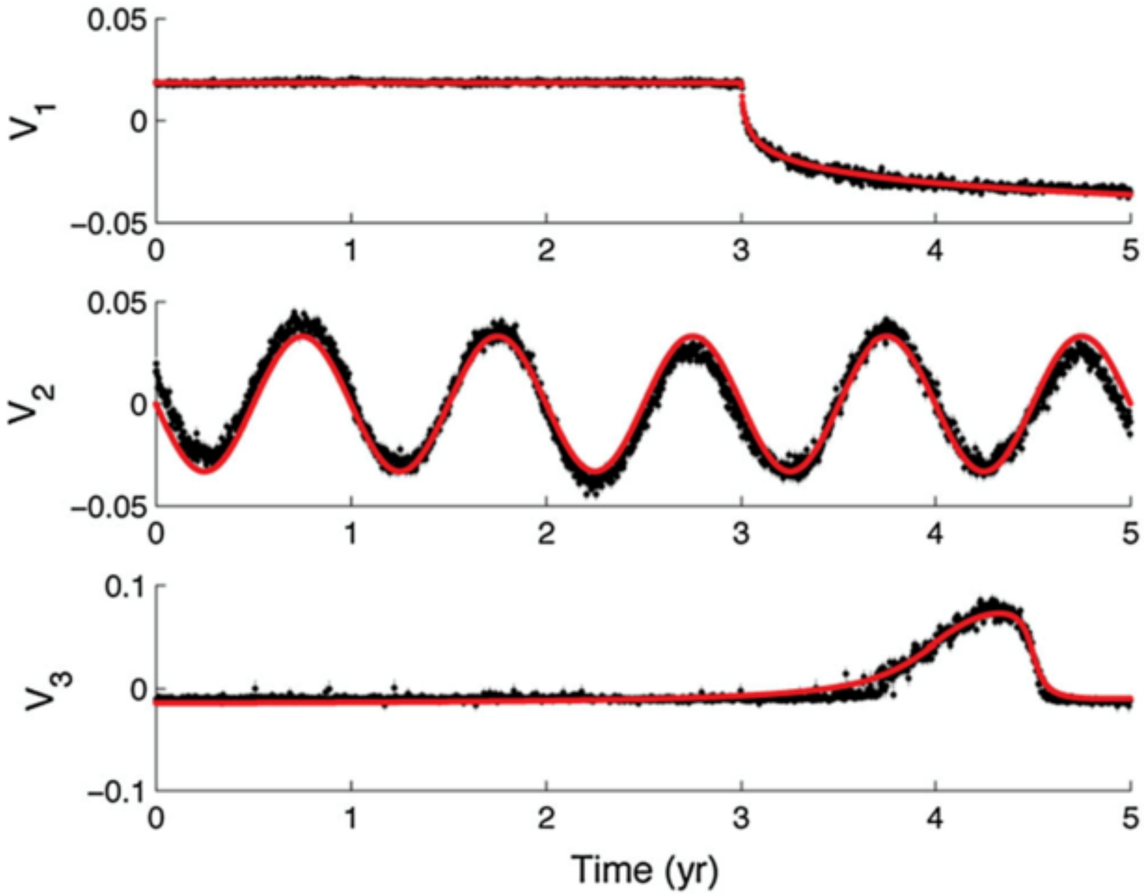
Signal Extraction From GNSS Position Times Series



Gualandi et al., 2016, *JOGE*

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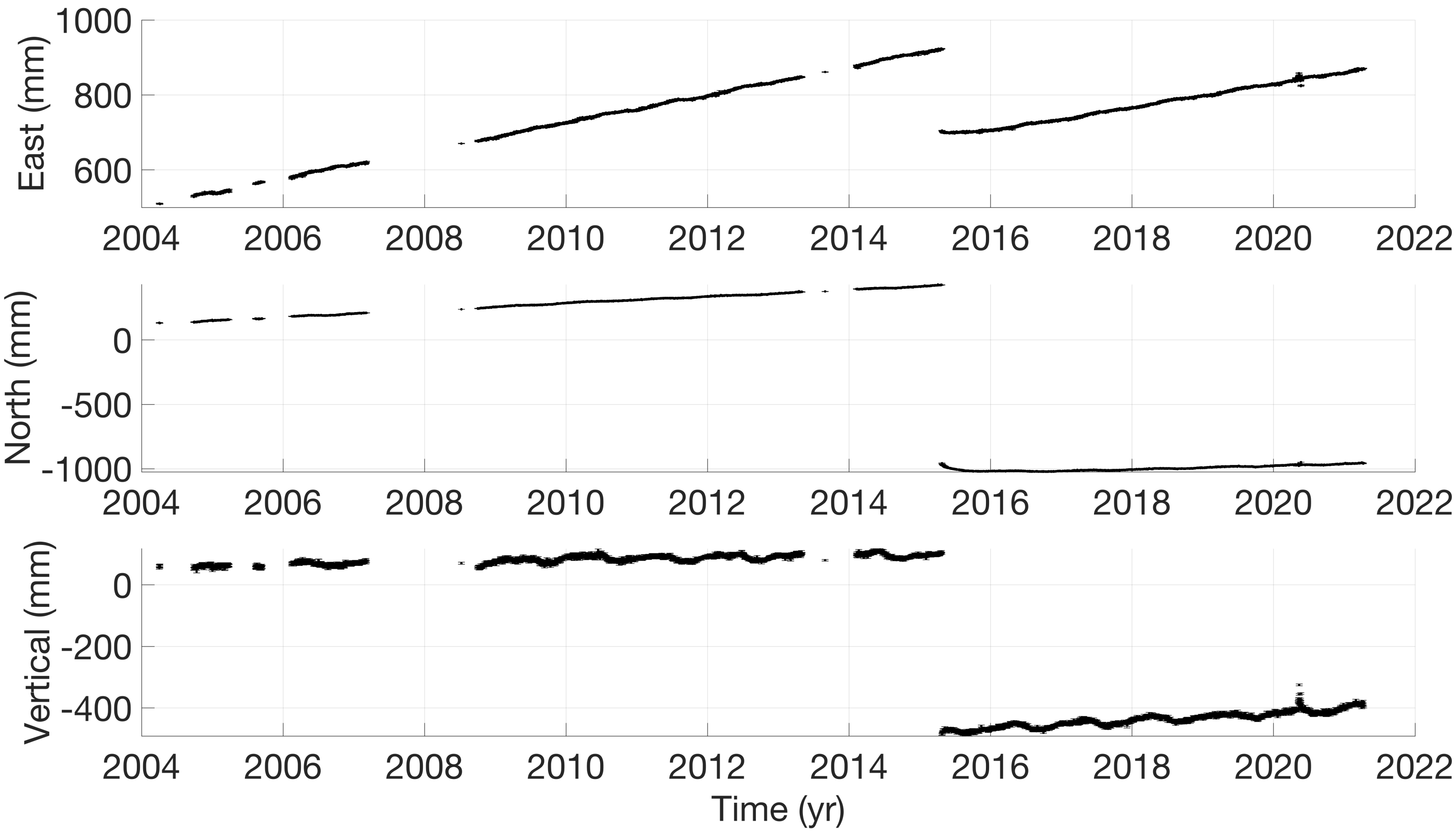
ICTP, Trieste

Workshop on
Mechanics of the Earthquake Cycle

23-24 Oct 2023

Goal

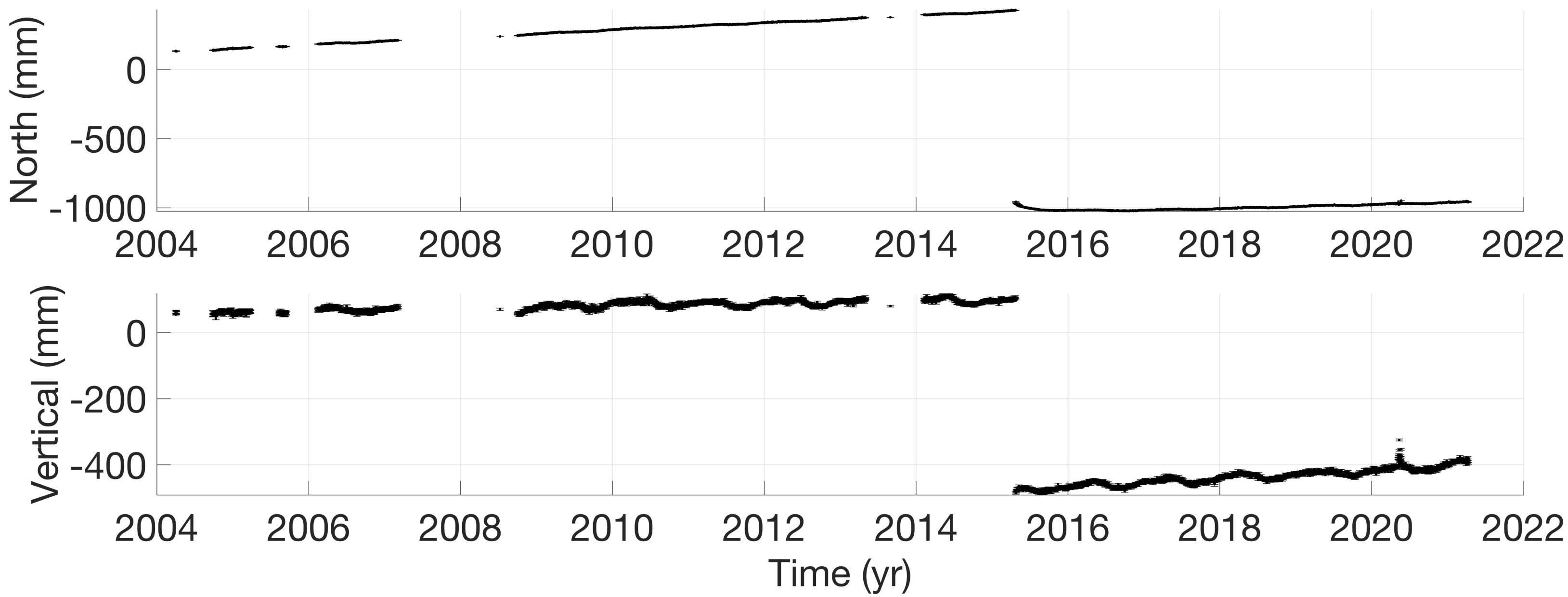
What model would you use to fit these time series ?



Trajectory Model

What model would you use to explain these time series ?

$$x(t) = x_{\text{linear}}(t) + x_{\text{offsets}}(t) + x_{\text{seasonal}}(t) + ?$$



Univariate Vs Multivariate Models

Univariate	Multivariate
Fit time series one-by-one	Fit all time series at once
Pre-determined functions	Data-driven modes

Trajectory Model

$$x(t) = x_{\text{linear}}(t) + x_{\text{offsets}}(t) + x_{\text{seasonal}}(t) + ?$$

$$\begin{aligned} x(t) = {} & q + m(t - t_0) + \\ & + \sum_{i=1}^{N^{offsets}} A_i^{\text{offsets}} \mathrm{H}(t - t_i^{\text{offsets}}) + \\ & + A \sin\left(\frac{2\pi t}{T} + \phi\right) + \\ & + ? \end{aligned}$$

Seasonal Signal

$$x_{\text{seasonal}}(t) = A_1 \sin\left(\frac{2\pi t}{T_1} + \phi_1\right) + A_2 \sin\left(\frac{2\pi t}{T_2} + \phi_2\right)$$

$$T_1 = 1 \text{ yr}$$

$$T_2 = \frac{1}{2} \text{ yr}$$

$$= S_1 \sin\left(\frac{2\pi t}{T_1}\right) + C_1 \cos\left(\frac{2\pi t}{T_1}\right) + S_2 \sin\left(\frac{2\pi t}{T_2}\right) + C_2 \cos\left(\frac{2\pi t}{T_2}\right)$$

Trajectory Model

$$x(t) = x_{\text{linear}}(t) + x_{\text{offsets}}(t) + x_{\text{seasonal}}(t) + ?$$

$$x(t) = q + m(t - t_0) +$$

$$+ \sum_{i=1}^{N^{offsets}} A_i^{\text{offsets}} \mathrm{H}(t - t_i^{\text{offsets}}) +$$

$$+ A \sin\left(\frac{2\pi t}{T} + \phi\right) +$$

$$+ ?$$

Trajectory Model

$$x(t) = x_{\text{linear}}(t) + x_{\text{offsets}}(t) + x_{\text{seasonal}}(t) + ?$$

$$x(t) = q + m(t - t_0) +$$

$$+ \sum_{i=1}^{N^{offsets}} A_i^{\text{offsets}} \mathrm{H}(t - t_i^{\text{offsets}}) +$$

$$+ S_1 \sin\left(\frac{2\pi t}{T_1}\right) + C_1 \cos\left(\frac{2\pi t}{T_1}\right) + S_2 \sin\left(\frac{2\pi t}{T_2}\right) + C_2 \cos\left(\frac{2\pi t}{T_2}\right) +$$

$$+ ?$$

Offsets

What can generate an offset ?

Earthquakes

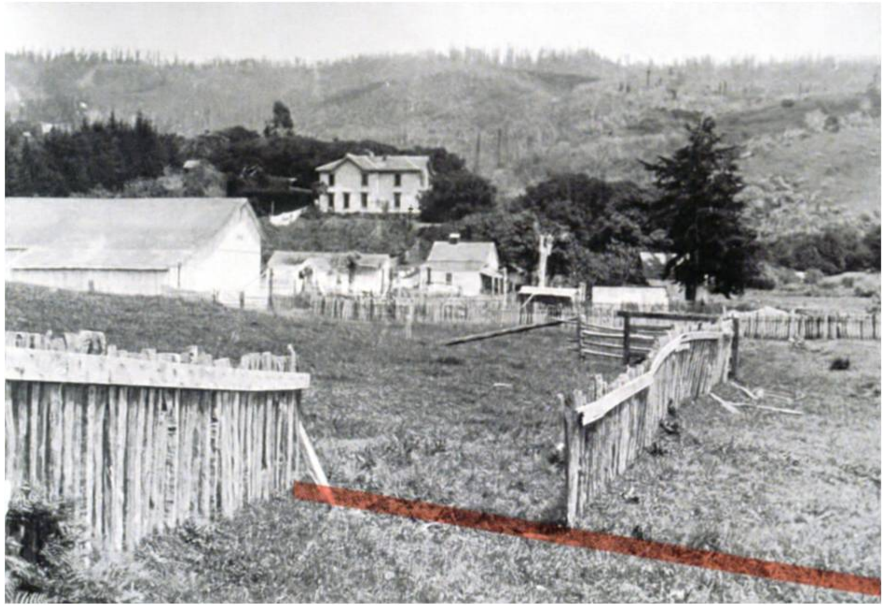


Photo by G.K. Gilbert

1906 San Francisco: $\sim$ 3 m

Antenna change / firmware update

Trajectory Model

$$x(t) = x_{\text{linear}}(t) + x_{\text{offsets}}(t) + x_{\text{seasonal}}(t) + ?$$

$$x(t) = q + m(t - t_0) +$$

$$+ \sum_{i=1}^{N^{offsets}} A_i^{\text{offsets}} \mathrm{H}(t - t_i^{\text{offsets}}) +$$

$$+ S_1 \sin\left(\frac{2\pi t}{T_1}\right) + C_1 \cos\left(\frac{2\pi t}{T_1}\right) + S_2 \sin\left(\frac{2\pi t}{T_2}\right) + C_2 \cos\left(\frac{2\pi t}{T_2}\right) +$$

$$+ ?$$

Trajectory Model

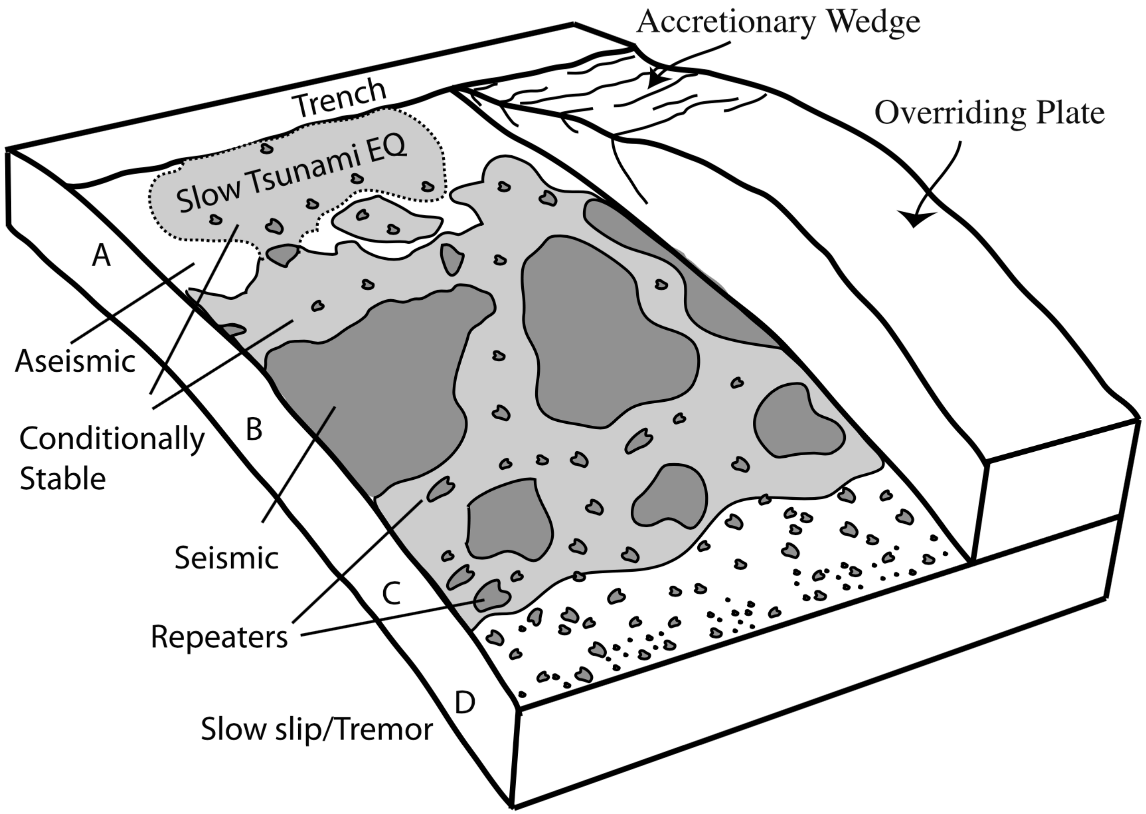
$$x(t) = x_{\text{linear}}(t) + x_{\text{offsets}}(t) + x_{\text{seasonal}}(t) + ?$$

$$x(t) = q + m(t - t_0) +$$
$$+ \sum_{i=1}^{N^{instr}} A_i^{\text{instr}} \mathrm{H}(t - t_i^{\text{instr}}) + \sum_{i=1}^{N^{eq}} A_i^{\text{eq}} \mathrm{H}(t - t_i^{\text{eq}}) +$$
$$+ S_1 \sin\left(\frac{2\pi t}{T_1}\right) + C_1 \cos\left(\frac{2\pi t}{T_1}\right) + S_2 \sin\left(\frac{2\pi t}{T_2}\right) + C_2 \cos\left(\frac{2\pi t}{T_2}\right) +$$
$$+ ?$$

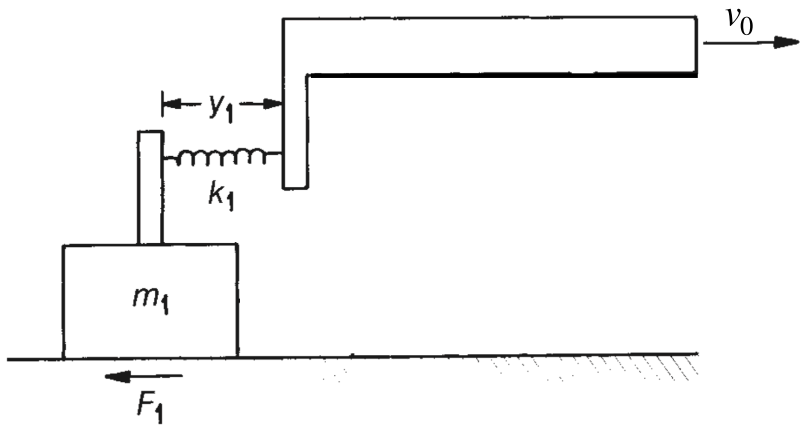
Post-seismic transients → Afterslip $\delta(t) \simeq v_0 t_r \ln\left(1 + \frac{t}{t_{as}}\right)$

Post-seismic transients → Viscoelastic relaxation $\delta(t) = v_0 t + \frac{\Delta\tau}{k}\left[1 - \exp\left(-\frac{k}{\eta} t\right)\right]$

Afterslip

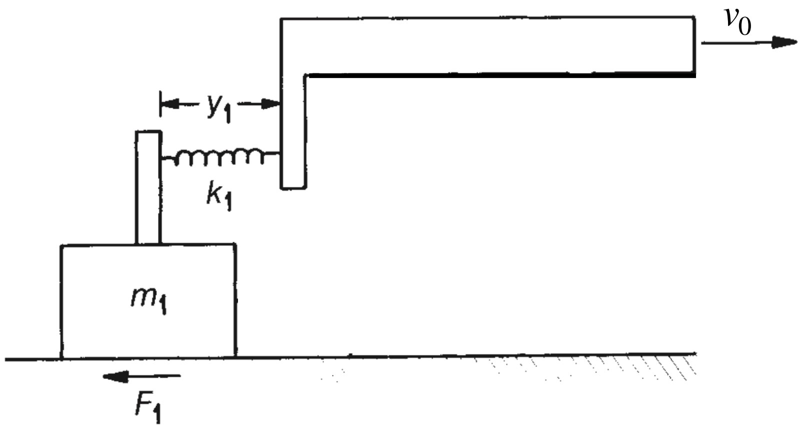


Lay et al., 2012, *JGR*.



Huang and Turcotte, 1990, *Nature*

Friction Law

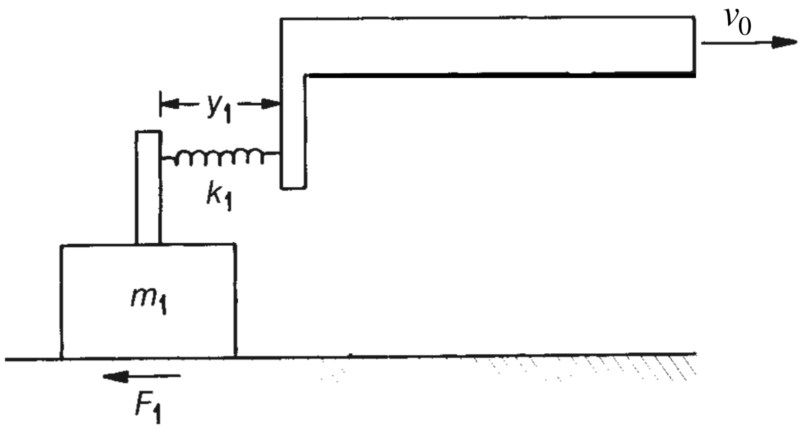


Huang and Turcotte, 1990, *Nature*

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = \sum_j \tau_j = \tau_i + \tau_l - \tau_f + \Delta\tau(t)$$

Perfettini and Avouac, 2004a, *JGR*

Friction Law



Huang and Turcotte, 1990, *Nature*

$$m\frac{dv}{dt} = \sum_j \tau_j = \tau_i + \tau_l - \tau_f + \Delta\tau H(t)$$

Quasi-static approximation:

$$0 = \tau_i + \tau_l - \tau_f + \Delta\tau H(t)$$

$$\tau_l = k(v_0 t - \delta)$$

$$\tau_f = \sigma_n \mu = \sigma_n \left[\mu_0 + a\ln\frac{v}{v_*} + b\ln\frac{v_*\theta}{L}\right]$$

$$\overset{\theta=\theta_{ss}}{=} \sigma_n\left[\mu_0 + (a-b)\ln\frac{v}{v_*}\right]$$

$$0 = \tau_i + k(v_0 t - \delta) - \sigma_n\left[\mu_0 + (a-b)\ln\frac{v}{v_*}\right] + \Delta\tau H(t)$$

Perfettini and Avouac, 2004a, *JGR*

Friction Law

$$0 = \tau_i + k(v_0 t - \delta) - \sigma_n \left[\mu_0 + (a-b)\ln\frac{v}{v_*}\right] + \Delta\tau \mathrm{H}(t)$$

$$0 = \tau_i + k(v_0 t_0 - \delta_i) - \sigma_n \left[\mu_0 + (a-b)\ln\frac{v_i}{v_*}\right] + \Delta\tau \mathrm{H}(t_0)$$

$$\tau_i = \sigma_n \left[\mu_0 + (a-b)\ln\frac{v_i}{v_*}\right]$$

$$0 = k(v_0 t - \delta) - (a-b)\sigma_n \ln\frac{v}{v_i} + \Delta\tau \mathrm{H}(t)$$

$$(a-b)\sigma_n \ln\frac{\dot{\delta}}{v_i} = k(v_0 t - \delta) + \Delta\tau \mathrm{H}(t)$$

$$t = t_0 = 0$$
$$\delta = \delta_i = 0$$
$$v = v_i$$

$$v = \dot{\delta}$$

Perfettini and Avouac, 2004a, *JGR*

Friction Law

$$(a-b)\sigma_n \ln \frac{\dot{\delta}}{v_i} = k(v_0 t - \delta) + \Delta\tau \mathrm{H}(t)$$

$$\ln \frac{\dot{\delta}}{v_i} = \frac{k}{(a-b)\sigma_n}(v_0 t - \delta) + \frac{\Delta\tau}{(a-b)\sigma_n}\mathrm{H}(t)$$

$$c = \frac{k}{(a-b)\sigma_n}$$

$$\dot{\delta} = v_i \exp\left[c(v_0 t - \delta) + \frac{\Delta\tau}{(a-b)\sigma_n}\mathrm{H}(t)\right]$$

$$t_r = \frac{1}{c v_0} = \frac{(a-b)\sigma_n}{k v_0}$$

$$\delta(t) = \frac{1}{c}\ln\left[1 + c v_i F(t)\right]$$

$$F(t) = \int_{0+}^{t} \exp\left[\frac{t'}{t_r} + \frac{\Delta\tau}{(a-b)\sigma_n}\mathrm{H}(t')\right] \mathrm{d}t'$$

Perfettini and Avouac, 2004a, *JGR*

Friction Law

$$\delta(t) = \frac{1}{c}\ln\left[1 + cv_i F(t)\right] \qquad F(t) = \int_{0+}^{t} \exp\left[\frac{t'}{t_r} + \frac{\Delta\tau}{(a-b)\sigma_n}\mathrm{H}(t')\right]\mathrm{d}t'$$

$$F(t) = \exp\left(\frac{\Delta\tau}{(a-b)\sigma_n}\right)\int_{0+}^{t}\exp\left(\frac{t'}{t_r}\right)\mathrm{d}t' = \exp\left(\frac{\Delta\tau}{(a-b)\sigma_n}\right)t_r\exp\left(\frac{t'}{t_r}\right)\Bigg|_{0+}^{t}$$

$$= \exp\left(\frac{\Delta\tau}{(a-b)\sigma_n}\right)t_r\left[\exp\left(\frac{t}{t_r}\right) - 1\right] \qquad t_r = \frac{1}{cv_0} = \frac{(a-b)\sigma_n}{kv_0}$$

$$\delta(t) = \frac{1}{c}\ln\left[1 + \frac{v_i}{v_0}\exp\left(\frac{\Delta\tau}{(a-b)\sigma_n}\right)\left(\exp\left(\frac{t}{t_r}\right) - 1\right)\right]$$

Perfettini and Avouac, 2004a, *JGR*

Friction Law

$$\delta(t) = \frac{1}{c} \ln \left[1 + \frac{v_i}{v_0} \exp\left(\frac{\Delta\tau}{(a-b)\sigma_n} \right) \left(\exp\left(\frac{t}{t_r} \right) - 1 \right) \right]$$

Perfettini and Avouac, 2004a, *JGR*

Friction Law

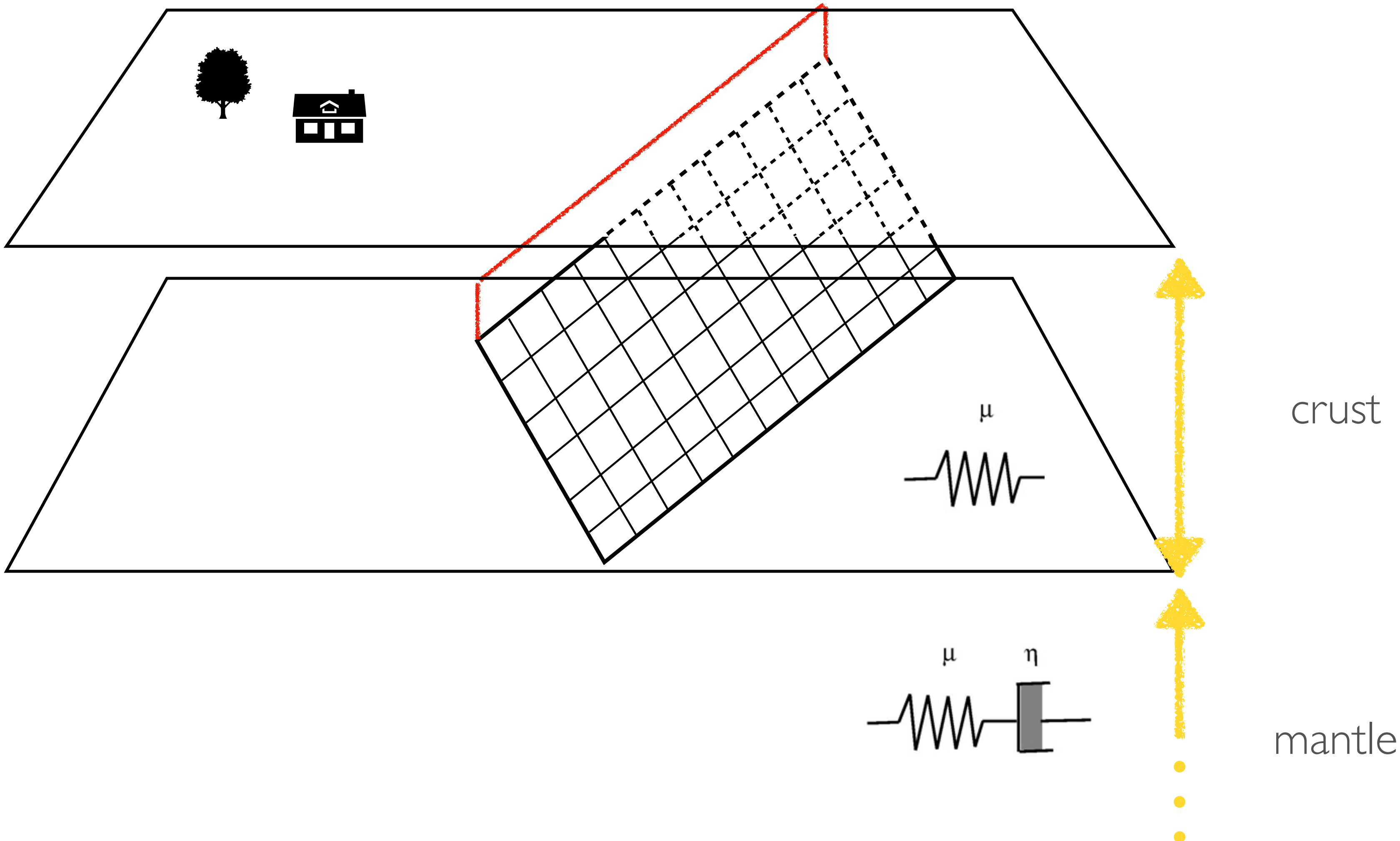
$$\delta(t) \overset{t \ll t_r}{\simeq} \frac{1}{c} \ln \left[1 + \frac{v_i}{v_0} \exp\left(\frac{\Delta\tau}{(a-b)\sigma_n} \right) \left(1 + \frac{t}{t_r} \quad - 1 \right) \right]$$

$$= \frac{1}{c} \ln \left[1 + \frac{v_i}{v_0} \frac{1}{t_r} \exp\left(\frac{\Delta\tau}{(a-b)\sigma_n} \right) t \right] = \frac{1}{c} \ln \left[1 + \frac{t}{t_{as}} \right] \qquad t_r = \frac{1}{c v_0} = \frac{(a-b)\sigma_n}{k v_0}$$

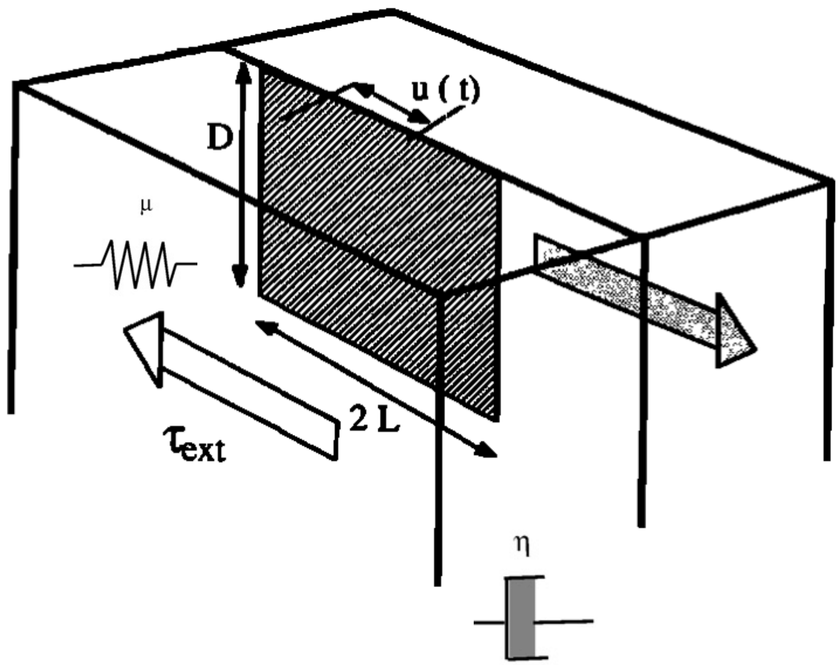
$$\delta(t) \simeq v_0 t_r \ln \left(1 + \frac{t}{t_{as}} \right) \qquad t_{as} = \frac{v_0 t_r}{v_i} \exp\left(-\frac{\Delta\tau}{(a-b)\sigma_n} \right)$$

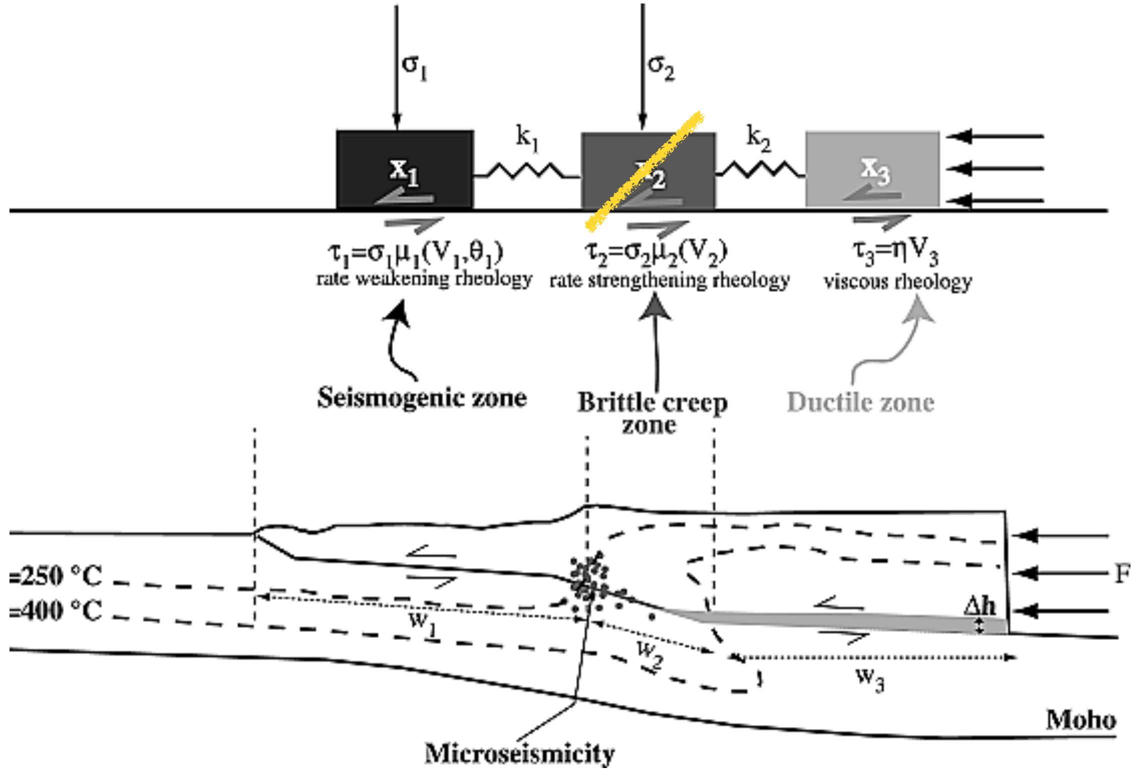
Perfettini and Avouac, 2004a, *JGR*

Viscoelastic Mantle



Viscous Resistance To Slip





$$\tau_f = \frac{\nu}{w} v = \eta v$$

Wesson, 1988, *JGR*

Perfettini and Avouac, 2004b, *JGR*

Viscous Resistance To Slip

Quasi-static approximation:

$$0 = \tau_i + \tau_l - \tau_f + \Delta\tau \mathrm{H}(t)$$

$$t = t_0 = 0$$
$$\delta = \delta_i = 0$$
$$v = v_i = v_0$$

$$0 = \tau_i + 0 - \eta v_0 + 0 \qquad \tau_i = \eta v_0$$

$$\tau_l = k(v_0 t - \delta)$$

$$\tau_f = \eta v$$

$$0 = \eta v_0 + k\left[\frac{\Delta\tau}{k}\mathrm{H}(t) + v_0 t - \delta\right] - \eta v$$

$$u = \frac{\Delta\tau}{k}\mathrm{H}(t) + v_0 t - \delta$$

$$\dot{u} = \frac{\Delta\tau}{k}\delta_{\mathrm{Dirac}}(t) + v_0 - v$$

$$0 = \eta v_0 + ku - \eta\left[\frac{\Delta\tau}{k}\delta_{\mathrm{Dirac}}(t) + v_0 - \dot{u}\right]$$

$$\eta\dot{u} + ku = \eta(v_0 - v_0) + \frac{\eta}{k}\Delta\tau\delta_{\mathrm{Dirac}}(t)$$

Viscous Resistance To Slip

$$\eta\dot{u} + ku = \eta(v_0 - v_0) + \frac{\eta}{k}\Delta\tau\delta_{\text{Dirac}}(t)$$

$$u = \frac{\Delta\tau}{k}\text{H}(t) + v_0 t - \delta$$

Viscous Resistance To Slip

$$\dot{u} + \frac{k}{\eta}u = \frac{\Delta\tau}{k}\delta_{\mathrm{Dirac}}(t)$$

$$u = \frac{\Delta\tau}{k}\mathrm{H}(t) + v_0 t - \delta$$

$$\dot{u} + \frac{k}{\eta}u = 0 \qquad \text{for } t > 0$$

$$u_{0+} = \frac{\Delta\tau}{k}$$

$$\frac{\mathrm{d}u}{u} = -\frac{k}{\eta}\mathrm{d}t$$

$$\int_{u_{0+}}^{u(t)} \frac{\mathrm{d}u'}{u'} = -\frac{k}{\eta}\int_{0+}^{t} \mathrm{d}t$$

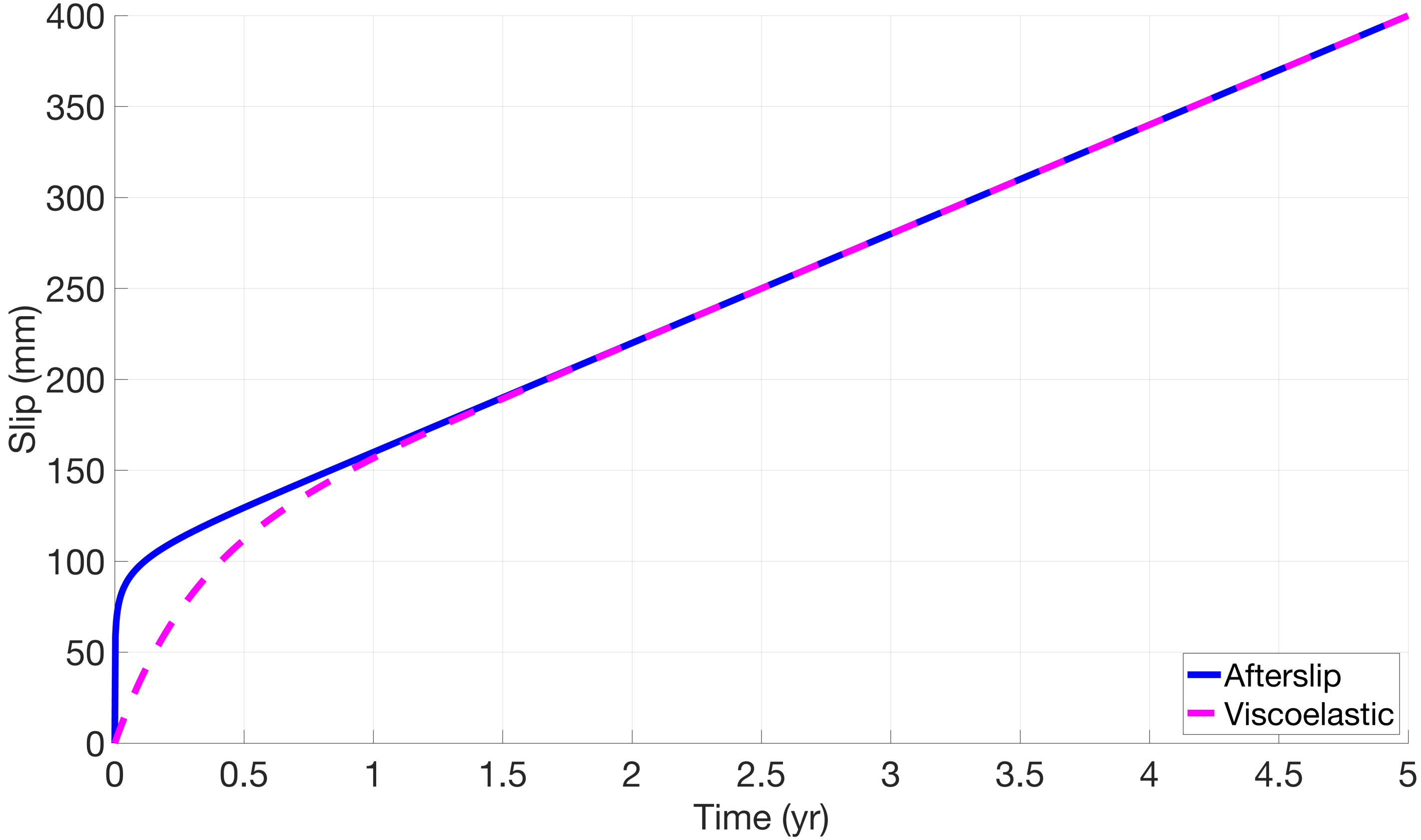
$$u = u_{0+}\exp\left(-\frac{k}{\eta}t\right)$$

$$u = \frac{\Delta\tau}{k}\exp\left(-\frac{k}{\eta}t\right)$$

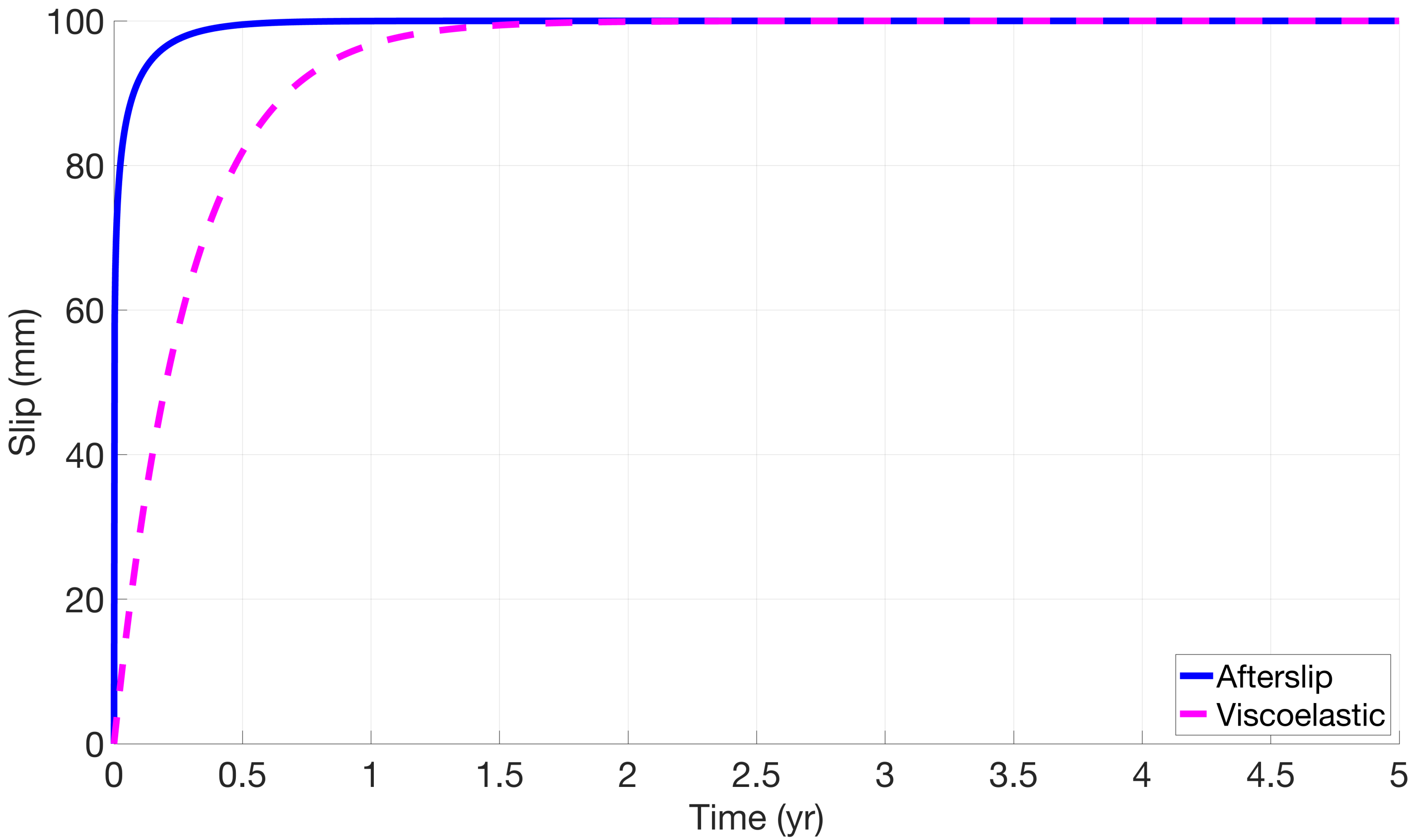
$$\delta(t) = \frac{\Delta\tau}{k}\mathrm{H}(t) + v_0 t - u \;= v_0 t + \frac{\Delta\tau}{k}\left[1 - \exp\left(-\frac{k}{\eta}t\right)\right]$$

$$t_r = \frac{\eta}{k}$$

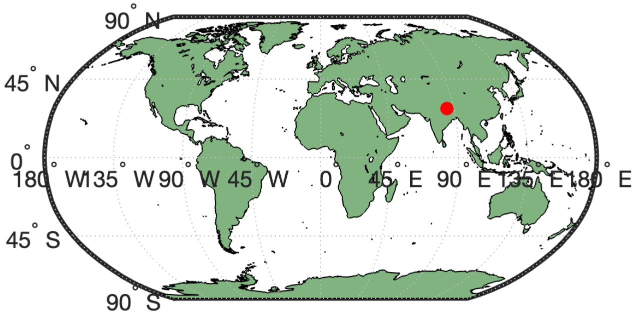
Afterslip vs Viscoelastic Relaxation

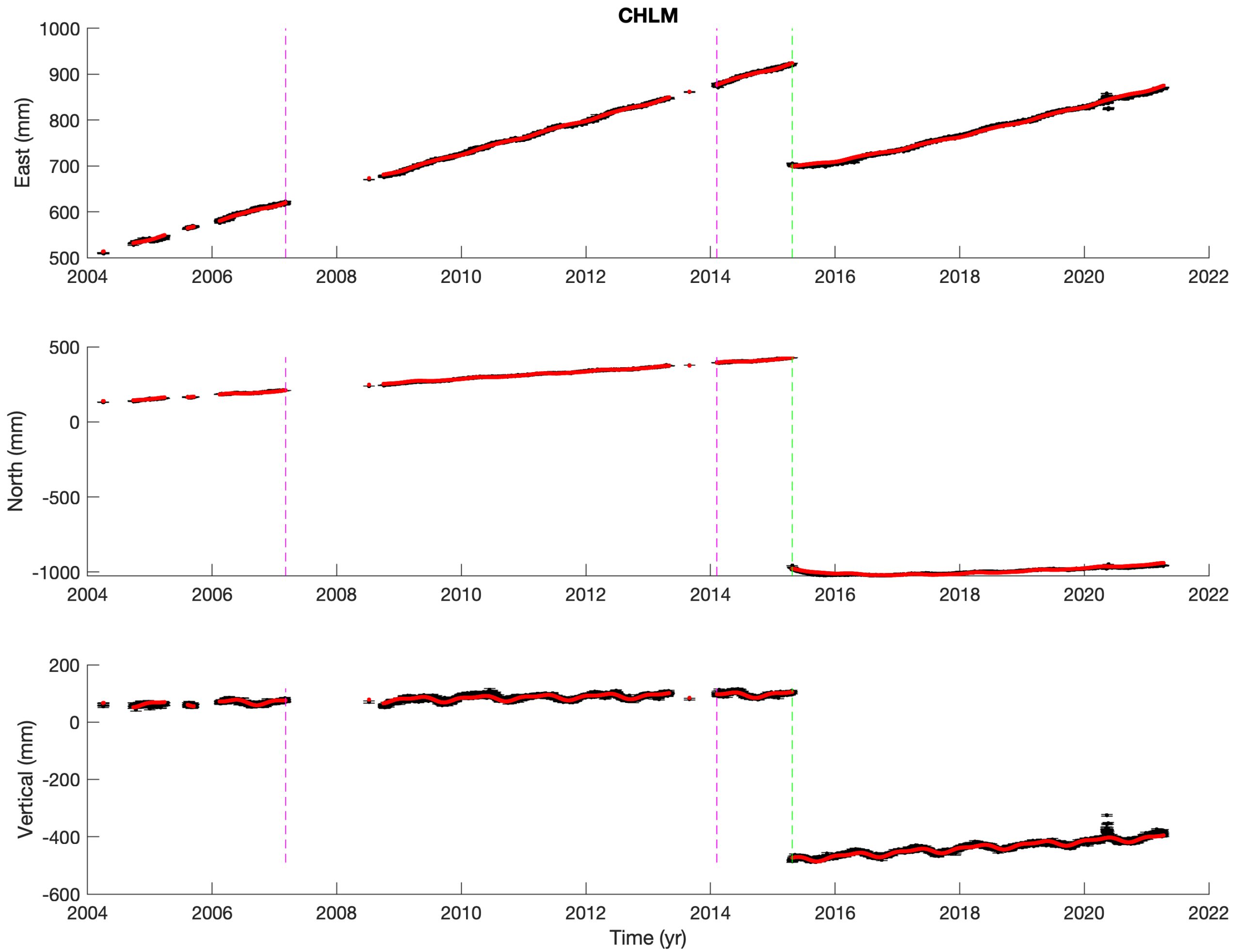


Afterslip vs Viscoelastic Relaxation

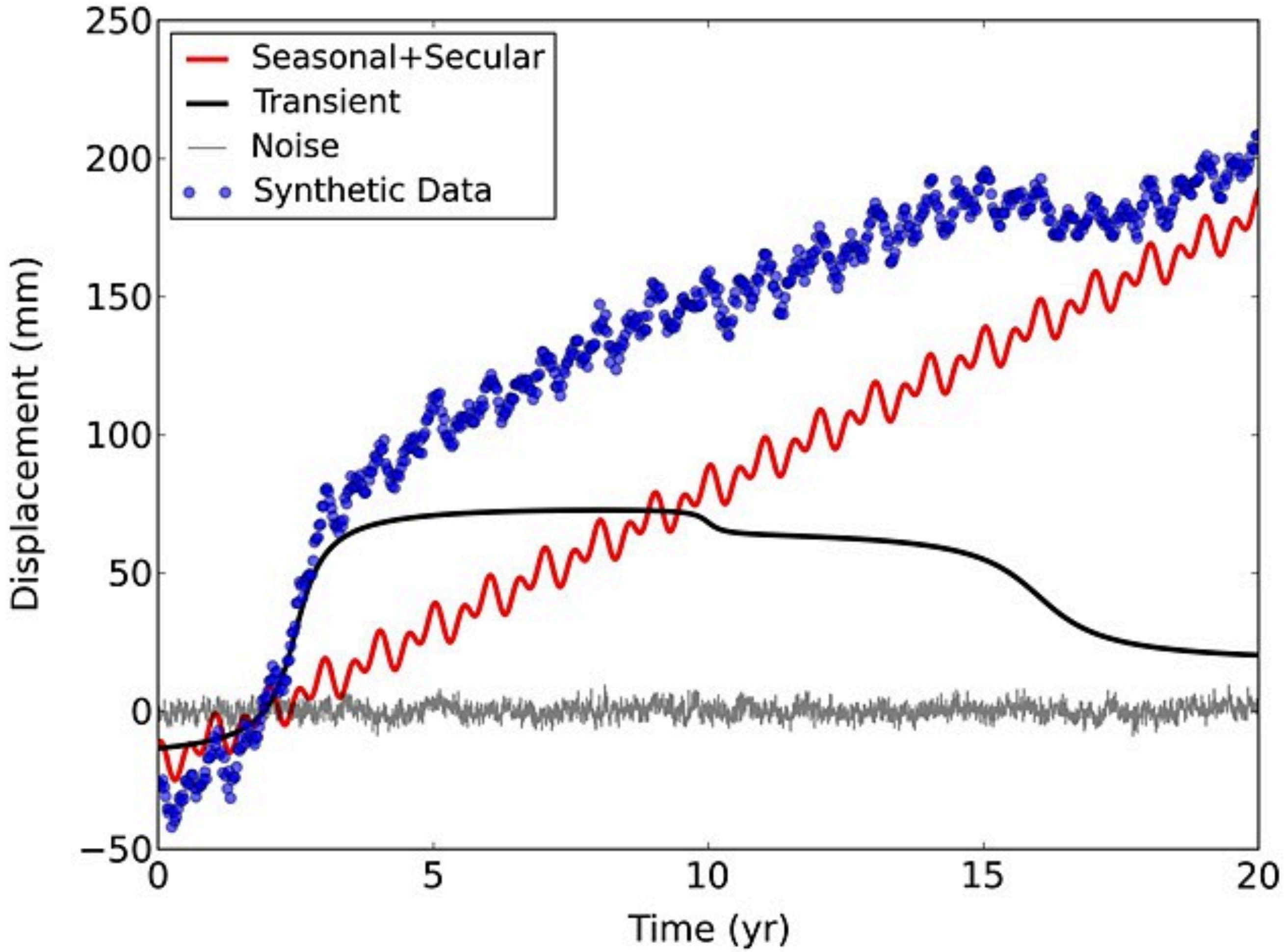


Trajectory Model



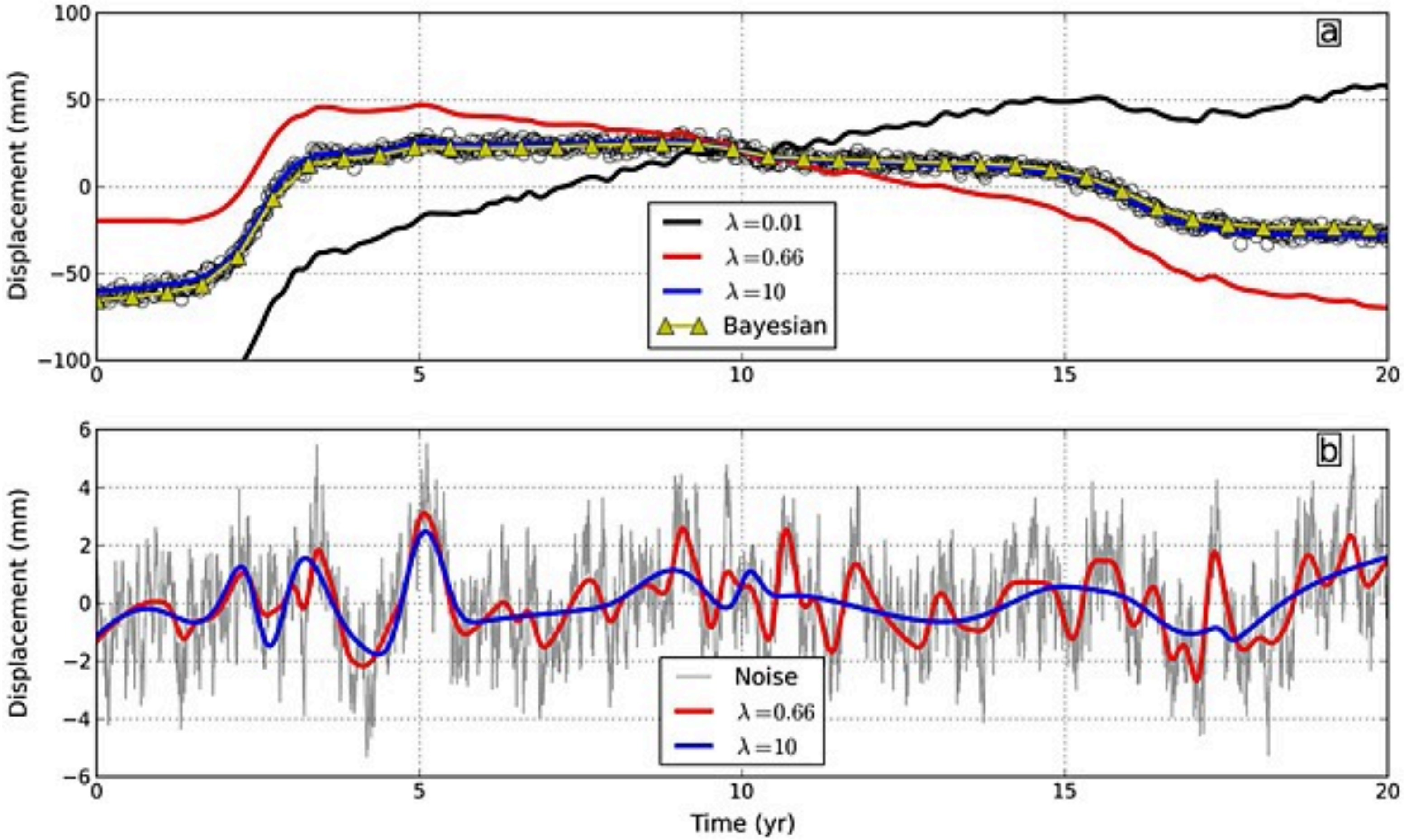


Advanced Trajectory Models



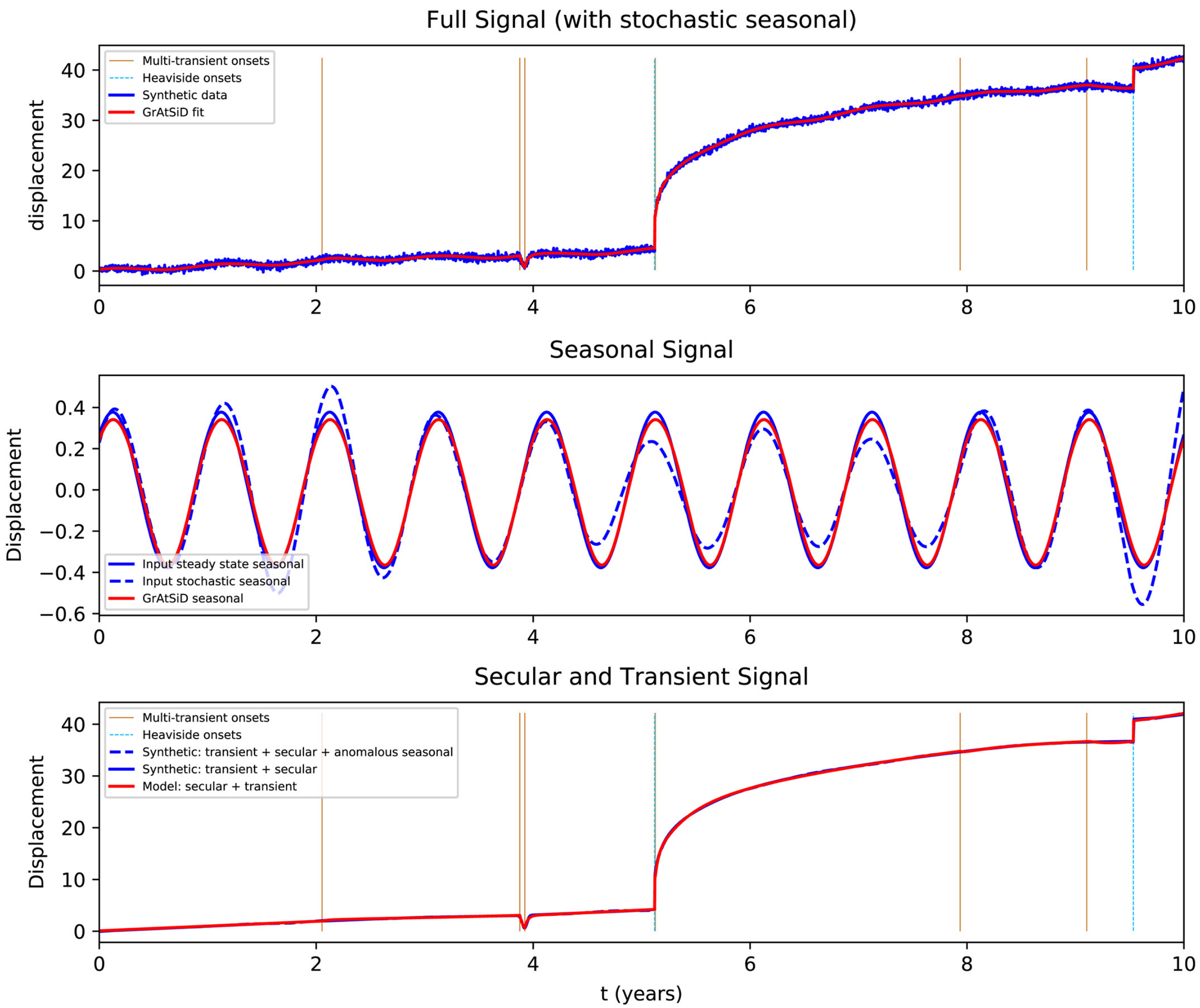
Riel et al., 2014, *JGR*

Advanced Trajectory Models



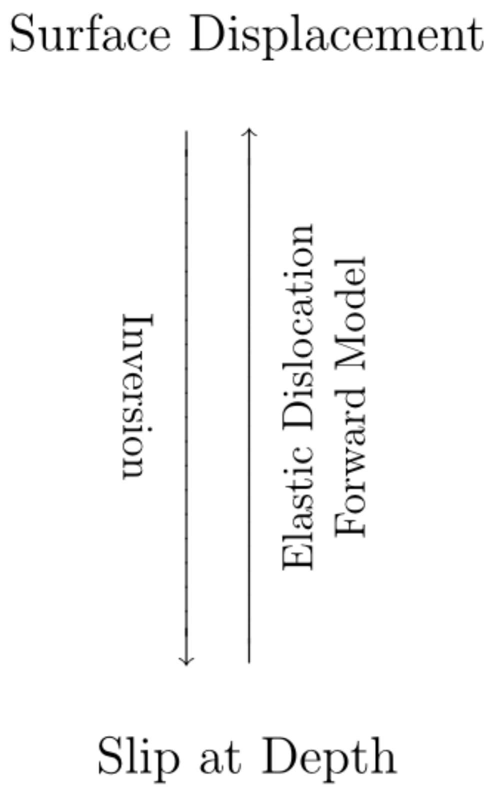
Riel et al., 2014, *JGR*

Advanced Trajectory Models

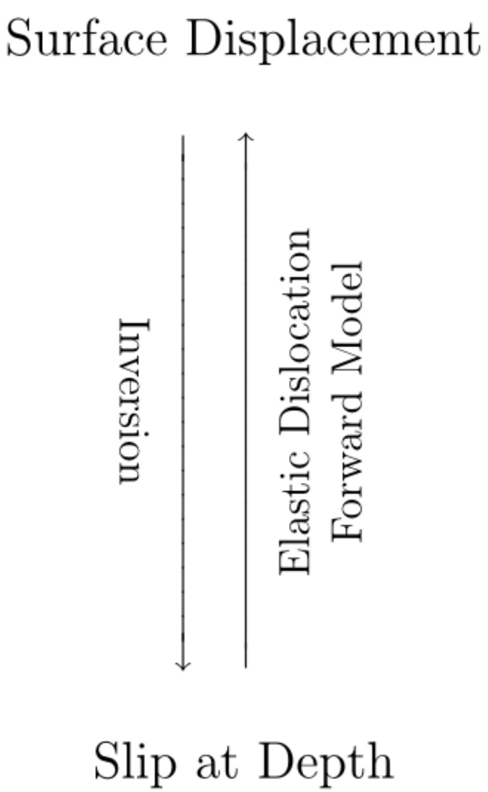


Bedford and Bevis, 2018, *JGR*

Multivariate Models



Extended Network Inversion Filter



$$X(t) = \sum_{i=1}^{N^{eq}} A_i^{\mathrm{co}} \, \mathrm{H}(t - t_i^{\mathrm{eq}})$$ earthquakes

$$+ \int_A s_p(\xi, t - t_0) \; G_{pq}^r(\mathbf{x}, \xi) \; \mathbf{n}_q(\xi) \; \mathrm{d}A(\xi)$$ tectonic transients

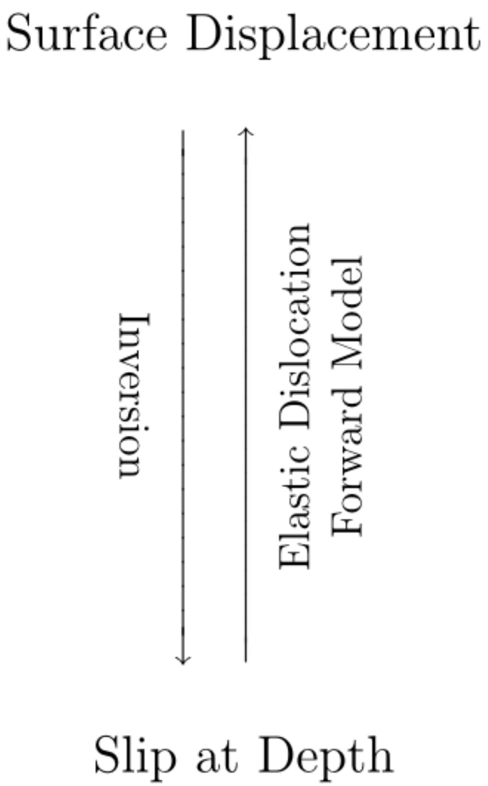
$$+ F\mathbf{f}(t)$$ Helmert transformation

$$+ L(\mathbf{x}, t - t_0)$$ Brownian motion

$$+ \epsilon$$ observational Gaussian noise

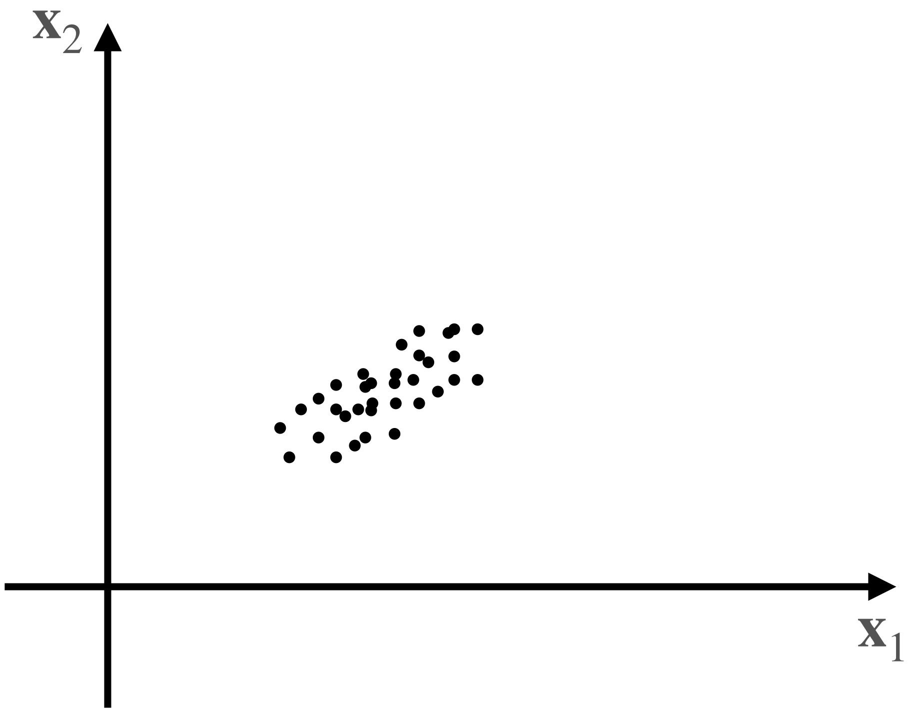
McGuire and Segall, 2003, *GJI*

Multivariate Statistics

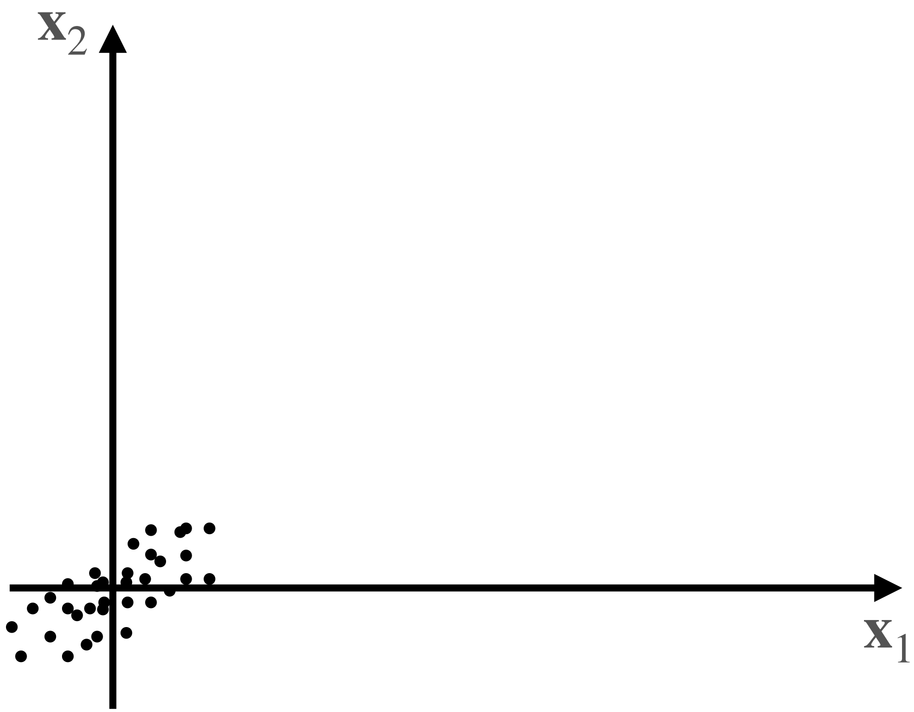


$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1T} \\ x_{21} & x_{22} & \cdots & x_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ x_{M1} & x_{M2} & \cdots & x_{MT} \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_M \end{pmatrix}$$

Centering



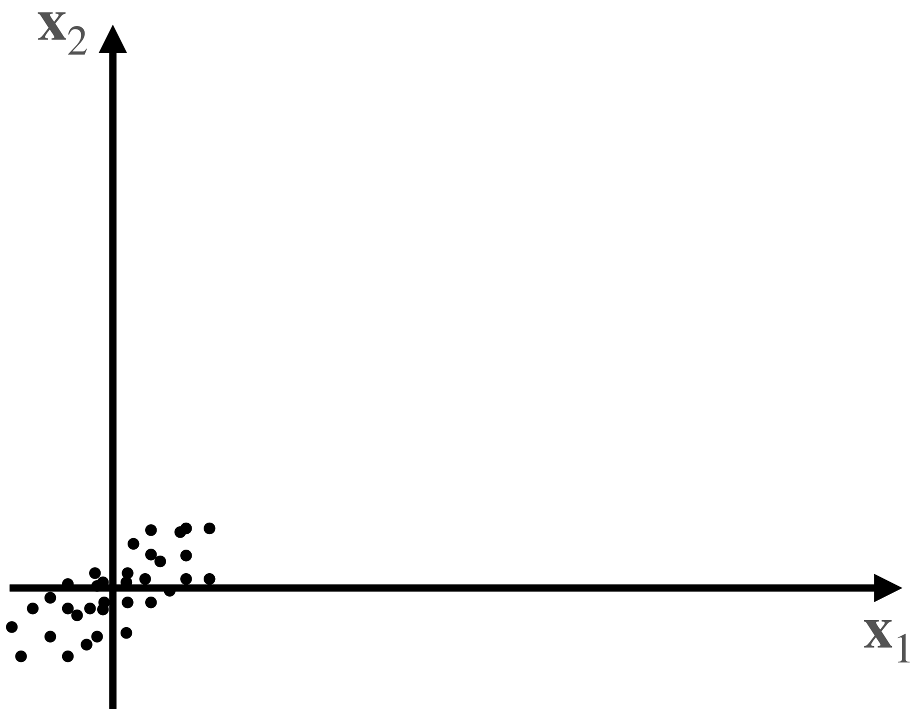
Centering



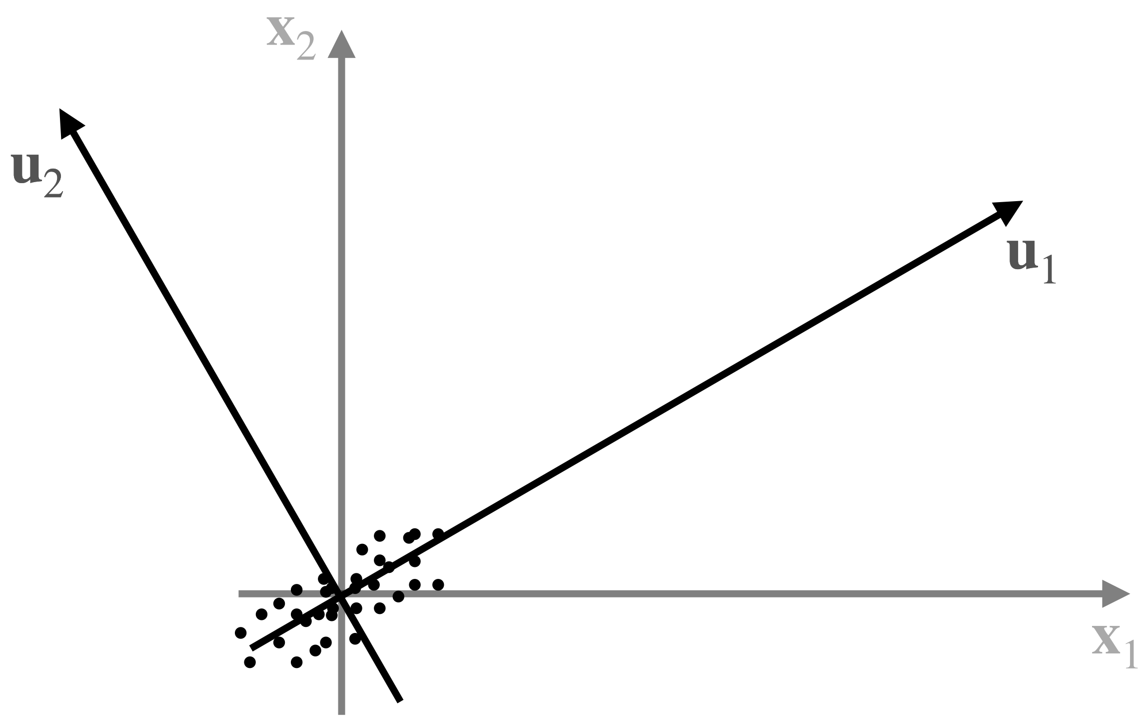
Multivariate Statistics

$$X = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_M \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_M \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_M \end{pmatrix}$$

Principal Component Analysis (PCA)



Principal Component Analysis (PCA)



Singular Value Decomposition (SVD)

$$X = U\Sigma V^T$$

$$UU^T = U^T U = I$$

$$VV^T = V^T V = I$$

Σ diagonal $\Rightarrow \Sigma = \Sigma^T$

$$\mathrm{var}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2] = \mathbb{E}[(X - \mathbb{E}(X))(X - \mathbb{E}(X))^T] = \frac{1}{M}(X-\mu)(X-\mu)^T \rightarrow \frac{1}{M}XX^T$$

$$\mathrm{var}(X) \propto XX^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T V \Sigma^T U^T = U\Sigma^2 U^T$$

$$XX^T U = U\Sigma^2 U^T U = U\Sigma^2$$

U eigenvectors

Σ^2 eigenvalues

Singular Value Decomposition (SVD)

$$XX^TU = U\Sigma^2U^TU = U\Sigma^2$$

U eigenvectors

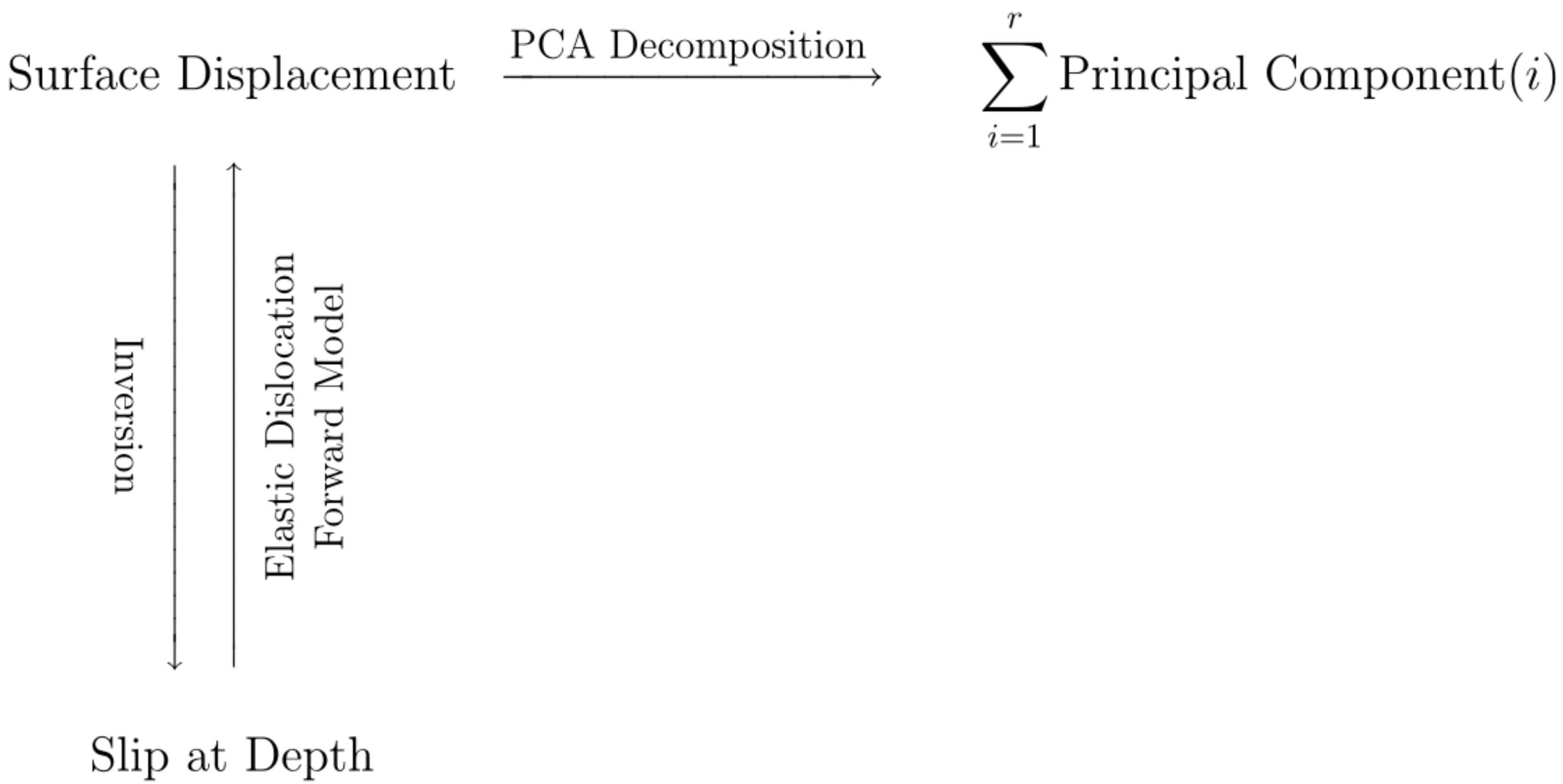
Σ^2 eigenvalues

$$X^TXV = V\Sigma^2$$

V eigenvectors

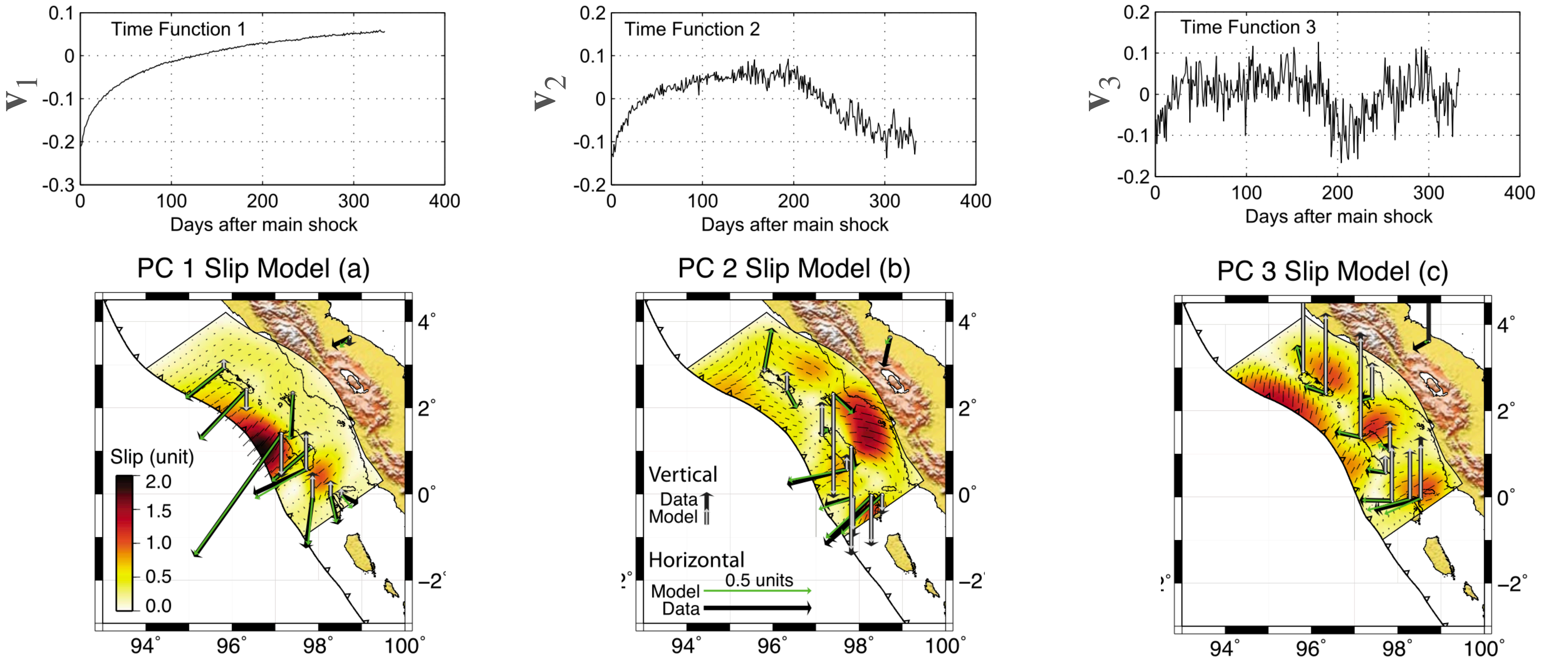
Σ^2 eigenvalues

PCA-Based Inversion Method



Kositsky and Avouac et al., 2010, *JGR*

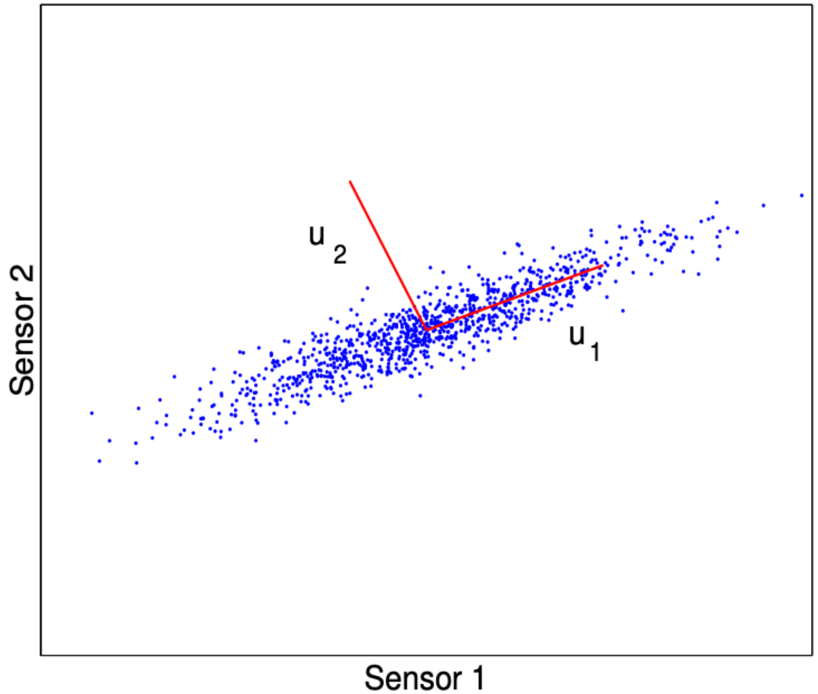
PCA-Based Inversion Method



Kositsky and Avouac et al., 2010, *JGR*

Gaussian Assumption and Whitening

$$X \qquad \mathbb{E}[XX^T] = \frac{1}{M}U\Sigma^2U^T \qquad X_w = \Sigma^{-1}U^TX$$



Choudrey, 2002, *PhD thesis*

Uncorrelation vs Independence

Y random variable $\sim \mathcal{N}(0,1)$

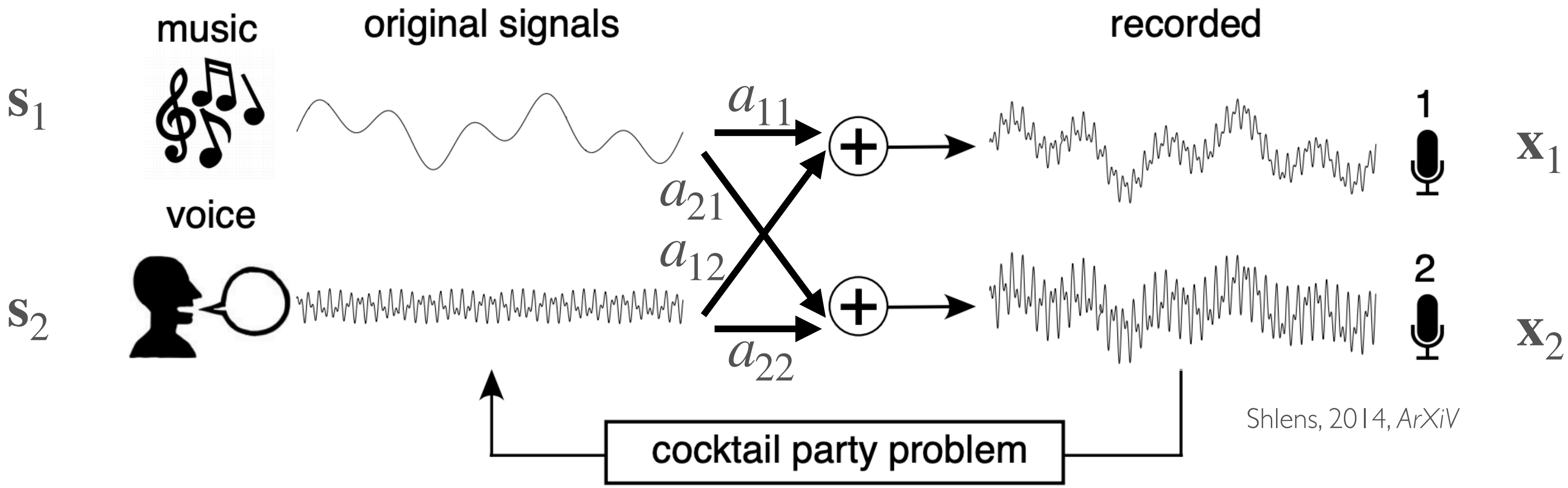
Z random variable such that $Z = Y^2$

$$\mathrm{cov}(Y,Z) = \mathbb{E}\left[(Y - \mathbb{E}[Y])(Z - \mathbb{E}[Z])\right] = \mathbb{E}[YZ] = \mathbb{E}\left[Y^3\right] = 0$$

Y and Z are uncorrelated, but clearly **not independent** !

Y and Z are independent if $P(Y,Z) = P(Y)P(Z)$

Independent Component Analysis



Shlens, 2014, *ArXiV*

$X = AS$ $\mathbf{s}_1$ and $\mathbf{s}_2$ are independent

Independent Component Analysis

$X = AS$ $\qquad$ $\mathbf{s}_1$ and $\mathbf{s}_2$ are independent $\qquad \Rightarrow \qquad \mathbb{E}[SS^T] = I$

$$A = U_A \Sigma_A V_A^T$$

$$\mathbb{E}[XX^T] = \frac{1}{M} U \Sigma^2 U^T$$

$$\begin{aligned}
\mathbb{E}[XX^T] &= \mathbb{E}[ASS^T A^T] \\
&= \mathbb{E}[U_A \Sigma_A V_A^T S S^T V_A \Sigma_A U_A^T] \\
&= U_A \Sigma_A V_A^T \; \mathbb{E}[SS^T] \; V_A \Sigma_A U_A^T \\
&= U_A \Sigma_A V_A^T V_A \Sigma_A U_A^T \\
&= U_A \Sigma_A^2 U_A^T
\end{aligned}$$

$$U_A = U$$

$$\Sigma_A = \frac{\Sigma}{\sqrt{M}}$$

$$\Rightarrow \qquad A = U \frac{\Sigma}{\sqrt{M}} V_A^T$$

Independent Component Analysis

$$X = AS$$

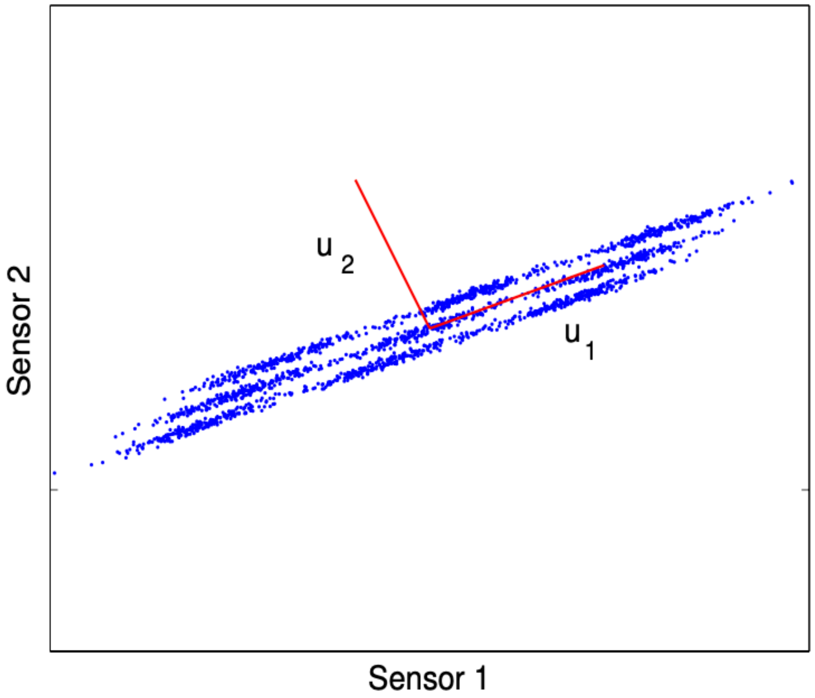
$$A = U\frac{\Sigma}{\sqrt{M}}V_A^T$$

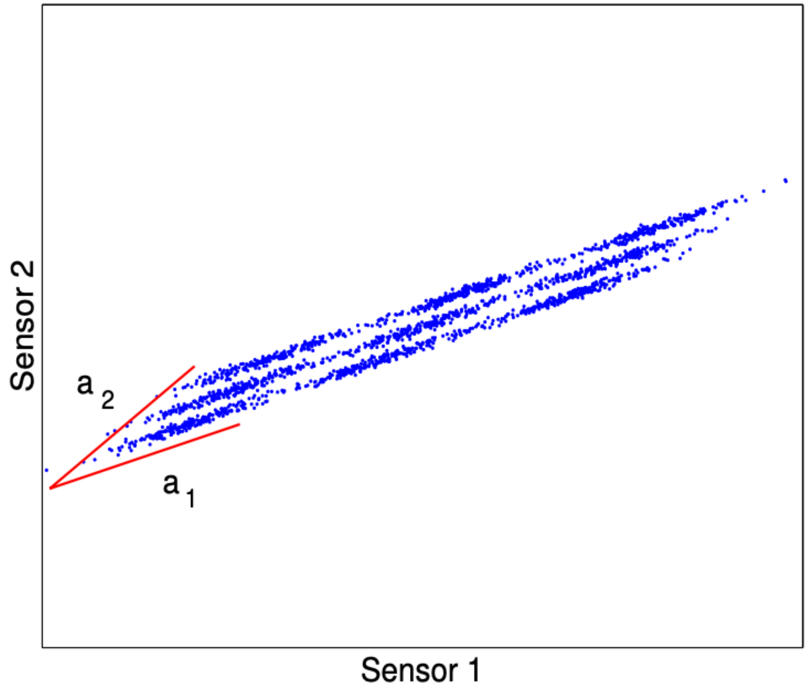
$$A^{-1} = V_A\sqrt{M}\Sigma^{-1}U^T \qquad X_w = \Sigma^{-1}U^T X$$

$$S = A^{-1}X = \sqrt{M}V_A\Sigma^{-1}U^T X = \sqrt{M}V_A X_w$$

Orthogonal rotation of pre-whitened data

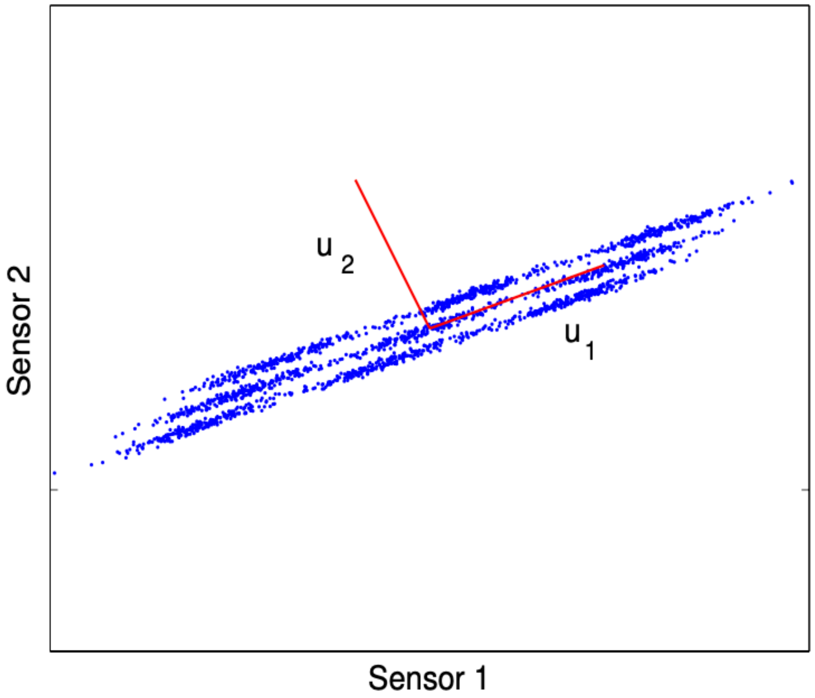
Geometrical Interpretation

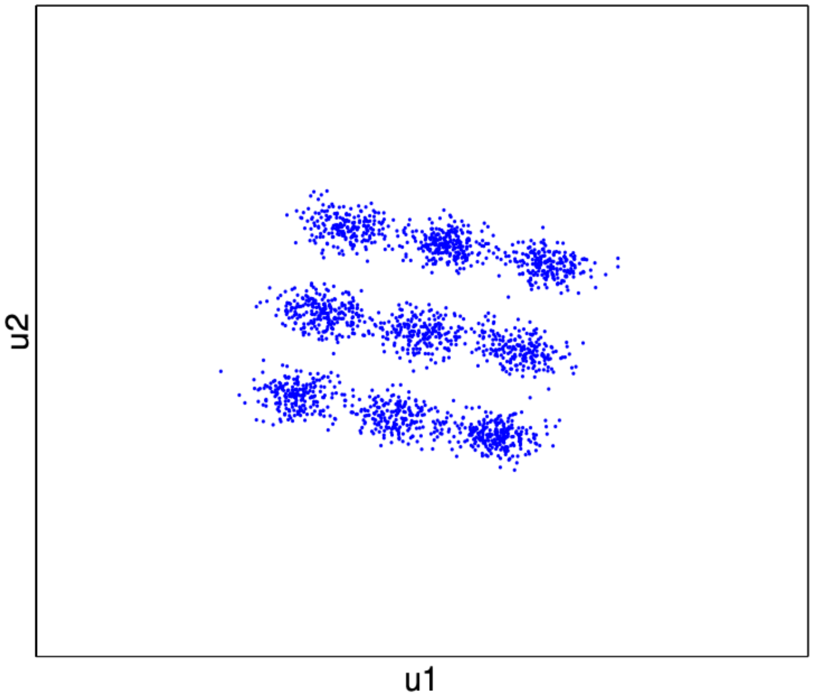




Choudrey, 2002, *PhD thesis*

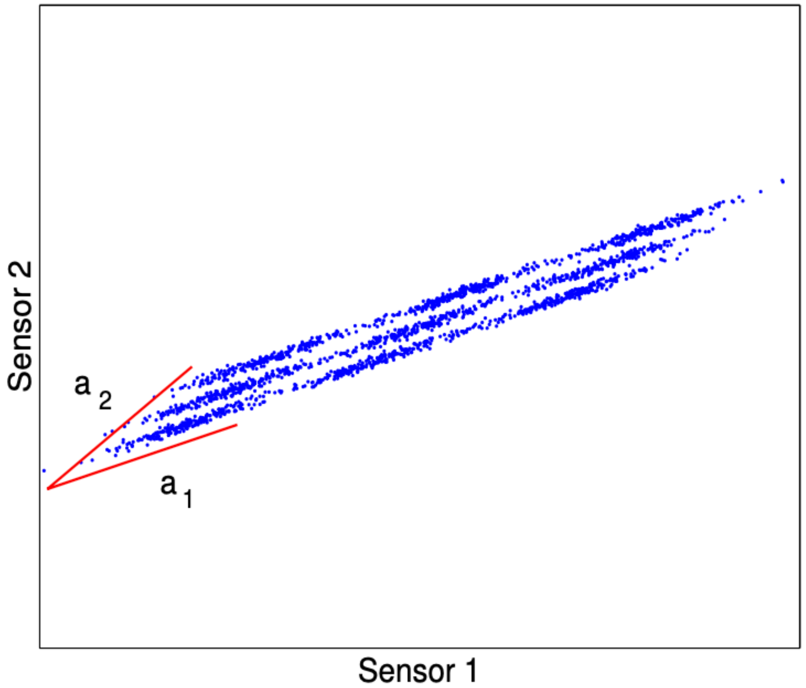
Geometrical Interpretation

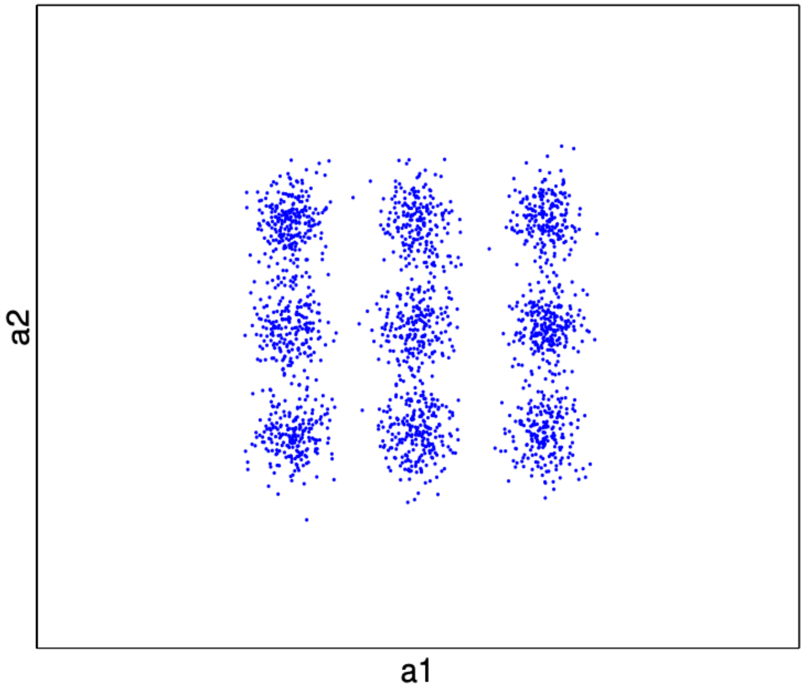




Choudrey, 2002, *PhD thesis*

Geometrical Interpretation





Choudrey, 2002, *PhD thesis*

Mutual Information

$$\mathrm{KL}\left(P_{\mathbf{s}_1\mathbf{s}_2} \,||\, P_{\mathbf{s}_1}P_{\mathbf{s}_2}\right) = \int\int P_{\mathbf{s}_1\mathbf{s}_2} \log \frac{P_{\mathbf{s}_1\mathbf{s}_2}}{P_{\mathbf{s}_1}P_{\mathbf{s}_2}} \,\mathrm{d}\mathbf{s}_2\mathrm{d}\mathbf{s}_2$$

$$= \int\int P_{\mathbf{s}_1\mathbf{s}_2} \log P_{\mathbf{s}_1\mathbf{s}_2} \mathrm{d}\mathbf{s}_1\mathrm{d}\mathbf{s}_2 - \int P_{\mathbf{s}_1} \log P_{\mathbf{s}_1} \mathrm{d}\mathbf{s}_1 - \int P_{\mathbf{s}_2} \log P_{\mathbf{s}_2} \mathrm{d}\mathbf{s}_2$$

$$= -\mathscr{H}[\mathbf{s}_1, \mathbf{s}_2] + \mathscr{H}[\mathbf{s}_1] + \mathscr{H}[\mathbf{s}_2] = \mathrm{MI}(\mathbf{s}_1; \mathbf{s}_2)$$

$\mathbf{s}_1$ and $\mathbf{s}_2$ are independent if $P(\mathbf{s}_1, \mathbf{s}_2) = P(\mathbf{s}_1)P(\mathbf{s}_2) \quad \Rightarrow \quad \mathrm{MI}(\mathbf{s}_1, \mathbf{s}_2)$

Mutual Information

$$\mathrm{KL}\left(P_{\mathbf{s}_1\mathbf{s}_2} \,||\, P_{\mathbf{s}_1}P_{\mathbf{s}_2}\right) = -\mathscr{H}[\mathbf{s}_1, \mathbf{s}_2] + \mathscr{H}[\mathbf{s}_1] + \mathscr{H}[\mathbf{s}_2]$$

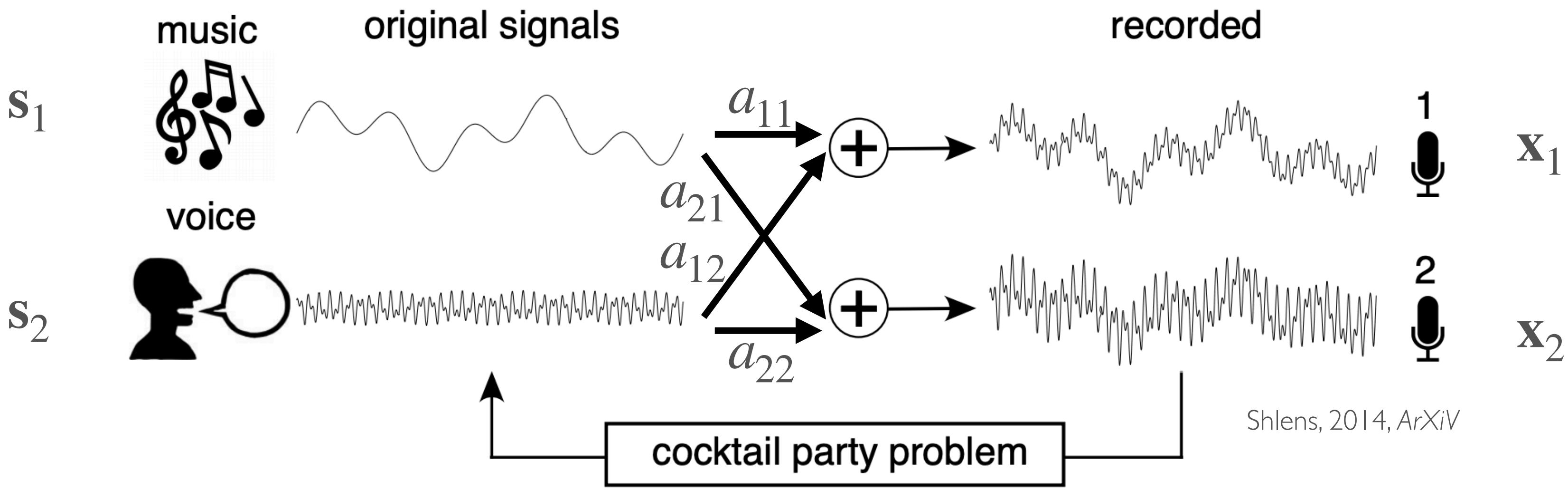
$$S = \sqrt{M}V_A X_w$$

$$\mathrm{KL}\left(P_{\mathbf{s}_1\mathbf{s}_2} \,||\, P_{\mathbf{s}_1}P_{\mathbf{s}_2}\right) = \sum_i \mathscr{H}[(\sqrt{M}V_A X_w)_i] - \boxed{\mathscr{H}[(\sqrt{M}V_A X_w)]}$$

$$\mathscr{H}[(\sqrt{M}V_A X_w)] = \mathscr{H}[X_w] + \log|\sqrt{M}V_A|$$ constant

$$V_A = \mathrm{argmin}_{V_A} \sum_i \mathscr{H}[(\sqrt{M}V_A X_w)_i]$$

Independent Component Analysis



Shlens, 2014, *ArXiV*

$X = AS$ $\mathbf{s}_1$ and $\mathbf{s}_2$ are independent

The sum of independent random variables (X) tends towards a Gaussian

central limit theorem

Maximize Non-Gaussianity

$X = AS$ $\mathbf{s}_1$ and $\mathbf{s}_2$ are independent

The sum of independent random variables (X) tends towards a Gaussian

central limit theorem

$WX = WAS = S$

Find $\mathbf{w}_i$ such that non-Gaussianity of projection $\mathbf{w}_i^T X$ is maximised

How can we maximise non-Gaussianity ?

Negentropy

$P_{\mathbf{y}} = \mathcal{N}(\mu, \sigma^2)$ $\qquad$ $P_{\mathbf{s}_i}$ arbitrary but with same μ and σ^2

$$0 \leq \mathrm{KL}\left(P_{\mathbf{s}_i} \,||\, P_{\mathbf{y}}\right) = \int_{-\infty}^{\infty} P_{\mathbf{s}_i}(x) \log \frac{P_{\mathbf{s}_i}(x)}{P_{\mathbf{y}}(x)} \mathrm{d}x = -\mathscr{H}[\mathbf{s}_i] - \boxed{\int_{-\infty}^{\infty} P_{\mathbf{s}_i}(x) \log(P_{\mathbf{y}}(x)) \mathrm{d}x}$$

$$\int_{-\infty}^{\infty} P_{\mathbf{s}_i}(x) \log(P_{\mathbf{y}}(x)) \mathrm{d}x = \int_{-\infty}^{\infty} P_{\mathbf{s}_i}(x) \log\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right) \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} P_{\mathbf{s}_i}(x) \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) \mathrm{d}x + \log e \int_{-\infty}^{\infty} P_{\mathbf{s}_i}(x) \left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \mathrm{d}x$$

$$= -\frac{1}{2} \log 2\pi\sigma^2 - \log e \frac{\sigma^2}{2\sigma^2} = -\frac{1}{2}\left(\log 2\pi e\sigma^2\right) = -\mathscr{H}[\mathbf{y}]$$

independent of $P_{\mathbf{s}_i}$

Negentropy

$$0 \leq \mathrm{KL}\left(P_{\mathbf{s}_i} \,||\, P_{\mathbf{y}}\right) = -\mathcal{H}[\mathbf{s}_i] - \int_{-\infty}^{\infty} P_{\mathbf{s}_i}(x)\log(P_{\mathbf{y}}(x))\mathrm{d}x$$

$$\mathcal{H}[\mathbf{y}] - \mathcal{H}[\mathbf{s}_i] \geq 0$$

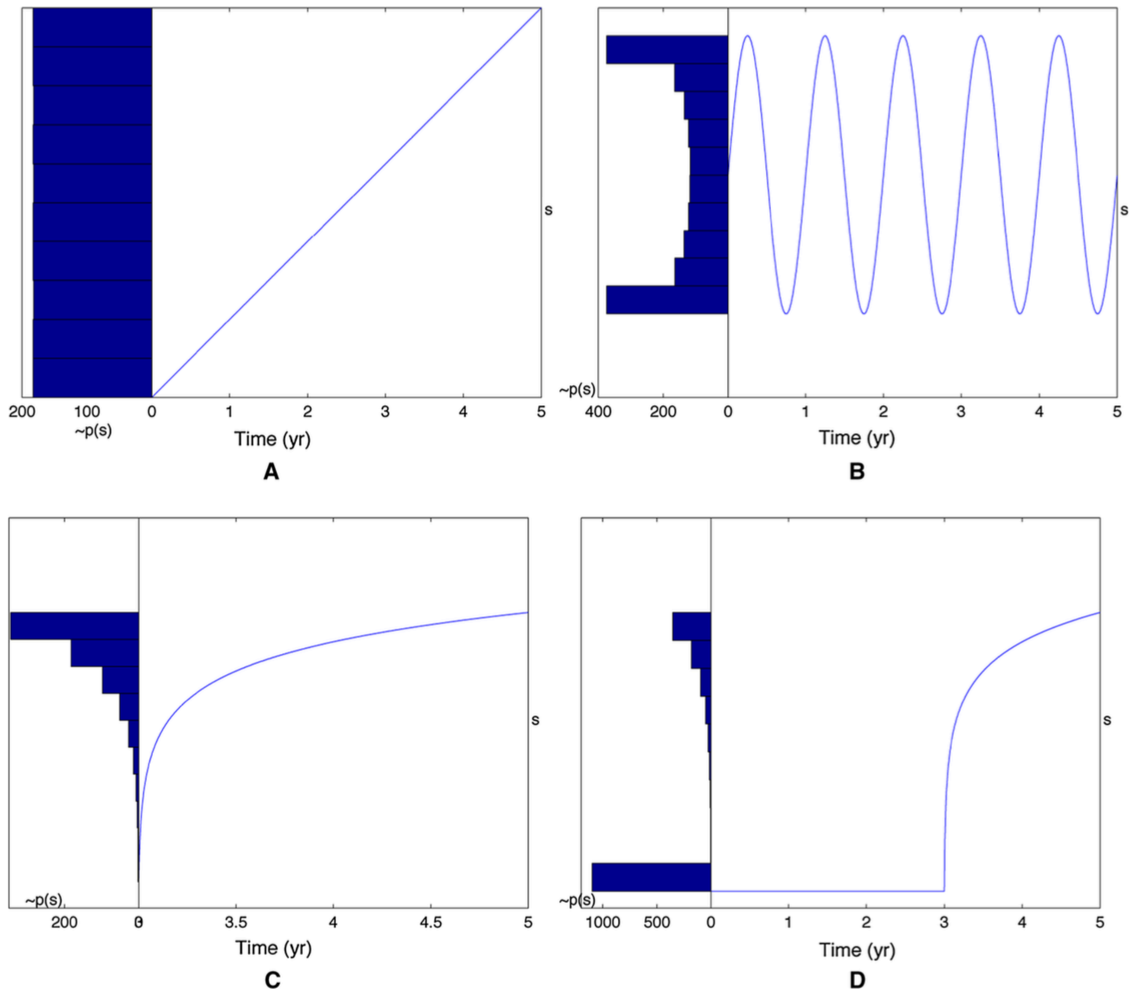
Gaussian distribution has max entropy

maximising non-Gaussianity $\Leftrightarrow$ minimising mutual information

$$V_A = \mathrm{argmin}_{V_A} \sum_i \mathcal{H}[\mathbf{s}_i]$$

$$-\mathcal{H}[\mathbf{y}]$$

Multimodal Distributions



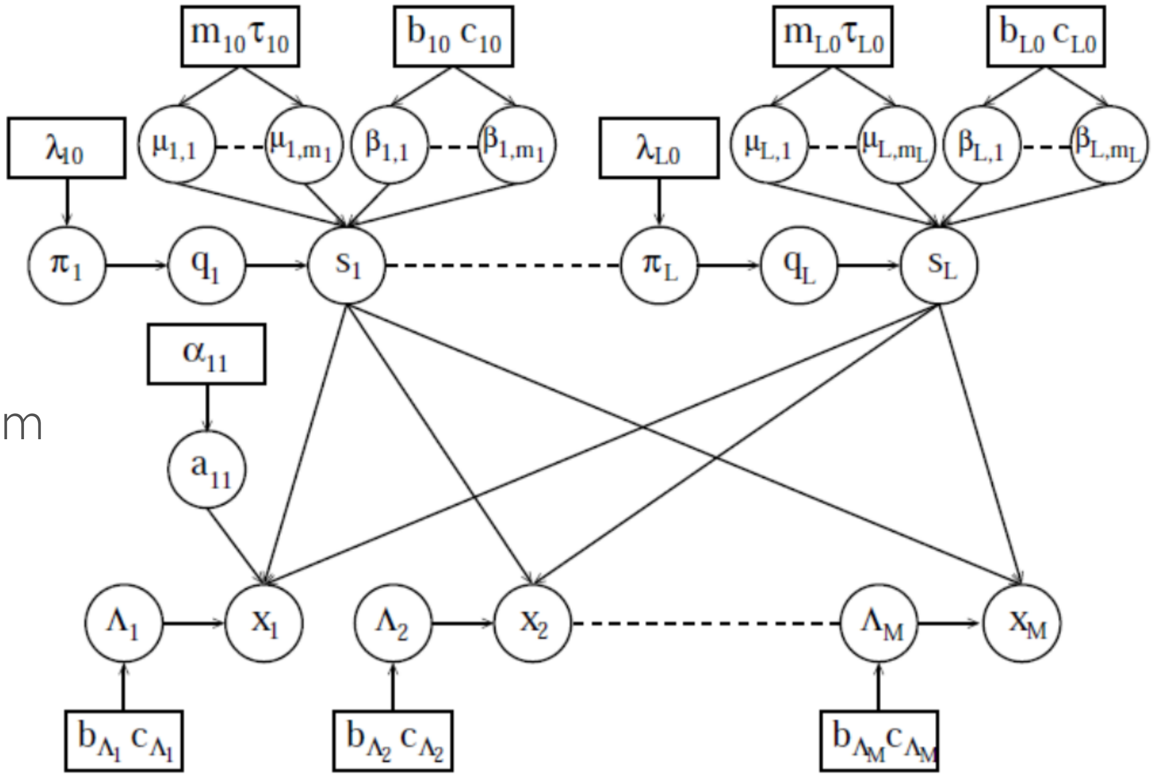
Gualandi et al., 2016, *JOGE*

Variational Bayesian ICA

X data

$W = \{A, S, \Lambda, q, \mu, \beta, \pi\}$ parameters

$$p(W|X) = \frac{p(X|W)p(W)}{p(X)}$$ Bayes' theorem



Choudrey and Roberts, 2003, *Neural Comput.*

Variational Approach

$$p(W|X) = \frac{p(X|W)p(W)}{p(X)} = \frac{p(X,W)}{p(X)}$$

usually intractable integral

$p'(W)$ approximation of $p(W|X)$

$$\mathrm{KL}(p'(W) \,||\, p(W|X)) = \int_W p'(W)\ln\frac{p'(W)}{p(W|X)}\mathrm{d}W = \int_W p'(W)\ln\frac{p'(W)p(X)}{p(X,W)}\mathrm{d}W$$

$$= \int_W p'(W)\ln\frac{p'(W)}{p(X,W)}\mathrm{d}W + \int_W p'(W)\ln p(X)\mathrm{d}W$$

$$= -\int_W p'(W)\ln\frac{p(X,W)}{p'(W)}\mathrm{d}W + \ln p(X)$$

Variational Approach

$$\frac{\mathrm{d}}{\mathrm{d}W}\,\mathrm{KL}(p'(W)\,||\,p(W|X)) + \frac{\mathrm{d}}{\mathrm{d}W}\int_W p'(W)\ln\frac{p(X,W)}{p'(W)}\mathrm{d}W = \frac{\mathrm{d}}{\mathrm{d}W}\ln p(X)$$

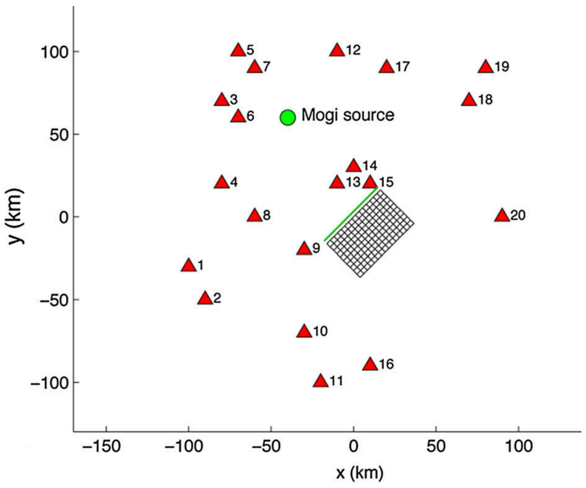
Negative Free Energy

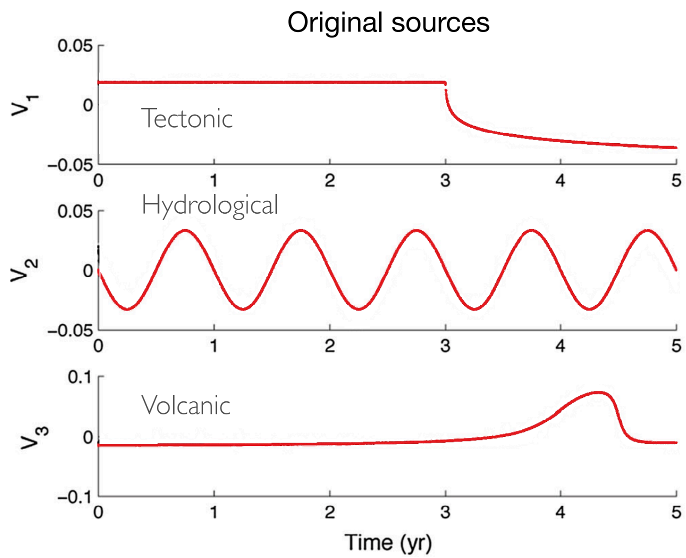
$p'(W)$ approximation of $p(W|X)$

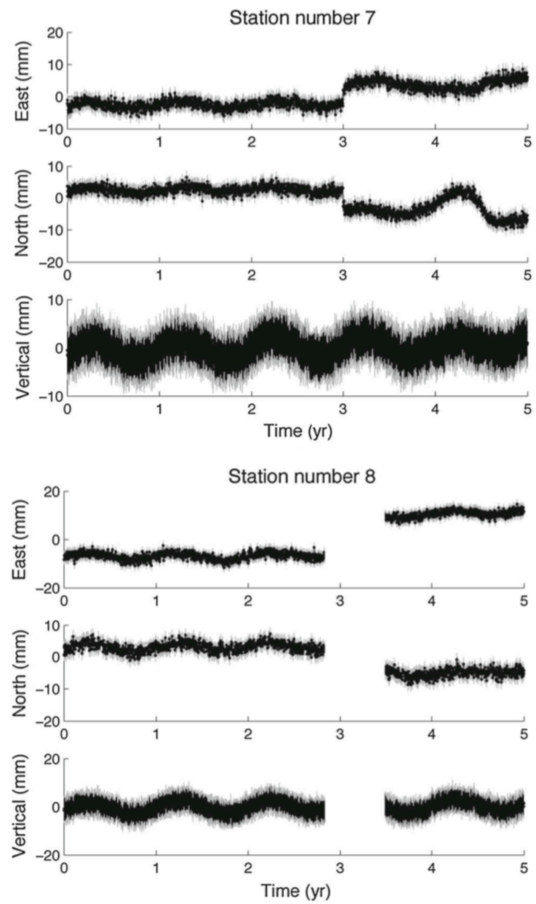
$$p'(W) = p'(A)p'(\Lambda)p'(S|q)p'(q)p'(\mu)p'(\beta)p'(\pi)$$

with independence among the sources: $p'(S) = \prod_{i=1}^{L} p'(S_i)$

vbICA on GNSS Synthetic Time Series

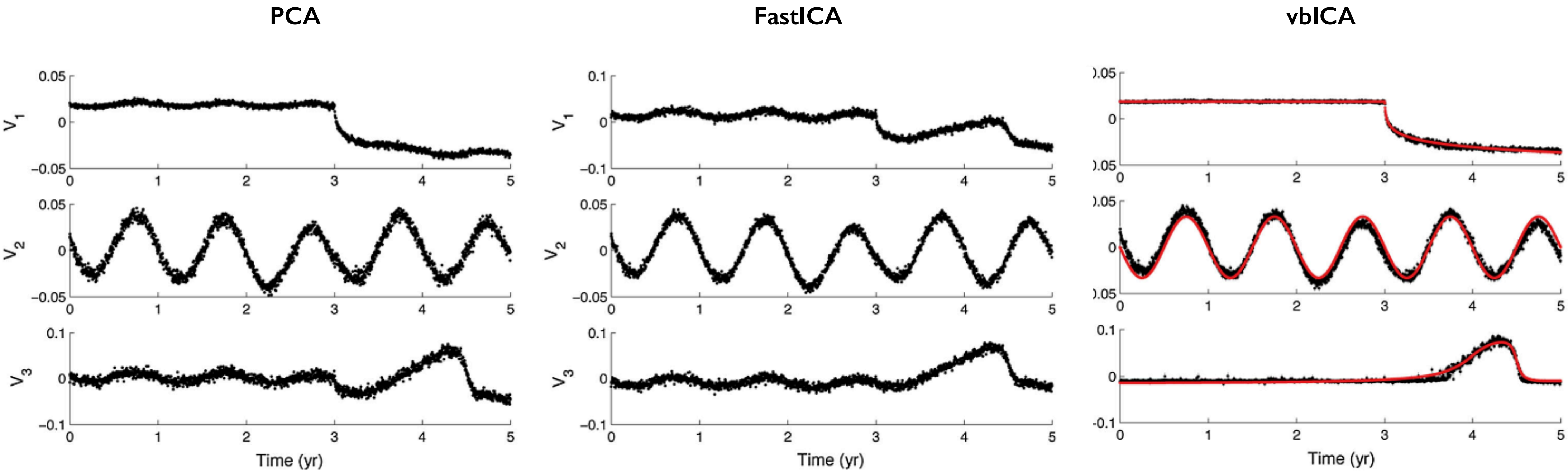






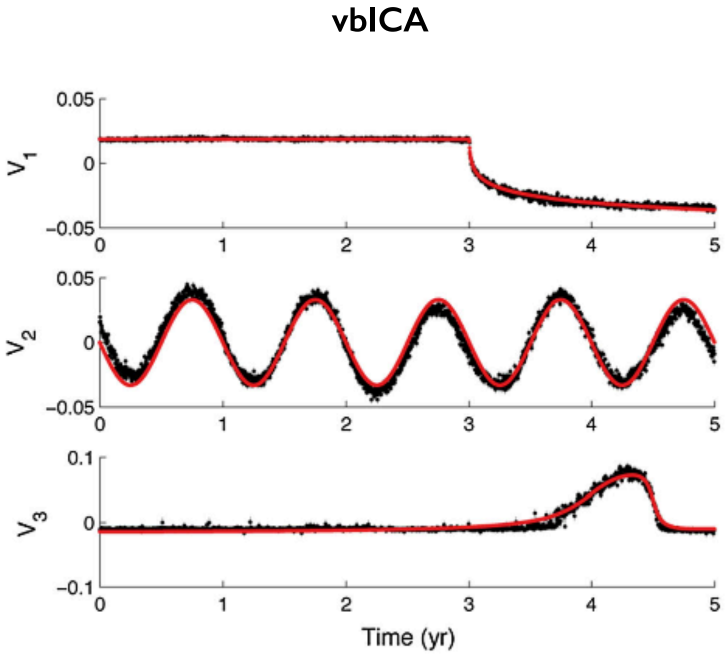
Gualandi et al., 2016, *JOGE*

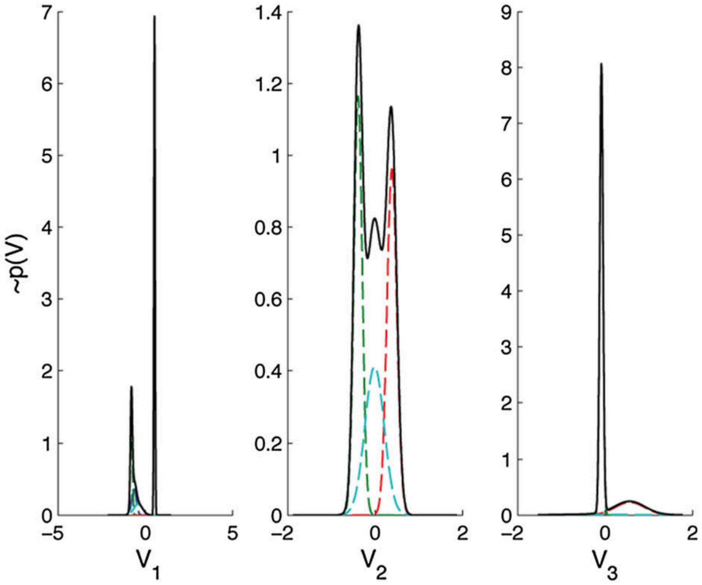
vbICA on GNSS Synthetic Time Series



Gualandi et al., 2016, *JOGE*

vbICA on GNSS Synthetic Time Series





Gualandi et al., 2016, *JOGE*

Conclusions

Let's play with some real data now !