Arnoux-Rauzy Interval Exchange Transformations: yesterday, today, tomorrow

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Interval Exchange Transformations: Definition

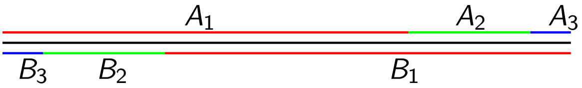
Definition

Let $I \subset \mathbb{R}$ be an interval and $\{A_i : i = 1 \cdots k\}$ be a partition of I. And *interval exchange transformation* is a bijective map $\phi : I \to I$ which is a translation of each A_i.

Restriction of map on each A_i is ϕ_i and $\phi_i(A_i) = B_i$. So, $\{B_i : i = 1 \cdots k\}$ is also a partition of I.

Motivation: measured foliations on oriented surface; rational billiards.

Two points $x, y \in I$ belong to the same *orbit* of IET if there exists a word consisting of ϕ_j and ϕ_j^{-1} that sends x to y.



Interval Exchange Transformations: What do we actually know?

Definition

IET is *minimal* if every orbit is everywhere dense.

Generic IET is minimal (M. Keane, 1975)

Definition

IET is *uniquely ergodic* if it admits exactly one invariant probability measure.

Almost every IET is uniquely ergodic (H. Masur - W. Veech, 1982).

Interval Exchange Transformations: even more!

IETs on n intervals do not admit more than $[\frac{n}{2}]$ invariant measures, and this boundary is sharp (A. Katok - E. Sataev/J. Fickensher).

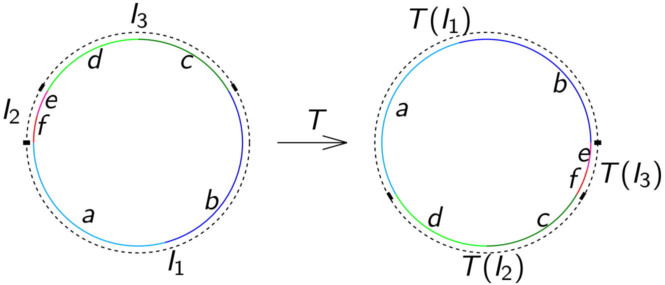
Definition

IET $\phi : I \to I$ is *weakly mixing* with respect to the Lebesgue measure m if for any measurable A and B

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |m(\phi^{-k}(A) \cap B) - m(A)m(B)| = 0.$$

Almost all irreducible IETs that are not rotations are weakly mixing (A. Avila - G. Forni, 2008).

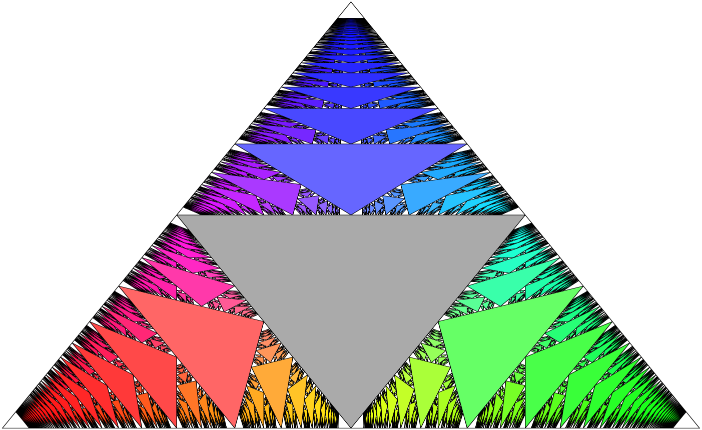
Arnoux-Rauzy IETs



Why do we care?

- Arnoux-Yoccoz example (1981): pseudo-Anosov diffeomorphism, the first example with odd degree stretch factors ;
- symbolic dynamics: Arnoux-Rauzy words as the first example of episturmian system;
- Novikov's problem: the simplest example of chaotic regimes for centrally symmetric surface of genus three with 2 double saddles;
- fully flipped IETs: set of parameters that gives rise to minimal FIETs.

The Rauzy Gasket: Photo



The Rauzy Gasket: Bio

- **Date of Birth**: 2011 (2008, 1993)
- **Creators**: P. Arnoux and S. Starosta
- **Motivation**: episturmian words, multidimensional fraction algorithms
- **Alternative origin**:
  - I. Dynnikov and R. De Leo (3-dimensional topology: Novikov's problem);
  - G. Levitt (geometric group theory: pseudogroups of rotations);

The Rauzy Gasket: Metric Characteristics

Theorem

(G. Levitt and J.-C. Yoccoz, I. Dynnikov and R. De Leo, P. Arnoux and S. Starosta):
The Rauzy Gasket has zero Lebesgue measure.

Open question (P. Arnoux): estimate Hausdorff dimension?
Numerical estimations (I. Dynnikov and R. De Leo, 2008): between 1.7 and 1.8.

Theorem

(Artur Avila, Pascal Hubert and SS, 2013):
The Hausdorff dimension of the Rauzy gasket is strictly less than 2.

Current estimations: 1.19 (C. Matheus - R. Gutiérrez-Romo), 1.74 (M. Policott- B. Sewell).

Key dynamical properties

Theorem (A. Avila - P. Hubert - SS, 2014)

There exists the measure of maximal entropy for the suspension flow of the Rauzy gasket, and this measure is unique.

The proof is based on thermodynamical formalism (O. Sarig' 2002).
Abramov's formula then gives us an invariant measure μ for the Rauzy gasket.

Theorem (P. Arnoux - J. Cassaigne - S. Ferenzsi - P. Hubert, I. Dynnikov - P. Hubert - SS, 2022)

Almost all (with respect to μ) minimal Arnoux-Rauzy IETs are uniquely ergodic.

How many measures and what about mixing?

Theorem (I. Dynnikov - P. Hubert - SS, 2022)

Minimal Arnoux-Rauzy IETs can not admit more than 2 invariant ergodic measures.

Non-uniquely ergodic examples of Arnoux - Rauzy IETs exist.

Theorem (P. Arnoux - J. Cassaigne - S. Ferenzsi - P. Hubert, 2022)

Arnoux-Rauzy IETs are almost never weakly mixing.

Weak mixing examples exist (J. Cassaigne - S. Ferenzsi - A. Messaoudi,2008).
Open question: how these exceptional sets are related?

SAF invariant

Definition

The Sah–Arnoux–Fathi (SAF) invariant of $T_{\pi,\mathbf{a}}$ is the following element of the rational vector space $\mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$:

$$SAF(T_{\pi,\mathbf{a}}) = \sum_{i=1}^{n} a_i \wedge_{\mathbb{Q}} (\widetilde{x}_{\pi^{-1}(i)} - x_i) = \sum_{i<j} (a_i \wedge_{\mathbb{Q}} a_j - a_{\pi(i)} \wedge_{\mathbb{Q}} a_{\pi(j)}).$$

The SAF invariant was introduced by P. Arnoux who also showed that $SAF(T_{\pi,\mathbf{a}})$ is an invariant of the measured foliation induced by $\omega_{\pi,\mathbf{a}}$ on $\Sigma_{\pi,\mathbf{a}}$ (in particular, it is invariant under the Rauzy induction).

I have a dream

Conjecture 1 (I. Dynnikov - SS, 2018): IETs with zero SAF invariant are never stably minimal. Numerically confirmed for genus 3. **Conjecture 2 (P. Hubert - SS. 2023):** IETs with zero SAF invariant are never weakly mixing. Holds for genus 3, counter-example in higher genus is work in progress.

Tanti auguri, caro Stefano!