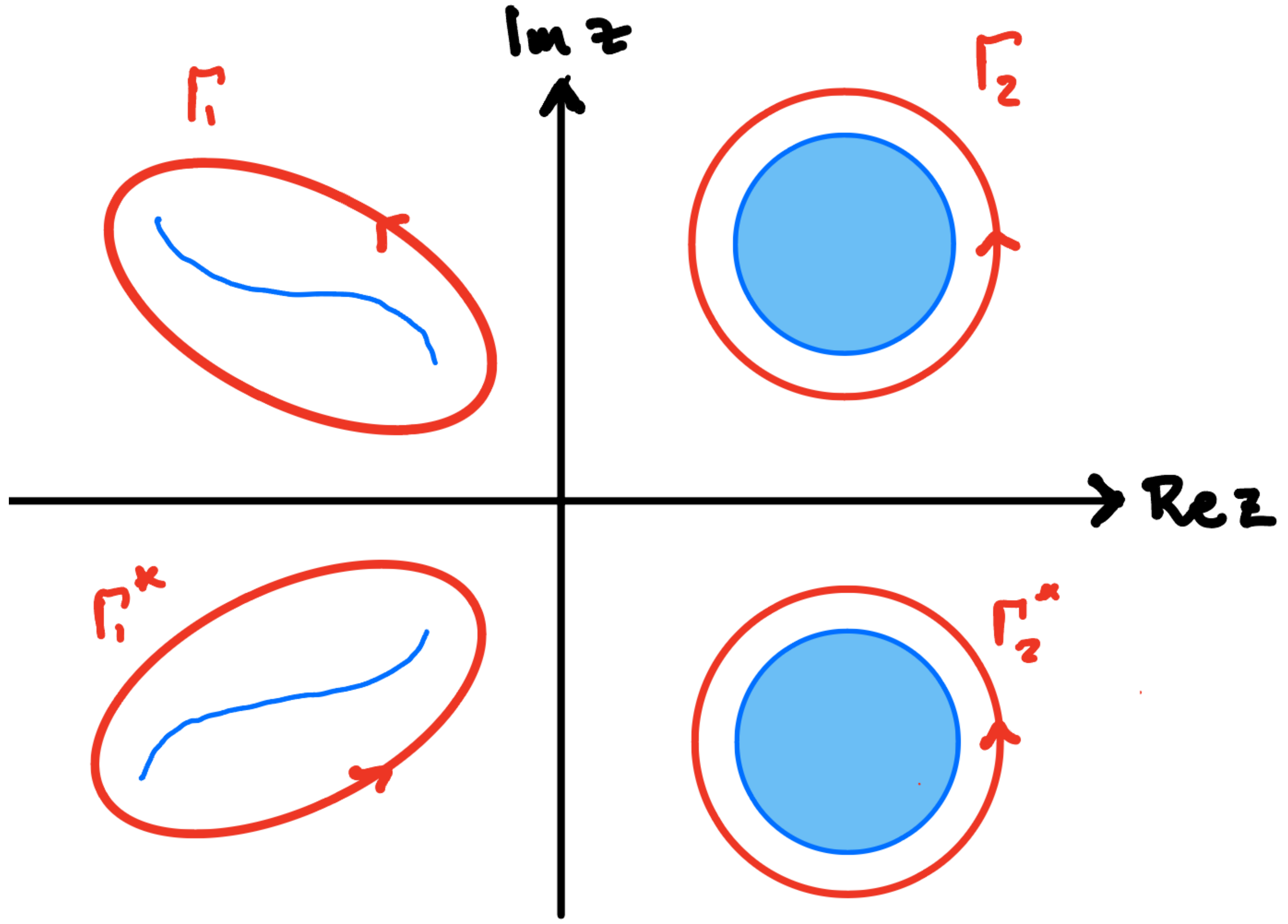


Taking it to the limit!

$$\sum_{k=1}^{n} \frac{c_k}{z - z_k} = \frac{1}{n} \sum_{k=1}^{n} \frac{f(z_k)}{z - z_k} = \int_{\mathcal{A}_1} \frac{\overbrace{f(w)\rho(w)}^{:= r(w)}}{w - z} \frac{dz}{2\pi i} + \int_{\mathcal{A}_2} \frac{\overbrace{f(w)\rho(w,\bar{w})}^{:= r(w,\bar{w})}}{w - z} \frac{dz \wedge d\bar{z}}{2\pi i} + \mathcal{O}\left(n^{-p/d}\right)$$

1. $\widetilde{\mathbf{H}}^{(\infty)}(z)$ analytic away for $z \in \mathbb{C} \setminus \Gamma$
2. $\widetilde{\mathbf{H}}^{(\infty)}(z) = \mathbb{I} + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.
3. $\widetilde{\mathbf{H}}^{(\infty)}(z) = \sigma_2 \mathbf{H}^{(\infty)}(z^*)^* \sigma_2$ for all $z \in \mathbb{C}$.
4. For $z \in \Gamma$, $\widetilde{\mathbf{H}}_+^{(\infty)}(z) = \widetilde{\mathbf{H}}_-^{(\infty)}(z)\mathbf{V}^{(\infty)}(z)$

$$\mathbf{V}^{(\infty)}(z) = \begin{cases} \begin{bmatrix} 1 & 0 \\ -e^{2i\theta(z;x,t)} \int_{\mathcal{A}_1} \frac{r(w)}{w-z} \frac{dw}{2\pi i} & 1 \end{bmatrix} & z \in \Gamma_1 \\ \begin{bmatrix} 1 & 0 \\ -e^{2i\theta(z;x,t)} \int_{\mathcal{A}_2} \frac{r(w,\bar{w})}{w-z} \frac{dw \wedge d\bar{w}}{2\pi i} & 1 \end{bmatrix} & z \in \Gamma_2 \end{cases}$$



The limit…a model of a soliton gas

The continuum limits $\int_{\mathscr{A}_1} \frac{r(w)}{w-z}\frac{dw}{w-z}$ and $\int_{\mathscr{A}_2} \frac{r(w,\bar{w})}{w-z}\frac{dw \wedge d\bar{w}}{w-z}$ are analytic away from $\mathscr{A}_1$ and $\mathscr{A}_2$.

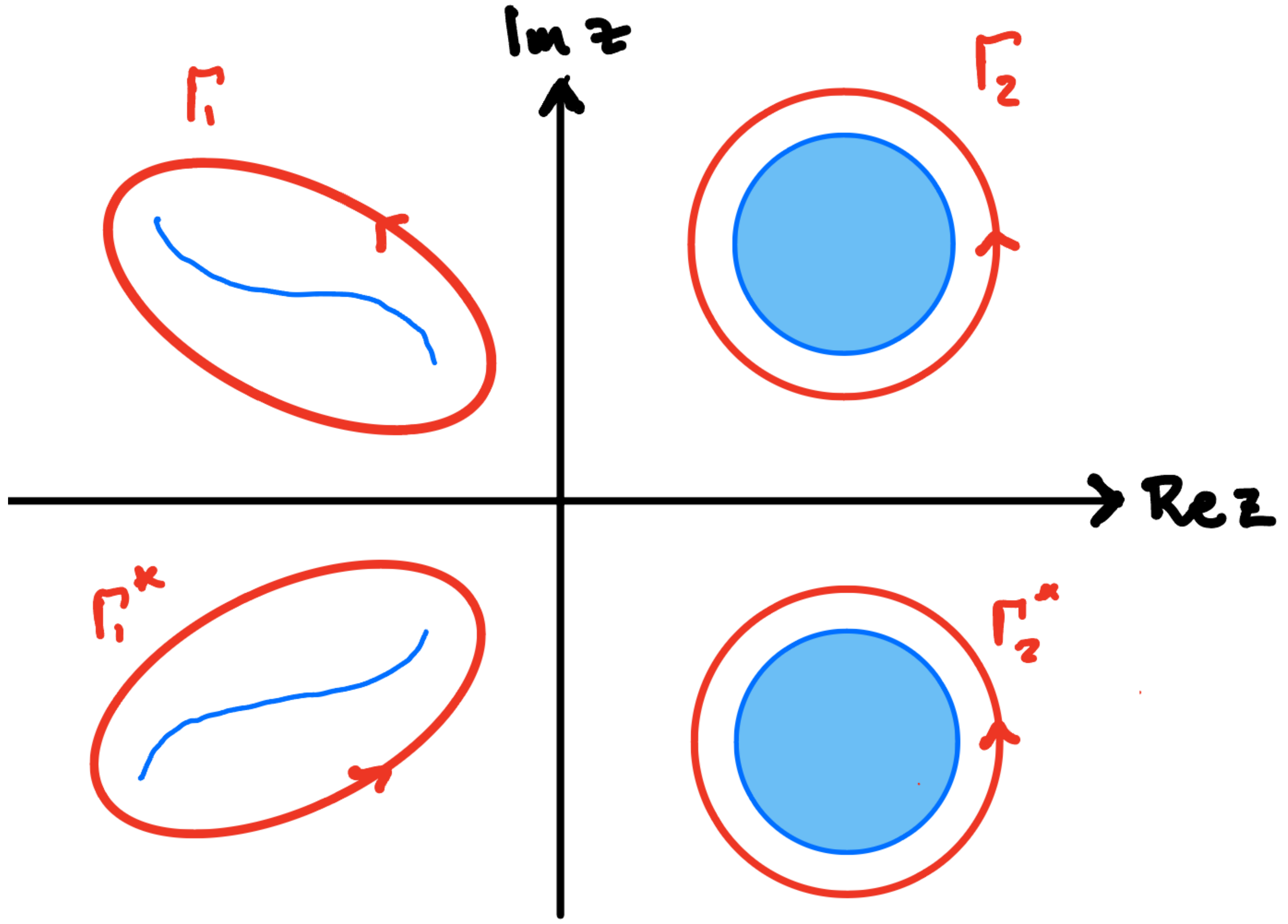
We can collapse the contours Γ_k, back onto the accumulation domains:

1. $\mathbf{H}^{(\infty)}(z)$ is continuous for $z \in \mathbb{C} \setminus \mathcal{A}_1$, analytic in $\mathbb{C} \setminus \mathcal{A}$.
2. $\mathbf{H}^{(\infty)}(z) = \mathbb{I} + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.
3. $\mathbf{H}^{(\infty)}(z) = \sigma_2 \mathbf{H}^{(\infty)}(z^*)^* \sigma_2$ for all $z \in \mathbb{C}$.
4. For $z \in \mathcal{A}_1$, $\mathbf{H}^{(\infty)}$ satisfies the jump relation

$$\mathbf{H}_+^{(\infty)}(z) = \mathbf{H}_-^{(\infty)}(z) \begin{bmatrix} 1 & 0 \\ r(z)e^{2i\theta(z;x,t)} & 1 \end{bmatrix}, \quad z \in \mathcal{A}_1$$

5. $\mathbf{H}^{(\infty)}$ is non-analytic in $\mathcal{A}_2$; it is locally a (weak) solution of the PDE:

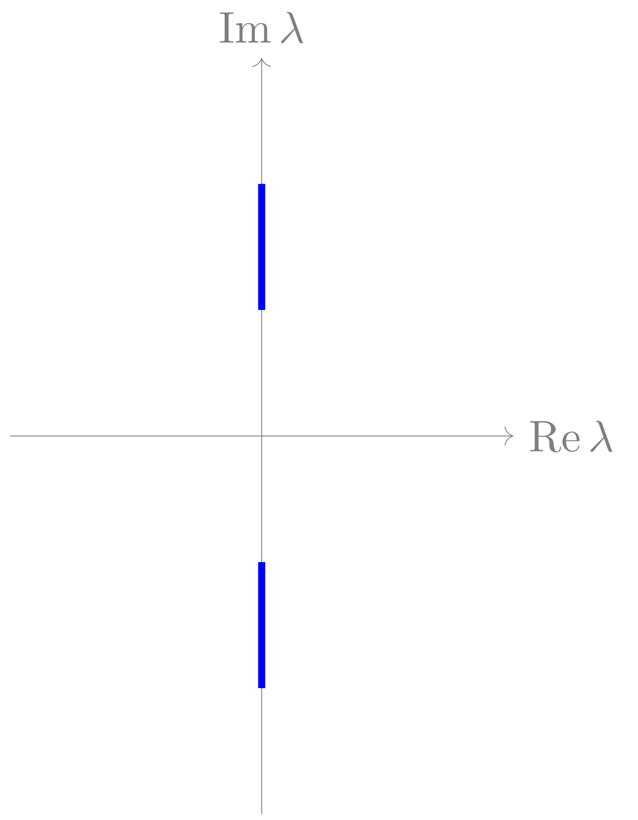
$$\bar{\partial}\mathbf{H}^{(\infty)} = \mathbf{H}^{(\infty)} \begin{bmatrix} 0 & 0 \\ r(z,\bar{z})e^{2i\theta(z;x,t)} & 0 \end{bmatrix}, \quad z \in \mathcal{A}_2.$$



A primitive potential

The resulting problem is a specialized case of the Zakharov et. al. primitive potential where one of the datum $R_k(z)$ is zero on each component of $\mathscr{A}_k$:

$$\mathbf{M}_+(z) = \mathbf{M}_-(z)\begin{bmatrix} \frac{1-R_1R_2}{1+R_1R_2} & \frac{2iR_2}{1+R_1R_2}e^{-2i\theta(z;x,t)} \\ \frac{2iR_2}{1+R_1R_2}e^{2i\theta(z;x,t)} & \frac{1-R_1R_2}{1+R_1R_2} \end{bmatrix}$$



1. $\mathbf{H}^{(\infty)}(z)$ is continuous for $z \in \mathbb{C} \setminus \mathcal{A}_1$, analytic in $\mathbb{C} \setminus \mathcal{A}$.
2. $\mathbf{H}^{(\infty)}(z) = \mathbb{I} + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.
3. $\mathbf{H}^{(\infty)}(z) = \sigma_2 \mathbf{H}^{(\infty)}(z^*)^* \sigma_2$ for all $z \in \mathbb{C}$.
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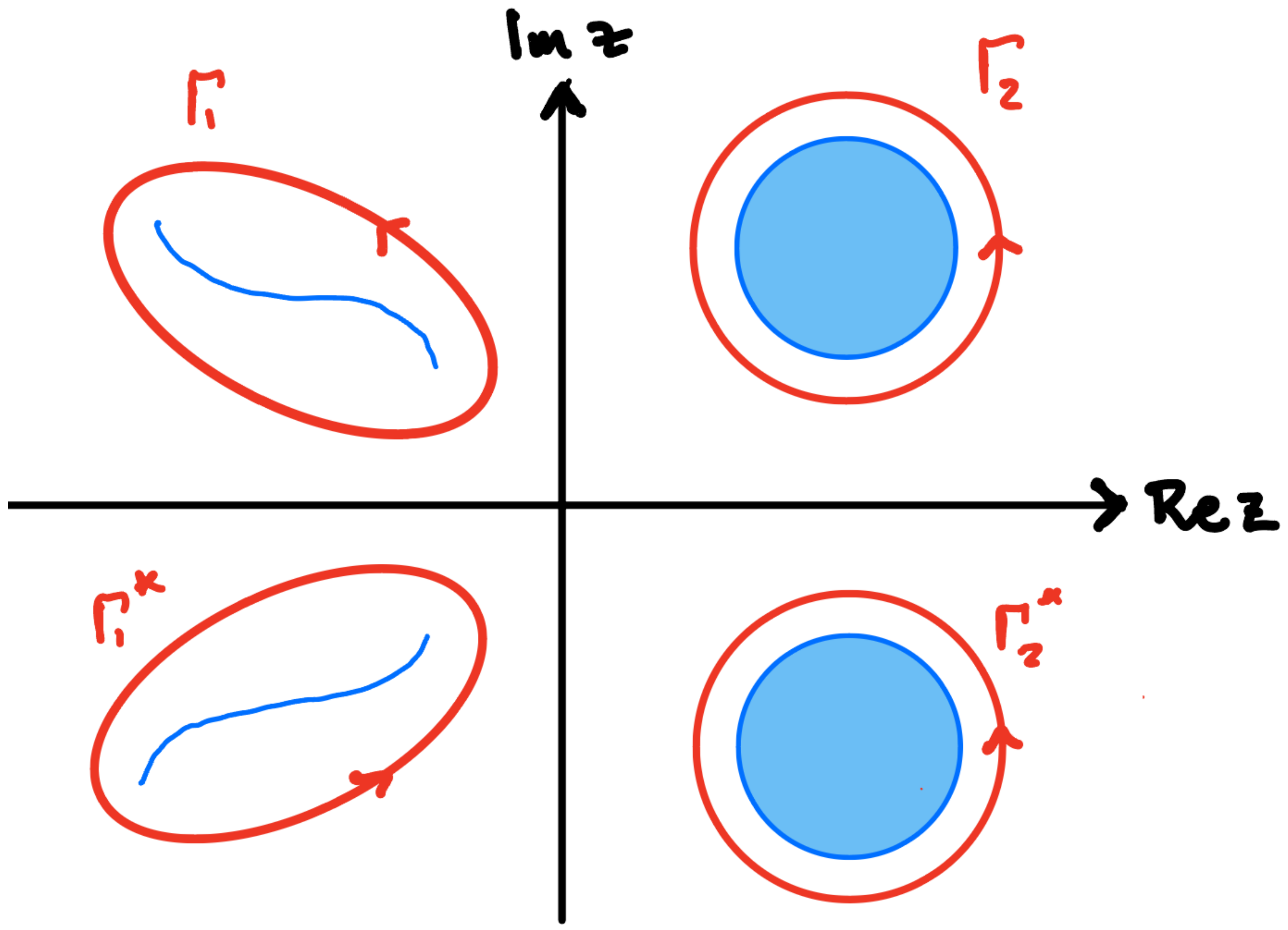
5. $\mathbf{H}^{(\infty)}$ is non-analytic in $\mathcal{A}_2$; it is locally a (weak) solution of the PDE:

$$\overline{\partial}\mathbf{H}^{(\infty)} = \mathbf{H}^{(\infty)}\begin{bmatrix} 0 & 0 \\ r(z,\bar{z})e^{2i\theta(z;x,t)} & 0 \end{bmatrix}, \quad z \in \mathcal{A}_2.$$



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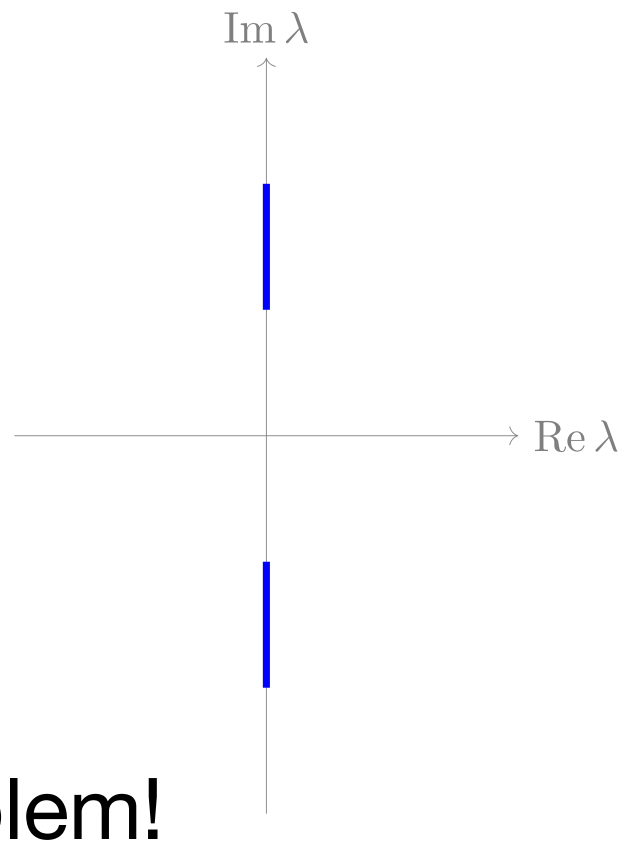

- Adapting this interpolation technique to recover full primitive potentials an open and challenging problem!

1. $\mathbf{H}^{(\infty)}(z)$ is continuous for $z \in \mathbb{C} \setminus \mathcal{A}_1$, analytic in $\mathbb{C} \setminus \mathcal{A}$.
2. $\mathbf{H}^{(\infty)}(z) = \mathbb{I} + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.
3. $\mathbf{H}^{(\infty)}(z) = \sigma_2 \mathbf{H}^{(\infty)}(z^*)^* \sigma_2$ for all $z \in \mathbb{C}$.
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$$\mathbf{H}_+^{(\infty)}(z) = \mathbf{H}_-^{(\infty)}(z) \begin{bmatrix} 1 & 0 \\ r(z)e^{2i\theta(z;x,t)} & 1 \end{bmatrix}, \quad z \in \mathcal{A}_1$$

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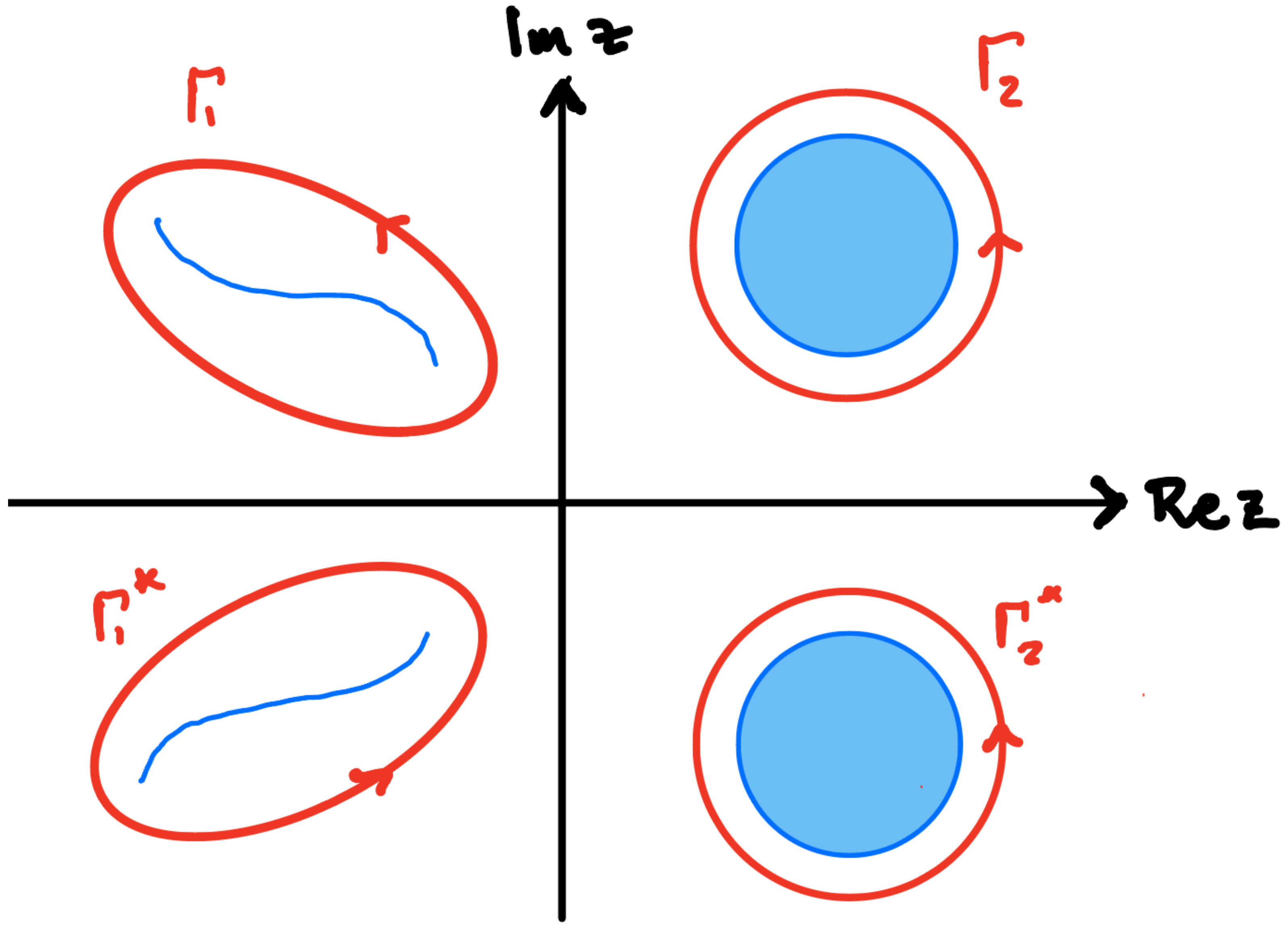
$$\bar{\partial}\mathbf{H}^{(\infty)} = \mathbf{H}^{(\infty)} \begin{bmatrix} 0 & 0 \\ r(z,\bar{z})e^{2i\theta(z;x,t)} & 0 \end{bmatrix}, \quad z \in \mathcal{A}_2.$$



What's hiding under the rug…

Question: In what sense does the exact solution $\widetilde{\mathbf{H}}^{(n)}(z;x,t)$ converge to $\widetilde{\mathbf{H}}^{(\infty)}(z;x,t)$?

Analyzing this question means understanding the ratio: $\mathbf{E}^{(n)}(z;x,t) := \widetilde{\mathbf{H}}^{(n)}(z;x,t)\widetilde{\mathbf{H}}^{(\infty)}(z;x,t)^{-1}$

$$\begin{aligned}\mathbf{E}_+^{(n)} &= \widetilde{\mathbf{H}}_+^{(n)}(\widetilde{\mathbf{H}}_+^{(\infty)})^{-1}\\ &= \widetilde{\mathbf{H}}_-^{(n)}\mathbf{V}^{(n)}(\widetilde{\mathbf{H}}_-^{(\infty)}\mathbf{V}^{(\infty)})^{-1}\\ &= \widetilde{\mathbf{H}}_-^{(n)}\mathbf{V}^{(n)}{\mathbf{V}^{(\infty)}}^{-1}(\widetilde{\mathbf{H}}_-^{(\infty)})^{-1}\\ &= \widetilde{\mathbf{E}}_-^{(n)}\widetilde{\mathbf{H}}_-^{(\infty)}\mathbf{V}^{(n)}{\mathbf{V}^{(\infty)}}^{-1}(\widetilde{\mathbf{H}}_-^{(\infty)})^{-1}\end{aligned}$$

$$\mathbf{E}_+^{(n)} = \widetilde{\mathbf{E}}_-^{(n)}\mathbf{v}_{\mathbf{E}}^{(n)}$$

$$\mathbf{v}_{\mathbf{E}}^{(n)}(z;x,t) := \widetilde{\mathbf{H}}_-^{(\infty)}\left[\mathbf{V}^{(n)}{\mathbf{V}^{(\infty)}}^{-1}\right](\widetilde{\mathbf{H}}_-^{(\infty)})^{-1}$$

$$\mathbf{V}^{(n)}(z) = \begin{bmatrix} 1 & 0 \\ -e^{2i\theta(z;x,t)}\sum_{k=1}^{n}\frac{c_k}{z-z_k} & 1\end{bmatrix}$$

$$\mathbf{V}^{(\infty)}(z) = \begin{bmatrix} 1 & 0 \\ -e^{2i\theta(z;x,t)}\int_{\mathcal{A}}\frac{r(w)}{z-w}\frac{dw}{2\pi i} & 1\end{bmatrix}$$

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$$\begin{aligned}
\mathbf{E}_+^{(n)} &= \widetilde{\mathbf{H}}_+^{(n)}(\widetilde{\mathbf{H}}_+^{(\infty)})^{-1} \\
&= \widetilde{\mathbf{H}}_-^{(n)}\mathbf{V}^{(n)}(\widetilde{\mathbf{H}}_-^{(\infty)}\mathbf{V}^{(\infty)})^{-1} \\
&= \widetilde{\mathbf{H}}_-^{(n)}\mathbf{V}^{(n)}{\mathbf{V}^{(\infty)}}^{-1}(\widetilde{\mathbf{H}}_-^{(\infty)})^{-1} \\
&= \widetilde{\mathbf{E}}_-^{(n)}\widetilde{\mathbf{H}}_-^{(\infty)}\mathbf{V}^{(n)}{\mathbf{V}^{(\infty)}}^{-1}(\widetilde{\mathbf{H}}_-^{(\infty)})^{-1}
\end{aligned}$$

$$\mathbf{E}_+^{(n)} = \widetilde{\mathbf{E}}_-^{(n)}\mathbf{v}_{\mathbf{E}}^{(n)}$$

$$\mathbf{v}_{\mathbf{E}}^{(n)}(z;x,t) := \widetilde{\mathbf{H}}_-^{(\infty)}\left[\mathbf{V}^{(n)}{\mathbf{V}^{(\infty)}}^{-1}\right](\widetilde{\mathbf{H}}_-^{(\infty)})^{-1}$$

$$\left\|\mathbf{V}^{(n)}(z)\mathbf{V}^{(\infty)}(z)^{-1} - \mathbb{I}\right\| = \left(\sum_{k=1}^{n}\frac{c_k}{w-z_k} - \int_{\mathcal{A}}\frac{r(w)}{z-w}\frac{dw}{2\pi i}\right) = e^{2i\theta(z;x,t)}\mathcal{O}\left(n^{-p/d}\right)$$

- Simple observation, but useful: model problem $\widetilde{\mathbf{H}}^{(\infty)}(z;x,t)$ is independent of n.

What's hiding under the rug…

$$\mathbf{v}_{\mathbf{E}}^{(n)}(z;x,t) := \widetilde{\mathbf{H}}_{-}^{(\infty)}\left[\mathbf{V}^{(n)}\mathbf{V}^{(\infty)^{-1}}\right](\widetilde{\mathbf{H}}_{-}^{(\infty)})^{-1}$$

$$\left\|\mathbf{V}^{(n)}(z)\mathbf{V}^{(\infty)}(z)^{-1} - \mathbb{I}\right\| = \left(\sum_{k=1}^{n}\frac{c_k}{w-z_k} - \int_{\mathcal{A}}\frac{r(w)}{z-w}\frac{dw}{2\pi i}\right) = e^{2i\theta(z;x,t)}\mathcal{O}\left(n^{-p/d}\right)$$

- For any fixed (x,t): $\Longrightarrow$ Convergence to primitive potential with algebraic rate of convergence.
- Uniform convergence for (x,t) in compact sets of $\mathbb{R}^2$
- Convergence on expanding domains is much more delicate:

  If solution of the gas model RHP $\mathbf{H}^{(\infty)}(z;x,t)$ is of exponential order in x and t, then convergence on sets

$$\|(x,t)\| \leq K\log n$$

To get convergence on larger domains with $\|(x,t)\| \gg \log n$ requires a different approach.