The Poisson boundary of hyperbolic groups without moment conditions

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From KAM to ETF: in honor of Stefano Marmi
October 12, 2023

Summary

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joint with K. Chawla, B. Forghani, and J. Frisch.

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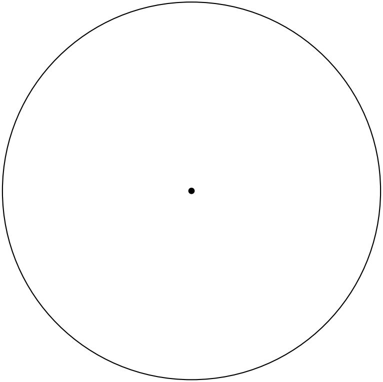

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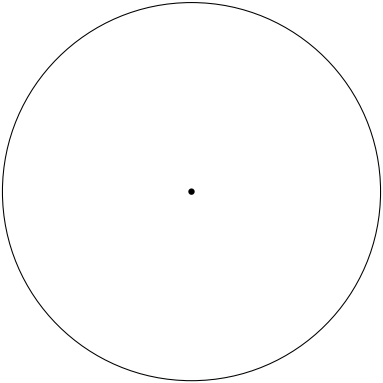

Theorem (Poisson representation)

There is a bijection

$$h^\infty(\mathbb{D}) \quad \leftrightarrow \quad L^\infty(S^1, \lambda)$$

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$$P_r(\theta) := \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}$$

is the Poisson kernel.

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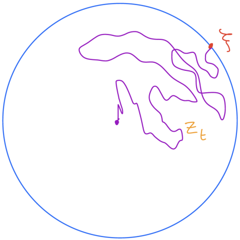

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Question. Can we generalize this to other groups $G \neq PSL_2(\mathbb{R})$?

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Let B be a space on which G acts (measurably). A measure ν on B is μ-stationary if

$$\nu = \int_G g\nu\, d\mu(g).$$

Random walks and μ-boundaries

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A space (B, ν) is a μ-boundary if there exists a measurable map

$$\text{bnd} : \Omega \to B$$

such that $\text{bnd} = \text{bnd} \circ T$.

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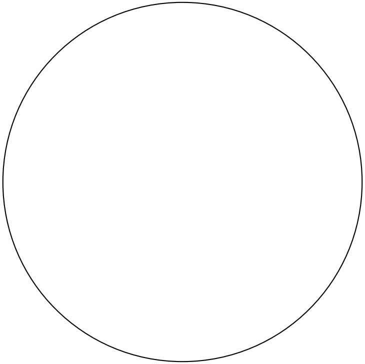

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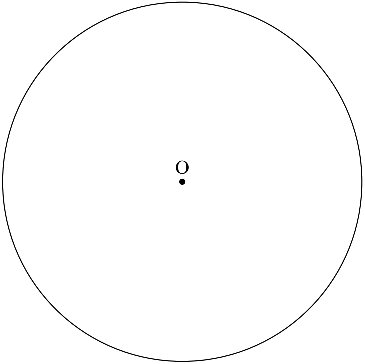

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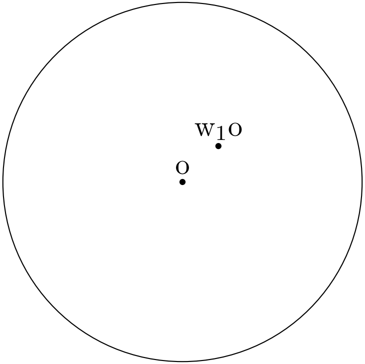

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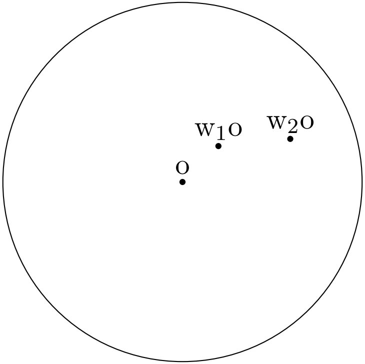

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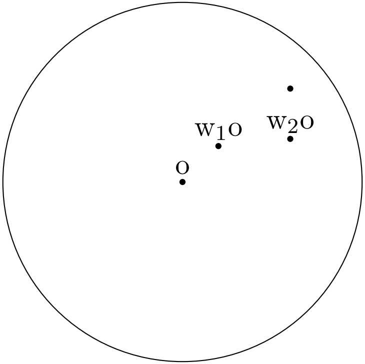

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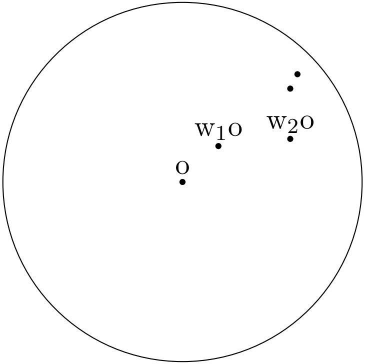

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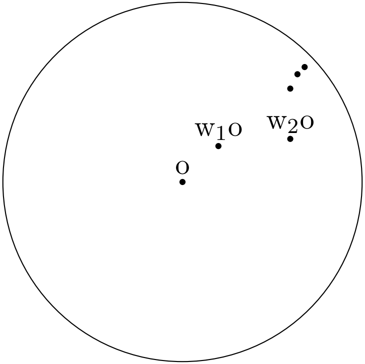

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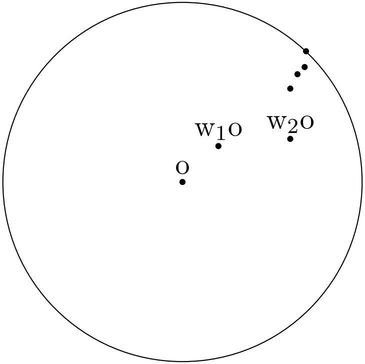

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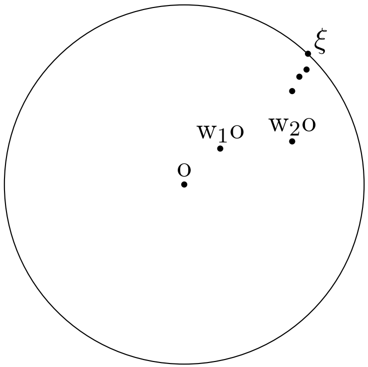

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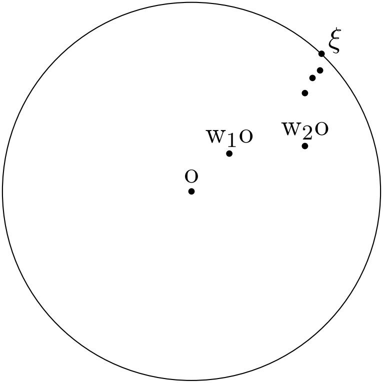

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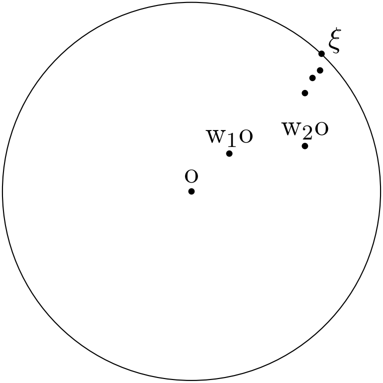

Moreover, $(\partial X, \nu)$ is a μ-boundary, given by the map

$$\Omega \ni \mathrm{bnd}(\omega) := \lim_{n\to\infty} w_n o \in \partial X$$

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Examples. Abelian groups; nilpotent groups

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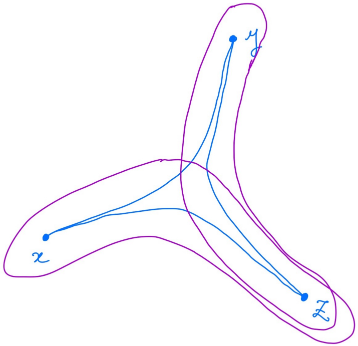
- Maher-T. '18: Cremona group

Hyperbolic spaces

A metric space (X, d) is hyperbolic if there exists $\delta > 0$ s.t.

$$[x, z] \subseteq N_\delta([x, y]) \cup N_\delta([y, z]) \qquad \text{for any } x, y, z \in X$$



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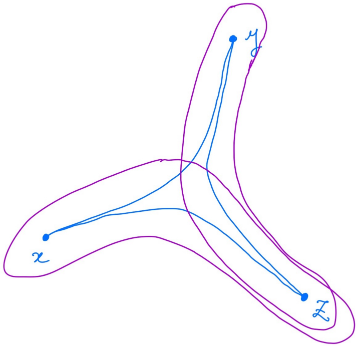

A group is hyperbolic if a Cayley graph of G is. E.g.:

- free groups
- (non-elementary) Fuchsian groups
- fundamental groups of negatively curved manifolds

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We say μ has finite logarithmic moment if

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Theorem (Kaimanovich '93)

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Conditional entropy

Fix $\xi \in \partial \mathrm{G}$. Consider

$$\mu_{\mathrm{n},\xi}(\mathrm{g}) := \mathbb{P}(\mathrm{w_n} = \mathrm{g} \mid \mathrm{w}_\infty = \xi)$$

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Theorem (Entropy criterion, Kaimanovich)

Let (B, ν) be a μ-boundary. Then (B, ν) is the Poisson boundary if and only if

$$h(\xi) = 0$$

for ν-almost every $\xi \in B$.

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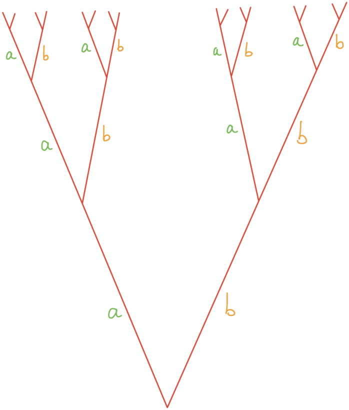
Toy example: the free semigroup

Consider the free semigroup $F_2^+ = \langle a, b \rangle$ in two generators.



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Consider the free semigroup $F_2^+ = \langle a, b \rangle$ in two generators.
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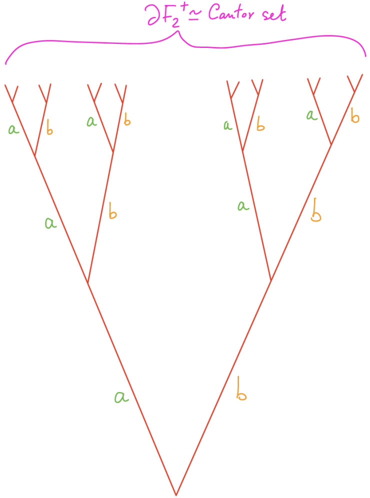
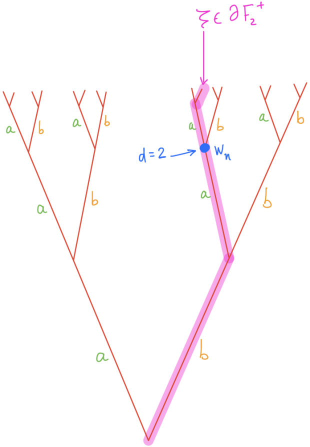


Toy example: the free semigroup

Let P_n be the partition given by specifying the distance from the origin: $d : F_2^+ \to \mathbb{N}$

$$P_n = \bigsqcup_{k \in \mathbb{N}} \{\omega \;:\; d(w_n) = k\}$$



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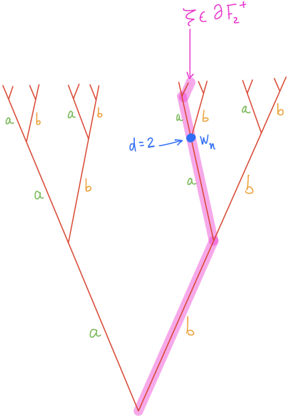

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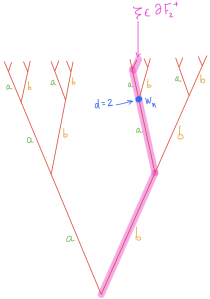

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Since the Poisson boundary of abelian (semi)-groups is trivial,

$$\frac{H(P_n)}{n} = \frac{H(\theta^{*n})}{n} = 0$$

$\Rightarrow$ End of proof for free semigroup

The end

Tanti auguri caro Stefano!

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   Key insight: since good times appear often, the entropy

   $$\mathrm{H}(\mathrm{A_n} \mid \mathrm{D_n} \text{ and } \mathrm{B_n} \text{ and } \xi)$$

   is still small!

⇒ Proof for all hyperbolic-like groups