Deviation spectrum of ergodic integrals for locally Hamiltonian flows on surfaces

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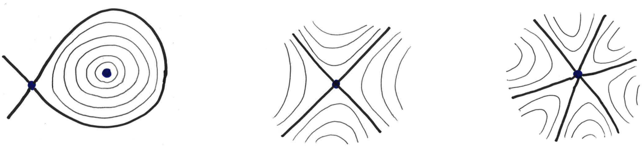
Let (M,ω) be a compact connected orientable surface with a fixed smooth area form ω. A *locally Hamiltonian flow* $\psi_{\mathbb{R}} = (\psi_t)_{t\in\mathbb{R}}$ on M is a smooth flow on M which preserves the area form ω. These flows are also called *multi-valued Hamiltonian flows*. The interest in the study of multi-valued Hamiltonians and the associated flows in higher genus ($g \geqslant 1$) was highlighted by Novikov in connection with problems arising in solid-state physics as well as in pseudo-periodic topology.

To see the Hamiltonian nature of such the flow let us consider the corresponding vector field $X : M \to TM$, $\frac{d}{dt}\varphi_t(x) = X(\varphi_t(x))$ and a real-valued differential 1-form η given by the contraction operator $\eta = i_X\omega = \omega(X,\cdot)$ $(\eta = -X_2dx + X_1dy)$. As $\psi_{\mathbb{R}}$ preserves the area form ω, η is closed $d(\eta) = 0$. So η is locally exact, this is $\eta = \omega(X,\cdot) = dH$, where H is defined locally. Moreover, if $\omega = dx \wedge dy$ (in local coordinates), then $X = (\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x})$, so the flow $\psi_{\mathbb{R}}$ is really "locally Hamiltonian".

Fixed points

A point $\sigma \in M$ is a fixed point if $X(\sigma) = 0$ or equivalently $\nabla H(\sigma) = 0$. We deal only with flows having isolated fixed points. Denote by $\mathrm{Fix}(\psi_{\mathbb{R}})$ the set of *fixed points* (also called *singularities*) of the flow $\psi_{\mathbb{R}}$. Then $\mathrm{Fix}(\psi_{\mathbb{R}})$ is a *finite set* and when $g \geqslant 2$, $\mathrm{Fix}(\psi_{\mathbb{R}})$ is always not empty. Since $\psi_{\mathbb{R}}$ is area-preserving, *singularities* in $\mathrm{Fix}(\psi_{\mathbb{R}})$, can be either centers, simple saddles or multi-saddles (i.e. saddles with $2k$ pronges, $k \geqslant 2$).



Fixed points

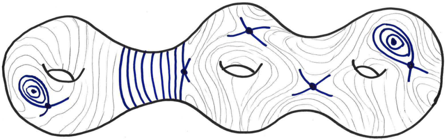
We distinguish so called non-degenerate fixed points, such that the Hessian of H at σ is non-zero. By Morse lemma, there exists a local chart (x, y) in a neighborhood of σ such that $H(x, y) = x^2 + y^2$ or $H(x, y) = 2xy(\text{or} = x^2 - y^2)$. It corresponds to a center or a simple saddle. Non-degenerate fixed points are topologically typical, this is there exists an open and dense subset of locally Hamiltonian flows such that all fixed points are non-degenerate. We permit the appearance of some *degenerate* fixed points σ, i.e. perfect saddles of multiplicity $m_\sigma > 2$ such that the corresponding Hamiltonian function is of the form $H(x, y) = \Im(x + iy)^{m_\sigma}$. Each such saddle has m_σ incoming and m_σ outgoing separatrices.

Minimality vs. decomposition into minimal and periodic components

Recall that a *saddle connection* is an orbit (separatrix) of $\psi_{\mathbb{R}}$ running from a saddle to a saddle. A *saddle loop* is a saddle connection joining the same saddle. For example, each center is surrounded by a saddle loop.

If there are no saddle connections then the flow $\psi_{\mathbb{R}}$ on M minimal (every orbit, except of fixed points, is dense in M).

In general, M splits into a finite number of $\psi_{\mathbb{R}}$-invariant surfaces (with boundary) so that every such surface is a *minimal component* of $\psi_{\mathbb{R}}$ or is a periodic component (is filled by periodic orbits, fixed points and saddle connections).



(Zero) measure class

Denote by $\mathcal{F}$ the set of smooth locally Hamiltonian flows on M with isolated fixed points. $\mathcal{F}$ has a natural stratification into subsets $\mathcal{F}_{\overline{m},c}$. For any vector $\overline{m} = (m_1, m_2, \dots, m_s)$ of natural numbers $\geqslant 2$ and any $c \leqslant \sum_{i=1}^{s}(m_i - 1)$, denote by $\mathcal{F}_{\overline{m},c}$ the set of smooth locally Hamiltonian flows with c centers and s saddles of multiplicity $m_1, m_2, \dots, m_s$. By the Poincaré-Hopf Theorem, $c - \sum_{i=1}^{s}(m_i - 1) = 2 - 2g$. A measure-theoretical notion of typicality on $\mathcal{F}$ (on each $\mathcal{F}_{\overline{m},c}$ separately) is defined by the cohomology class of the 1-form η, so called *Katok fundamental class*. Let $\gamma_1, \dots, \gamma_n$ be a base of $H_1(M, \mathrm{Fix}(\psi_{\mathbb{R}}), \mathbb{Z})$, where $n = 2g + s + c - 1$. Let us consider the period map

$$\Theta(\psi_{\mathbb{R}}) = \Big(\int_{\gamma_1} \eta, \dots, \int_{\gamma_n} \eta \Big) \in \mathbb{R}^n,$$

which is well-defined in a neighbourhood of $\psi_{\mathbb{R}} \in \mathcal{F}_{\overline{m},c}$.

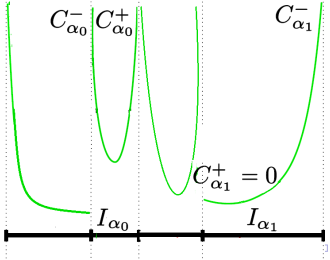
The Θ-pullback of the Lebesgue measure class (i.e. class of sets with zero measure) gives the desired measure class on $\mathcal{F}_{\overline{m},c}$. When we use the expression *a.e. locally Hamiltonian flow* below we mean full measure in each $\mathcal{F}_{\overline{m},c}$ with respect to the corresponding measure class. We distinguish a subset $\mathcal{F}_{\min} = \bigcup_{\overline{m}} \mathcal{F}_{\overline{m},0} \subset \mathcal{F}$ and the corresponding measure class.

Theorem (Masur, Veech)

Almost every flow $\psi_{\mathbb{R}}$ in $\mathcal{F}_{\min}$ is ergodic (with respect to the area measure ω). Moreover, every ergodic measure is either ω or the delta Dirac measure at a fixed point.
For almost every $\psi_{\mathbb{R}} \in \mathcal{F} \setminus \mathcal{F}_{\min}$, the flow $\psi_{\mathbb{R}}$ restricted to any is minimal component is ergodic and ect.

Special representation

Locally Hamiltonian flows are represented as special flows. Let us consider a restriction of a locally Hamiltonian flow $\psi_{\mathbb{R}}$ on M to its minimal component $M' \subset M$. Let $I \subset M'$ be any transversal smooth curve. By minimality, I is a global transversal and the first return map $T : I \to I$ is an interval exchange transformation (IET) (in so called standard coordinates on I). Moreover, $\psi_{\mathbb{R}}$ restricted to M' is isomorphic to the special flow $T^g_{\mathbb{R}}$, where $g : I \to \mathbb{R}_{>0} \cup \{+\infty\}$ is the first return time map. The roof function has logarithmic (polynomial) singularities derived from non-degenerate (degenerate) saddles.



Special representation

Recall that every IET $T : I \to I$ exchanging d intervals is determined by a pair (π, λ), where $\lambda = (\lambda_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{R}^d_{>0}$ ($\#\mathcal{A} = d$ and elements of $\mathcal{A}$ label the exchanged intervals) collects the length of exchanged intervals $I_\alpha = [l_\alpha, r_\alpha)$, $\alpha \in \mathcal{A}$ which are rearranged according to the permutation π. Then we write $T = T_{(\pi,\lambda)}$. This gives a natural Lebesgue measure on the space of all IETs.
We say that a function $\varphi : I \to \mathbb{R}$ ($|I| = 1$) for an IET $T_{(\pi,\lambda)}$ has *logarithmic singularities* if there exist constants $C^+_\alpha, C^-_\alpha \in \mathbb{R}$, $\alpha \in \mathcal{A}$, and a function g_φ absolutely continuous on the interior of each interval I_α, $\alpha \in \mathcal{A}$ such that

$$\varphi(x) = -\sum_{\alpha\in\mathcal{A}} C^+_\alpha \log\{x - l_\alpha\} - \sum_{\alpha\in\mathcal{A}} C^-_\alpha \log\{r_\alpha - x\} + g_\varphi(x).$$

The space of such functions is denoted by $LOG(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. This is a Banach space equipped with the norm

$$\|\varphi\|_{LV} = \sum_{\alpha\in\mathcal{A}} (|C^+_\alpha| + |C^-_\alpha|) + \sum_{\alpha\in\mathcal{A}} Var_{I_\alpha}(g_\varphi) + \|g_\varphi\|_{sup}.$$

Special representation

For every $0 < a < 1$ denote by $P_a(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ the space of functions with *polynomial singularities* of degree at most a, i.e. the space of piecewise C^1 maps such that

$$p_a(\varphi) := \max_{\alpha\in\mathcal{A}} \min\left\{ \sup_{x\in I_\alpha} |\varphi'(x)(x-l_\alpha)^{1+a}|, \sup_{x\in I_\alpha} |\varphi'(x)(r_\alpha-x)^{1+a}|\right\} < +\infty$$

and for every $\alpha \in \mathcal{A}$ the limits

$$C_\alpha^+ := -\lim_{x\searrow l_\alpha} \varphi'(x)(x-l_\alpha)^{1+a} \text{ and } C_\alpha^- := \lim_{x\nearrow r_\alpha} \varphi'(x)(r_\alpha - x)^{1+a}$$

exist. $P_a(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ equipped with the norm

$$\|\varphi\|_a = p_a(\varphi) + \|\varphi\|_{L^1(I)}$$

is also a Banach space.

Proposition

Let $\psi_{\mathbb{R}}$ be a locally Hamiltonian flow, $M' \subset M$ its minimal component and $I \subset M'$ a transversal curve.
If all its saddles are non-degenerate then $g \in LOG(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$.
If $\psi_{\mathbb{R}}$ has degenerate perfect saddles in M', then $g \in P_a(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$, where $a = \frac{m-2}{m}$ with $m := \max\{m_\sigma : \sigma \in \mathrm{Fix}(\psi_{\mathbb{R}}) \cap M'\}$.
Moreover, if $\psi_{\mathbb{R}}$ is minimal on M then the singularities of g are of symmetric type, this is

$$\sum_\alpha C_\alpha^+ = \sum_\alpha C_\alpha^-.$$

General approach to further results: To prove that a dynamical property is satisfied for a.e. locally Hamiltonian flow in $\mathcal{F}_{c,\bar{m}}$ it is enough to show this property for special flows T^g for almost all IETs T and for all roof functions g from the appropriate function class. This reduces many of the problems regarding flows on surfaces to roof functions analysis.

Mixing

Theorem (Kochergin, Khanin-Sinai, Ulcigrai)

Non-degrenerated case. *Almost every flow* $\psi_{\mathbb{R}} \in \mathcal{F}_{0,\bar{2}}$ *(minimal and non-degenerate case) is weakly mixing but not strongly mixing. Almost every flow* $\psi_{\mathbb{R}} \in \mathcal{F}_{c,\bar{2}}$ *with* $c > 0$ *(non-minimal and non-degenerate case) restricted to any its minimal component is strongly mixing.*

Degrenerated perfect case. *Almost every flow* $\psi_{\mathbb{R}} \in \mathcal{F}_{c,\bar{m}}$ *with at least one* $m_i > 2$ *(degenerate perfect saddle) restricted to any its minimal component is strongly mixing.*

Deviation spectrum - the beginning of the story

The phenomenon of deviation spectrum and its relation with so called Lyapunov exponents of the Kontsevich-Zorich cocycle were first observed by Zorich in the context of studying deviations of Birkhoff (ergodic) sums for piecewise constant observables for almost all interval exchange translations:

$$\varphi^{(n)}(x) = \sum_{0\leqslant k<n}\varphi(T^k x) \sim n^{\nu} \text{ for some } 0 \leqslant \nu \leqslant 1.$$

Inspired by this result and numerical experiments, Kontsevich and Zorich in 1997 formulated the following conjecture: there exist Lyapunov exponents $0 < \nu_i \leqslant 1$, $1 \leqslant i \leqslant g$ so that for almost every locally Hamiltonian flow $\psi_{\mathbb{R}}$ with non-degenerate fixed points and for every smooth map $f : M \to \mathbb{R}$ there exists $1 \leqslant i \leqslant g+1$ such that

$$\limsup_{T\to+\infty} \frac{\log\left|\int_0^T f(\psi_t(x))\,dt\right|}{\log T} = \nu_i \text{ for almost every } x \in M.$$

$(\nu_{g+1} = 0)$ The exponents $0 < \nu_g < \ldots < \nu_1 = 1$ are the positive Lyapunov exponents of the Kotsevich-Zorich cocycle.

Deviation spectrum - the beginning of the story

This conjecture was essentially positively verified by Forni in his seminal paper (2002). More precisely, for almost every locally Hamiltonian flow $\psi_{\mathbb{R}} \in \mathcal{F}_{min}$ (here we do not demand that all saddles are non-degenerate) and a class of function vanishing at $Fix(\psi_{\mathbb{R}})$ Forni constructed g (genus of M) invariant distributions $D_1, \ldots, D_g$ such that if $D_1(f) = \ldots = D_i(f) = 0$ and $D_{i+1}(f) \neq 0$ then

$$\limsup_{T\to+\infty} \frac{\log\left|\int_0^T f(\psi_t(x))\,dt\right|}{\log T} = \nu_{i+1} \text{ for almost every } x \in M,$$

where $\nu_{g+1} = 0$. The first distribution is obvious $D_1(f) = \int_M f d\omega$, but the others are not so directly defined.
Forni used the fact that the flow $\psi_{\mathbb{R}} \in \mathcal{F}_{min}$ after a smooth change of speed on $M \setminus Fix$ is a translation flow $h_{\mathbb{R}}$. Suppose that $W : M \setminus Fix \to \mathbb{R}_{>0}$ describes that change of speed. As the translation flow has constant speed and $\psi_{\mathbb{R}}$ slows down quickly around fixed points, W has singularities at Fix.

Next, Forni developed a huge and powerful machinery using so called Teichmüller flow on the moduli space to prove deviation spectrum for a.e. translation flow $h_{\mathbb{R}}$ and smooth (Sobolev) observables $f : M \to \mathbb{R}$. By passing through the inverse change of velocity, one can obtain deviation spectrum for $\psi_{\mathbb{R}}$ and observables of the form f/W. They must vanish at Fix.

The next step was taken by Bufetov (2014), who proved the deviation spectrum in an improved form. He proved the existence of g cocycles $u_i : \mathbb{R} \times M \to \mathbb{R}$ ($u_i(t+s,x) = u_i(t,x) + u_i(s,\psi_t x)$ for all $t,s \in \mathbb{R}$) such that for every observable $f : M \to \mathbb{R}$ such that Wf is smooth (weakly Lipschitz) we have

$$\int_0^T f(\psi_t(x))dt = \sum_{i=1}^{g} D_i(f)u_i(T,x) + err(f,T,x),$$

where for a.e. $x \in M$ we have

$$\limsup_{T\to+\infty} \frac{\log\left|u_i(T,x)\right|}{\log T} = \nu_i, \quad \lim_{T\to+\infty} \frac{\log|err(f,T,x)|}{\log T} \leqslant 0.$$

Further problems with deviation spectrum

Problems:
1. The tools developed by Forni and Bufetov work only for locally Hamiltonian flows which are the minimal strata, i.e. there are no centers. But the existence of a non-degenerate center is an open condition (it is stable under a small perturbation). Therefore the set of flows with centers is topologically big.
2. The tools developed by Forni and Bufetov work only for observables which vanish at saddle points. In fact, also, some higher-order derivatives must also vanish when we deal with multiple saddles.

The first problem one can solve by passing to the special representation of $\psi_{\mathbb{R}}$ and applying a Marmi-Mousa-Yoccoz (2005) approach. Let us consider a restriction of the locally Hamiltonian flow $\psi_{\mathbb{R}}$ on M to its minimal component $M' \subset M$. Let $I \subset M'$ be any transversal smooth curve. Then $\psi_{\mathbb{R}}$ restricted to M' is isomorphic to the special flow $T^g_{\mathbb{R}}$, where $T : I \to T$ is and IET and $g : I \to \mathbb{R}_{>0} \cup \{+\infty\}$ is the first return time map. Moreover, to any smooth observable $f : M \to \mathbb{R}$ we assign the map $\varphi_f : I \to \mathbb{R}$ given by

$$\varphi_f(x) := \int_0^{g(x)} f(\psi_t x)dt \text{ for every } x \in I.$$

Heuristic observation: the problem of studying the asymptotic of the growth of ergodic integrals for the flow $\psi_{\mathbb{R}}$ boils down to studying the asymptotic of the growth of ergodic sums $\varphi_f^{(n)} = \sum_{0\leqslant k<n} \varphi_f(T^k x)$, since $\varphi_f^{(n)}(x) := \int_0^{g^{(n)}(x)} f(\psi_t x)dt$.

Properties of φ_f

1. If $f : M \to \mathbb{R}$ is zero on a neighborhood of any saddle point then $\varphi_f \in AC^{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$, i.e. φ_f is absolutely continuous on each I_α and its derivative is of bounded variation on I_α. This is the case considered by Marmi-Mousa-Yoccoz.
2. For general observables f if all saddles of $\psi_\mathbb{R}$ are non-degenerate, then $\varphi_f \in LOG(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$, i.e.

$$\varphi_f(x) = -\sum_{\alpha\in\mathcal{A}} C_\alpha^+ \log\{x - l_\alpha\} - \sum_{\alpha\in\mathcal{A}} C_\alpha^- \log\{r_\alpha - x\} + g_\varphi(x).$$

3. If $\psi_\mathbb{R}$ has degenerate saddles, then $\varphi_f \in P_a(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$, where $a = \frac{m-2}{m}$ with $m := \max\{m_\sigma : \sigma \in \mathrm{Fix}(\psi_\mathbb{R}) \cap M'\}$.

Moreover, $AC^{BV} \subset LOG \subset P_a$ and in each case the linear operator

$$f \mapsto \varphi_f$$

is bounded.

Renormalization procedure

Let $T = T_{(\pi,\lambda)} : I \to I$ is a minimal and ergodic IET exchanging intervals I_α, $\alpha \in \mathcal{A}$. Suppose that there exists a nested sequence $(I^{(k)})_{k\geqslant 0}$, ($I^{(0)} = I$) such that the induced map $T^{(k)} : I^{(k)} \to I^{(k)}$ on $I^{(k)}$ is an IET exchanging d-intervals and $T^{(k)} = T_{(\pi^{(k)},\lambda^{(k)})}$. Moreover, assume that there is a subset $A \subset S_d \times \mathbb{R}^d_{>0}$ and a map $R : A \to A$ such that

$$(\pi^{(k)}, \lambda^{(k)}) = R^k(\pi, \lambda)$$

and the projectivization of R is ergodic. Such sequences of intervals are usually obtained by accelerations of so called Rauzy-Veech induction.

By the definition of the induced map, $T^{(k)}x = T^{\tau^{(k)}(x)}x$, where $\tau^{(k)} : I^{(k)} \to \mathbb{N}$ is the first return time map to $I^{(k)}$. In fact, $\tau^{(k)}$ is piecewise constant and denote by $\tau^{(k)}_\alpha$ the common first return time on $I^{(k)}_\alpha$. Finally, we can define the renormalization operator

$$S(k) : L^1(I) \to L^1(I^{(k)}), \qquad S(k)\varphi(x) = \varphi^{(\tau^{(k)}_\alpha)}(x) \text{ for } x \in I^{(k)}_\alpha.$$

Renormalization operator

$$S(k)(AC^{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)) \subset AC^{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$$
$$S(k)(LOG(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)) \subset LOG(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$$
$$S(k)(P_a(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)) \subset P_a(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)}).$$

If $\Gamma^{(k)}$ denotes the space of functions constant on intervals $I_\alpha^{(k)}$, $\alpha \in \mathcal{A}$, then $S(k)\Gamma^{(0)} = \Gamma^{(k)}$.
As $\Gamma^{(k)}$ can be identified with $\mathbb{R}^d$, the restricted operator can be identified with a matrix $Q(k) \in SL(d,\mathbb{Z})$. Moreover, $Q(k)$ can be treated as an $SL(d,\mathbb{Z})$-valued cocycle over the transformation $R : P(A) \to P(A)$ called a renormalization. This is an acceleration of so called Kontsevich-Zorich cocycle. By the ergodic Oseledets theorem, sympecticity of the cocycle and Avila-Viana (2007) about the simplicity of the spectrum, it has Lyapunov exponents of the form

$$-\lambda_1 < \ldots < -\lambda_g < 0 = \ldots = 0 < \lambda_g < \ldots < \lambda_1.$$

Accelerated KZ-cocycle

Using an invertible version of the renormalization map $R : P(A) \to P(A)$ we can construct a spliting

$$\Gamma = \bigoplus_{1 \leqslant i \leqslant g} \Gamma_{-i} \oplus \Gamma_0 \oplus \bigoplus_{1 \leqslant i \leqslant g} \Gamma_i (=: \Gamma_u)$$

such that $\dim \Gamma_{\pm i} = 1$ for $1 \leqslant i \leqslant g$ and

$$\lim_{k \to +\infty} \frac{\log \|Q(k)h\|}{k} = \lambda_i \text{ if } h \in \Gamma_i \text{ for some } -g \leqslant i \leqslant g \ (\lambda_0 = 0).$$

Let $h_1, \ldots, h_g$ be a basis of the unstable subspace such that

$$\lim_{k \to +\infty} \frac{\log \|S(k)h_i\|_{\sup}}{k} = \lambda_i.$$

In 1997, Zorich observed that for a.e. IET T we also have

$$\limsup_{n \to +\infty} \frac{\log \|h_i^{(n)}\|_{\sup}}{\log n} = \nu_i := \frac{\lambda_i}{\lambda_1}.$$

On the other hand, for every $h_i \in \Gamma_u$ there exists a C^∞ map $\xi_i : M \to \mathbb{R}$ vanishing on a neighborhood of any fixed point such that $\varphi_{\xi_i} = h_i$. Then

$$\limsup_{T\to+\infty} \frac{\log \left| \int_0^T \xi_i(\psi_t x)dt \right|}{\log T} \leqslant \nu_i \text{ for every regular orbit starting form } x$$

$$\limsup_{T\to+\infty} \frac{\log \left| \int_0^T \xi_i(\psi_t x)dt \right|}{\log T} = \nu_i \text{ for a.e. } x$$

$$\limsup_{T\to+\infty} \frac{\log \left\| \int_0^T \xi_i \circ \psi_t dt \right\|_{L^1}}{\log T} = \nu_i.$$

So $\int_0^T \xi_i(\psi_t x)dt$ is a good candidate to play the role of $u_i(T,x)$ in the decomposition

$$\int_0^T f(\psi_t(x))dt = \sum_{i=1}^{g} D_i(f)u_i(T,x) + err(f,T,x).$$

Marmi-Mousa-Yoccoz constructed the following (correction) operator (their original construction is different from what I present) $\mathfrak{h} : AC^{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)) \to \Gamma_u$ given by

$$\mathfrak{h}(\varphi) = \lim_{k\to\infty} Pr_{\Gamma_u} \circ Q(k)^{-1} \circ M^{(k)} \circ S(k)(\varphi),$$

where $M^{(k)} : AC^{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)}) \to \Gamma^{(k)}$ is the mean-value operator given by

$$M^{(k)}(\varphi)(x) = \frac{1}{|I_\alpha^{(k)}|} \int_{I_\alpha^{(k)}} \varphi(y)dy \text{ if } x \in I_\alpha^{(k)}.$$

One of the important challenges here is to show that this operator is well defined as well as bounded. Moreover,

$$\lim_{k\to+\infty} \frac{\log \|S(k)(\varphi - \mathfrak{h}(\varphi))\|_{\sup}}{k} \leqslant 0.$$

Correction operator

Note that if $\varphi = h \in \Gamma_u$ then we can take $\mathfrak{h}(h) = h$.
As $h_1, \dots h_g$ is a basis of Γ_u, we can define g functionals $d_i : AC^{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \to \mathbb{R}$, $i = 1, \dots g$ such that

$$\mathfrak{h}(\varphi) := \sum_{i=1}^{g} d_i(\varphi) h_i.$$

Finally, we define D_i as $D_i(f) := d_i(\varphi_f)$.
Moreover, for every smooth observable $f : M \to \mathbb{R}$ vanishing on a neighborhood of saddles we use the following decomposition

$$f(x) = \sum_{i=1}^{g} D_i(f)\xi_i(x) + f_e(x).$$

By passing to ergodic integrals, this gives

$$\int_0^T f(\psi_t x)dt = \sum_{i=1}^{g} D_i(f)\int_0^T \xi_i(\psi_t x)dt + \int_0^T f_e(\psi_t x)dt$$
$$= \sum_{i=1}^{g} D_i(f)u_i(T,x) + err(f,T,x).$$

On the other hand, applying the operator $f \mapsto \varphi_f$, we have

$$\varphi_f = \sum_{i=1}^{g} d_i(\varphi_f)h_i + \varphi_{f_e}.$$

Next, we apply the operator $\mathfrak{h}$ to obtain

$$\mathfrak{h}(\varphi_f) = \sum_{i=1}^{g} d_i(\varphi_f)\mathfrak{h}(h_i)+\mathfrak{h}(\varphi_{f_e}) = \sum_{i=1}^{g} d_i(\varphi_f)h_i+\mathfrak{h}(\varphi_{f_e}) = \mathfrak{h}(\varphi_f)+\mathfrak{h}(\varphi_{f_e}),$$

so $\mathfrak{h}(\varphi_{f_e}) = 0$.

Error term

As $\mathfrak{h}(\varphi_{f_e}) = 0$, we have

$$\lim_{k\to+\infty} \frac{\log \|S(k)(\varphi_{f_e})\|_{\sup}}{k} \leqslant 0.$$

This gives

$$\limsup_{n\to+\infty} \frac{\log \|\varphi_{f_e}^{(n)}\|_{\sup}}{\log n} \leqslant 0$$

and finally

$$\limsup_{T\to+\infty} \frac{\log \|err(f, T, \cdot)\|_{\sup}}{\log T} = \limsup_{T\to+\infty} \frac{\log \| \int_0^T f_e \circ \psi_t \, dt\|_{\sup}}{\log T} \leqslant 0.$$

This completes the proof of deviation formula (deviation spectrum) for smooth observables f vanishing around fixed points.

General observables - non-degenerate case

Suppose that $\psi_{\mathbb{R}}$ has only non-degenerate fixed points and let us consider smooth observables $f : M \to \mathbb{R}$ which can be non-zero at some saddles in $Fix(\psi_{\mathbb{R}}) \cap M'$. Then $\varphi_f \in LOG \setminus AC^{BV}$.

Theorem (A) (Ulcigrai-F)

The correction operator $\mathfrak{h}$ can be extended to $\mathfrak{h} : LOG(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \to \Gamma_u$ so that for every $\varphi \in LOG(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ we have

$$\lim_{k\to+\infty} \frac{\log \frac{1}{|I^{(k)}|}\|S(k)(\varphi - \mathfrak{h}(\varphi))\|_{L^1(I^{(k)})}}{k} \leqslant 0.$$

Moreover, if φ has logarithmic singularities of symmetric type $\sum C_\alpha^+ = \sum C_\alpha^-$, then

$$\frac{1}{|I^{(k)}|}\|S(k)(\varphi - \mathfrak{h}(\varphi))\|_{L^1(I^{(k)})} \text{ is bounded along a subsequence.}$$

General observables - non-degenerate case

Using the overall strategy outlined earlier, based on the first part of Theorem (A), we have confirmed completely the Konstevich-Zorich conjecture.

Theorem (Ulcigrai-F)

For a.e. locally Hamiltonian flow $\psi_{\mathbb{R}}$ with non-degenerate fixed points restricted to its minimal component $M' \subset M$ there are g invariant distribution $D_i : C^2(M) \to \mathbb{R}$ and cocycles $u_i(t,x)$ for $i = 1, \dots, g$ such that for every $f \in C^2$ we have

$$\int_0^T f(\psi_t(x))dt = \sum_{i=1}^{g} D_i(f) u_i(T,x) + err(f,T,x),$$

where for a.e. $x \in M$ we have

$$\limsup_{T \to +\infty} \frac{\log |u_i(T,x)|}{\log T} = \nu_i, \quad \lim_{T \to +\infty} \frac{\log |err(f,T,x)|}{\log T} = 0.$$

General observables - non-degenerate minimal case

If $\psi_{\mathbb{R}}$ is additionally minimal, based on the second part of Theorem (A), we provide much more accurate information on the behavior of the error term. We have the following dichotomy:

Theorem (Ulcigrai-F)

Suppose additionally that the locally Hamiltonian flow $\psi_{\mathbb{R}}$ is minimal on M. If $f \in C^2$ and $f(\sigma) = 0$ for every $\sigma \in Fix$ then the error term is uniformly bounded, i.e. there exists $C > 0$ such that

$$|err(f,t,x)| \leqslant C \text{ for every } x \in M \text{ and } t \in \mathbb{R}.$$

If $f \in C^2$ and $f(\sigma) \neq 0$ for some $\sigma \in Fix$ then the error term is equidistributed on $\mathbb{R}$ for a.e. $x \in M$, i.e. for a.e. $x \in M$, for any pair of finite intervals $I, J \subset \mathbb{R}$ we have

$$\lim_{T\to\infty} \frac{|\{t \in [0,T] : err(f,t,x) \in I\}|}{|\{t \in [0,T] : err(f,t,x) \in J\}|} = \frac{|I|}{|J|}.$$

General observables - degenerate case

Suppose that the locally Hamiltonian flow $\psi_{\mathbb{R}}$ has multiple saddles and f is non-trivial around them. Then $\varphi_f \in P_a \setminus LOG$ for some $0 < a < 1$. Unfortunately (or fortunately), the correction operator $\mathfrak{h} : LOG(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \to \Gamma_u$ cannot be extended to $P_a(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. However, by modifying the previous construction we can construct another operator

$$\mathfrak{h}_j : P_a(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \to \bigoplus_{1\leqslant i\leqslant j} \Gamma_i \subset \Gamma_u,$$

whenever $\lambda_{j+1} \leqslant a\lambda_1 < \lambda_j$. In fact, we construct (with Minsung Kim) a one-parameter family of such correction operators. They help us for prove deviation spectrum in much more complicated form than in the non-degenerate case. In the general case, there are new exponents that are not derived from Lyapunov exponents for the KZ-cocycles.

Theorem (Kim-F)

For a.e. $\psi_{\mathbb{R}}$ *for every* $f \in C^m$ $(m = \max_\sigma m_\sigma)$ *we have*

$$\int_0^T f(\psi_t(x))dt = \sum_{\sigma \in \mathrm{Fix}(\psi_{\mathbb{R}}) \cap M'} \sum_{\substack{\alpha \in \mathbb{Z}^2_{\geqslant 0} \\ |\alpha| < m_\sigma - 2}} \partial_\sigma^\alpha(f) c_{\sigma,\alpha}(T,x)$$

$$+ \sum_{i=1}^{g} D_i(f) u_i(T,x) + err(f,T,x)$$

with

$$\limsup_{T\to\infty} \frac{\log |c_{\sigma,\alpha}(T,x)|}{\log T} = \frac{m_\sigma - 2 - |\alpha|}{m_\sigma} \text{ for a.e. } x \in M';$$

$$\limsup_{T\to\infty} \frac{\log |u_i(T,x)|}{\log T} = \nu_i \text{ for a.e. } x \in M';$$

$$\limsup_{T\to\infty} \frac{\log |err(f,T,x)|}{\log T} \leqslant 0 \text{ for a.e. } x \in M'.$$

Thank you for your attention!

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