Bifurcations in spaces of meromorphic maps

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Joint work with M. Astorg, N. Fagella

Dynamics and Finance: from KAM Tori to ETFs
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For Stefano Marmi's Birthday

(Dynamics of) Meromorphic maps

A meromorphic map is a holomorphic map $f : \mathbb{C} \to \hat{\mathbb{C}}$

- ∞ is an essential singularity
- It has poles (!)
- A covering if you remove critical values (images of critical points) and asymptotic values like 0 for e^z
- As an individual dynamical system, we consider $f^n = f \circ \ldots \circ f$ and look at orbit of points

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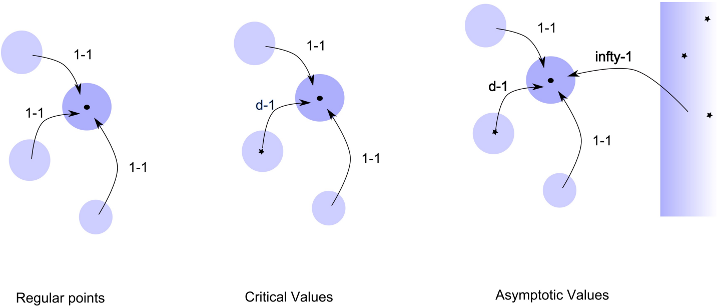
Covering properties of transcendental maps

A singular value is a value near which not all inverse branches of the inverse all well defined. They can be critical, asymptotic, or an accumulation thereof.



- f is a covering outside the set of singular values ($f : \mathbb{C} \setminus f^{-1}(S(f)) \to \mathbb{C} \setminus S(f)$ is a covering)
- We will always assume we have finitely many singular values (finite type)

av/∞; homeo

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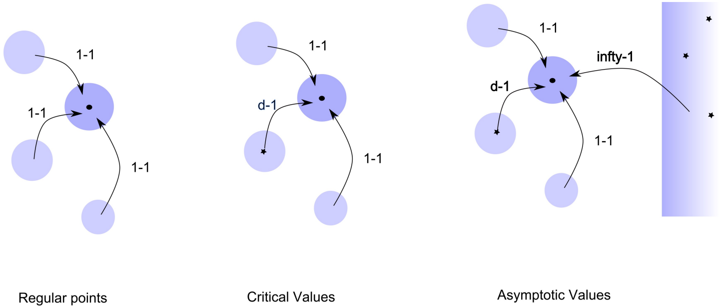

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Fatou and Julia set

In holomorphic dynamics, the plane splits into two completely invariant subsets:

- The Fatou set is the maximal open set on which the dynamics is stable, it contains for example attracting basins, rotation domains;
- The Julia set is the set on which the dynamic is chaotic, and is the closure of repelling periodic points.

The beauty of going natural

Definition

A natural family of meromorphic maps $(f_\lambda)_{\lambda\in M}$, M complex manifold, is a family of meromorphic maps with the same covering properties, depending holomorphically on $\lambda \in M$.

Theorem (ABF)

Locally, being a natural family is equivalent to the fact that all singular values and their preimages can be expressed as holomorphic functions of λ and that they do not collide (fig)

Setup; Eremenko-Lyubich/Epstein; Rempe Schleicher Fagella Keen Kotus Chen; Examples; Rat_d, cubic

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A map f in a (natural) family of meromorphic maps is structurally stable if f is conjugate to every map in a neighborhood thereof (and the conjugacy depends continuously on the parameter).

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What's special about structural stability in complex dynamics?

- It is usually replaced by J-stability rigidity: centers
- It is related to
  - Bifurcations of periodic points
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Mañé-Sad-Sullivan

We say that a map is J-stable if it is conjugate to all maps in a neighborhood of itself, when restricted to the Julia set

Theorem (MSS'83, Lyu'84, EL'92)

Let $\{f_\lambda\}_{\lambda \in M}$ be a natural family of rational or entire maps with finitely many singular values. Then, the following are equivalent.

(a) *f_{λ_0} is J-stable;*

(b) *There are no bifurcations of periodic points;*

(c) *All singular values are passive in a nbhd of λ_0.*

Corollary

J- stable parameters form an open dense set; equivalently, the bifurcation locus has empty interior; structurally stable parameters are dense.).

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What are bifurcations

Bifurcations:

Let $\mathcal{E}_\lambda(x) = 0$ be an equation (differential, functional, algebraic, variational problems...) depending on a parameter λ. Solutions (or lack thereof) usually depend nicely on the parameter. When this dependence breaks at some parameter, we think of it as a bifurcation. (parameters at which local uniqueness is lost)

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Bifurcations of periodic points

Let $(f_\lambda)_{\lambda \in M}$ be a natural family. Fix k and consider

$$\mathcal{E}_\lambda : f_\lambda^k(z) = z$$

Set of Solutions:

$$\mathrm{Per}_k := \{(\lambda, z) : f_\lambda^k(z) = z\} \subset M \times \hat{\mathbb{C}}$$

Goal: Study a new type of bifurcation of periodic points that only happens for meromorphic maps (as opposed to polynomials, rational and transcendental entire maps)

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Bifurcations II

$$\text{Per}_k := \{(\lambda, z) : f_\lambda^k(z) = z\} \subset M \times \hat{\mathbb{C}}$$

Definition

We have a bifurcation at λ_0 if we cannot extend each solution $z(\lambda_0)$ as a holomorphic function $z(\lambda)$ in a neighborhood of λ_0.

Other points of view:

- λ_0 is a singular value for the projection $\pi : \text{Per}_k \to M$
- Think about the implicit function theorem:

$$G(\lambda, z) = f_\lambda^k(z) - z = 0$$

  to write $z = z(\lambda)$ near (λ_0, z_0) we need
  - $\partial_z G|_{\lambda_0, z_0} = (f_{\lambda_0}^k)'(z_0) - 1$ is well defined (???)
  - $\partial_z G|_{\lambda_0, z_0} \neq 0$, that is, $(f_{\lambda_0}^k)'(z_0) \neq 1$ (parabolic cycles)

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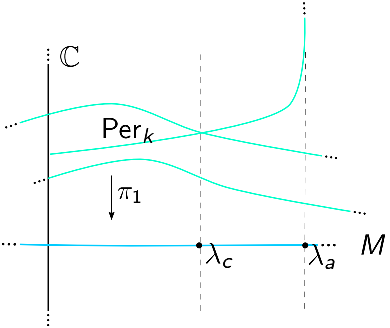
Periodic points



Proposition

The only possible bifurcations are cycle collisions (parabolic), cycles disappearing to ∞ (or accumulation thereof).

Different bifurcations in different families

Theorem (Eremenko-Lyubich)

For transcendental entire functions, all bifurcations are of parabolic type.

(For rational maps this is easy)

As opposed to it

Theorem (Astorg-B-Fagella)

For meromorphic maps (with at least one non omitted pole) bifurcations of the asymptotic type are dense in the bifurcation locus (whatever that is)

Why do we need meromorphic maps for this type of bifurcations? (fig)

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Why periodic points?

What is the relationship with dynamics?

No bifurcations of periodic points at λ_0,

$\Downarrow$ Λ-Lemma (MSS, Lyubich)

the Julia set moves holo respecting the dynamics

$\Downarrow$

all maps in a neighborhood of λ_0 are topologically conjugate to each other on the Julia set

Special feature in complex dynamics: Bifurcations of periodic points can be related in a precise way to a sudden change in the dynamics of singular values (Levin, MSS/Lyubich, Astorg-B-Fagella)

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Mane-Sad-Sullivan for meromorphic maps

Theorem (ABF)

Let $(f_\lambda)_{\lambda\in M}$ be a natural family of finite type meromorphic maps, $U \subset M$. Then the following are equivalent:

- *Maps in U are J-stable;*
- *Periodic points do not bifurcate in U;*
- *There are no active singular values in U;*
- *There are no (non-persistent) parabolic parameters in U.*

Showing that an asymptotic value is involved (virtual cycle) when cycles go to infinity is not too difficult, but showing that it- or someone else- needs to be active is very delicate. EL: ∞ is a persistent asymptotic value

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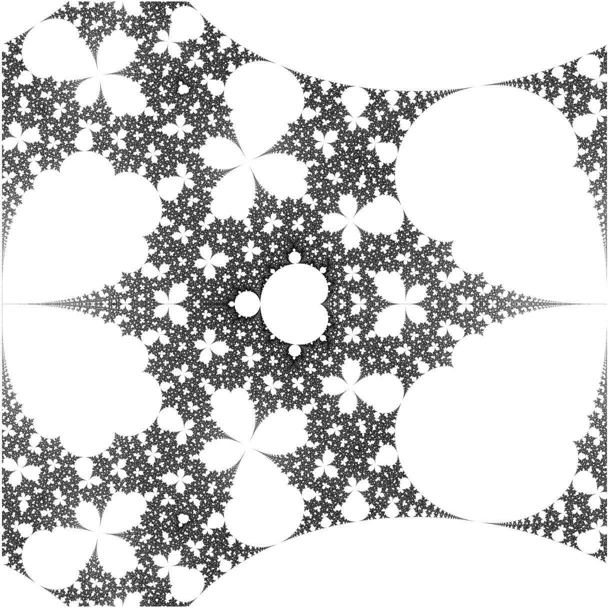
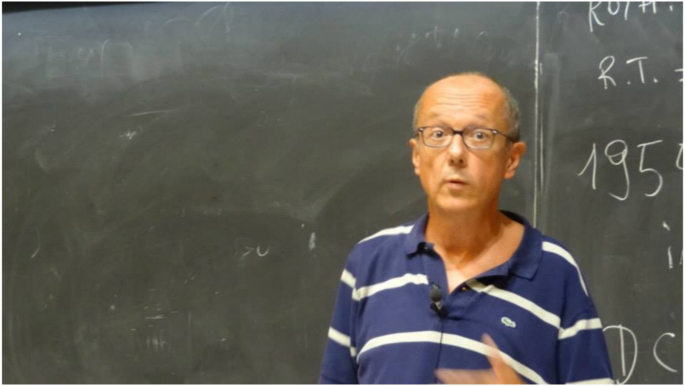
Could we do more?

- When there are infinitely many singular values, a Newhouse-type phenomena has been announced by Epstein-Rempe, which would lead to bifurcation locus with non-empty interior
- Our results apply to the class of finite type maps: holomorphic maps from $W \to \hat{\mathbb{C}}$ with finitely many singular values Epstein, cpt Riemann Surface, need tameness



$f_\lambda(z) = \pi \tan^2(z) + \lambda$ (centers and virtual centers)

Happy Birthday Stefano!



Active and passive

Definition

Let (f_λ) be a natural family, $v_\lambda = \phi_\lambda(v)$ a singular value for f_λ. Then v_λ is active at λ_0 if either

- $(f_\lambda^n(v_\lambda))_{n\in\mathbb{N}}$ *is (nonpersistently) not well defined at λ_0 (NEW!)*
  OR
- $(f_\lambda^n(v_\lambda))_{n\in\mathbb{N}}$ *is not normal in a nbhd of λ_0.*

Explanation:

- $f_{\lambda_0}^k(v_{\lambda_0}) = \infty$ for some k, NON PERSISTENTLY. We say that λ is a singular parameter (critical or asymptotic parameter).
- Examples:
  - If v_{λ_0} tends to an attracting fixed point a_{λ_0}, the attracting point is persistent in a λ-nbhod of λ_0, and $f_\lambda^n(v_\lambda) \to a_\lambda$ in that nbhd.
  - If v_{λ_0} tends to a parabolic fixed point a_{λ_0}, in a λ-nbhod of λ_0 we can find local 1D disk on which a_{λ_0} becomes attracting and repelling in different directions, hence $f_\lambda^n(v_\lambda)$ cannot have convergent subsequences.

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Pushing results to the boundary

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