From Thermal 2pt Functions To Conformal Fishnet Graphs

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- Overview
- New results
- Outlook

Upcoming work with:

M. Karydas, S. Li and M. Vilatte

$\Longleftrightarrow$

$\Rrightarrow$ CFTs at finite size/temperature

Even if one "knows fully" the CFT_d in the plane $\mathbb{R}^d$, it is not clear whether one "knows" as well the same CFT_d in $S^1_\beta \times \mathbb{R}^{d-1}$ i.e. at finite size/temperature.

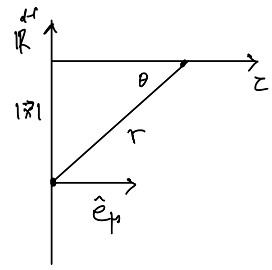
* $d=2$ is special as $S^1_\beta \times \mathbb{R}^1$ is conformal to $\mathbb{R}^2$ and the "knowledge" from the plane can be transfered to the "thermal geometry" with the help of the conformal anomaly.

In $d>2$ the problem shows up at the level of 2pt functions. For a quasiprimary scalar operator $Q(x)$ with dimension Δ_Q, the thermal 2pt function takes the general form:

$$\langle Q(x)Q(0)\rangle = \frac{1}{r^{\Delta_Q}} \sum_{\{Q_s\}} r^{\Delta_{Q_s}} C_s^{d/2-1}(\cos\theta)\, a_{Q_s}$$

- $x^\mu = (\tau, \vec{x})$, $r = \tau^2 + |\vec{x}|^2$
- $\theta \in [0,\pi]$ is a polar angle in $\mathbb{R}^{d-1}$

  $\tau = r\cos\theta$, $|\vec{x}| = r\sin\theta$



- $\beta = 1$ (mostly)
- $C_s^{d/2}(\cos\theta)$ are Gegenbauer polynomials

The sum goes over all operators that appear in the O.P.E. with spin-s and dimensions Δ_s.

- $a_{Q_s} = \frac{s!}{2^s (d/2-1)_s} \frac{g_{QQQ_s}}{C_{Q_s}} b_{Q_s}$ → 2- and 3-pt function coefficients

- $\langle Q_s(x)\rangle_\beta = b_{Q_s}(\hat{e}_{\mu_1}\hat{e}_{\mu_2}\cdots\hat{e}_{\mu_s} - \text{traces})$, $\hat{e}_\mu = (\cos\theta, \vec{0})$

⇒ As the presence of finite size/temperature breaks translation invariance, the thermal one-point functions are non-zero.

$$T_{11} + (d-1)\, T_{ii} = 0$$

$$\Rightarrow \hat{x}^\mu \hat{x}^\nu \langle T_{\mu\nu} \rangle = \frac{d}{d-1}\left(\hat{e}_1 \hat{e}_1 - \frac{1}{d}\right) \langle T_{11} \rangle \propto C_2^{d/2-1}(\cos\theta) \langle T_{11} \rangle$$

The "thermal one-point functions" **$a_{\mathcal{O}_s}$** appear to be additional information that is required to "know" the thermal CFT_d.

There is currently **no thermal bootstrap.**

What is known so far

Massless free scalars

The standard calculation for the thermal 2pt function of a massless scalar field **$\varphi(x)$** in **d-dimensions** is:

$$\boxed{\omega_n = 2\pi n}$$

$$\langle \varphi(x) \varphi(0) \rangle \equiv g_d(r, \cos\theta)$$

$$= \sum_{n=-\infty}^{\infty} \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \frac{e^{-i\omega_n \tau - i\vec{p}\cdot\vec{x}}}{\omega_n^2 + \vec{p}^{\,2}}$$

$$= \frac{\Gamma(d/2-1)}{4\pi^{d/2}} \sum_{k=-\infty}^{+\infty} \frac{1}{\left[(\tau+k)^2 + \vec{x}^{\,2}\right]^{d/2-1}}$$

$$= \frac{\Gamma(d/2-1)}{4\pi^{d/2}} \left[\frac{1}{r^{d-2}} + \sum_{k=1}^{\infty} \left(\frac{1}{(r^2+k^2+2rk\cos\theta)^{d/2-1}} + \frac{1}{(r^2+k^2-2rk\cos\theta)^{d/2-1}} \right) \right]$$

$$\frac{1}{(1+x^2-2tx)^a} = \sum_{n=0}^{\infty} t^n C_n^a(x)$$

$$\boxed{g_d(r,\cos\theta) = \frac{\Gamma(d/2-1)}{4\pi^{d/2}} \frac{1}{r^{d-2}} \left[1 + \sum_{k=0}^{\infty} r^{d-2+2k} \, C_{2k}^{d/2-1}(\cos\theta) \; 2 \cdot J(d-2+2k) \right]}$$

- We notice the contribution from the infinite class of **higher-spin conserved currents** – <u>even spin</u> with dimensions $\Delta_s = d-2+s \; : \; s=0,2,4\ldots$
- Their corresponding thermal one-point functions are:

$$a_{\mathcal{O}_s} = 2J(d-2+s)$$

QUESTIONS:

- Where do these numbers come from?
- Could one follow the thermal one-point functions along the **RG-flow**?

Ultimately, thermal one-point functions give us info for the CFT spectrum

Massive free scalars

One can do the correspondin calculation for massive free scalars in any d. CFT?

$$g_d(r,\cos\theta;m) = \sum_{n=-\infty}^{+\infty} \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \frac{e^{-i\omega_n \tau - i\vec{p}\cdot\vec{x}}}{\omega_n^2 + \vec{p}^{\,2} + m^2}$$

$$g_d(r,\cos\theta;\omega) = \frac{1}{\sqrt{2\pi}}\left(\frac{m}{r}\right)^{d/2-1} K_\nu[mr]$$

zero temperature result

$\nu = d/2 - 1$

$J = \tau + i|\vec{x}|$

$\bar{J} = \tau - i|\vec{x}|$

$$+\frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}\left[\left(\frac{m}{[(k-J)(k-\bar{J})]^{1/2}}\right)^{\nu} K_\nu\left[m[(k-J)(k-\bar{J})]^{1/2}\right]\right.$$

$$\left. + \left(\frac{m}{[(k+J)(k+\bar{J})]^{1/2}}\right)^{\nu} K_\nu\left[m[(k+J)(k+\bar{J})]^{1/2}\right]\right]$$

From now on consider $d = 2L+1$, $L = 1, 2, 3 \ldots$

works for $L=0 \Leftrightarrow d=1$

Bessels become polynomial.

- It is actually possible to obtain a **thermal CFT**

$$\left(\frac{m}{r}\right)^{\nu} K_\nu(mr) \simeq \frac{1}{r^{d-2}}\left[1 + (mr)^{\#} + \ldots\right]$$

Such terms can be combined with the contributions from the thermal part to fix $m \longrightarrow m_*$ $\Rightarrow$ only powers of $\left(\frac{r}{\beta}\right)$ appear.

Example:

In $d=3$, we find:

$$\langle \varphi^2 \rangle \propto m + 2\ln(1 - e^{-m})$$

If we impose the absence of the φ^2 operator we fix m to the famous value:

$$m_* = 2\ln\left(\frac{1+\sqrt{5}}{2}\right)$$

which gives the critical frequency of a quantum harmonic oscillator with zero free energy.

- It is also possible to expand the thermal part in Gegenbauer polynomials [one way to do it is using "inversion" methods].

The result for the **even-spin higher spin currents** is:

$$\boxed{\underset{(s\neq 0)}{a(\Delta_s)} = \frac{1}{2^{2s+\frac{d-3}{2}}} \frac{1}{s!} \frac{\Gamma(d/2-1)}{\Gamma(d-2+s)} \sum_{n=0}^{\frac{d-3}{2}+s} \frac{2^{n+1}}{n!} \frac{(d-3+2s-n)!}{\left(\frac{d-3}{2}+s-n\right)!} m^n \, \mathrm{Li}_{d-2+s-n}(e^{-m})}$$

For $s=0$ we add the term

$$\boxed{\tilde{a}(\Delta,0) = \frac{1}{2^{\Delta-\frac{d-5}{2}}} \frac{1}{\sqrt{\pi}} m^{\Delta} \, \Gamma(-\Delta/2)}$$

These are the thermal one-point functions of even-spin, higher spin operators with $\Delta_s = d-2+s$, $s=0,2,\ldots$

A novel observation

- Motivated by the curiosity to understand the intriguing mathematical formulae that give the thermal one-point functions, I focused on two of them: $\langle \varphi^2 \rangle_\ell$ and $\langle Q \rangle_\ell$ → spin-1 operator
  global "$U(1)$" charge.

  These one-point functions arise as derivatives of thermal partition functions of complex free massive scalars.

  An exactly calculable thermal partition function is the **grand canonical partition function in the presence of imaginary chemical potential.**

  (equivalently: in the presence of the temporal component of a real gauge CS field).

  We are in $d = 2L+1$ dimensions:

- The calculation is:

$$Z_{gc}(\beta; m, \mu) = \int [\mathcal{D}\bar{\varphi}][\mathcal{D}\varphi]\, e^{-S_E} = e^{-\beta F_{gc}(\beta; m, \mu)}$$

$$S_E = \int_0^\beta d\tau \int d^{d-1}x \left[|(\partial_\tau - i\mu)\varphi|^2 + |\vec{\partial}\varphi|^2 + m^2 |\varphi|^2 \right]$$

Standard scaling gives:

$$\boxed{\begin{aligned} F_{gc}(\beta; m, \mu) &= \frac{V_{d-1}}{\beta^{d-1}}\, C_d(\beta m, \beta\mu) \\ &= \frac{V_{d-1}}{\beta^{d-1}}\, C_d(z, \bar{z}) \\ z = e^{-\beta m - i\beta\mu}&, \quad \bar{z} = e^{-\beta m + i\beta m} \end{aligned}}$$

- We essentially want to calculate the thermal partition functions of the form:

$$Z_{gc} = \mathrm{Tr}\left(e^{-\beta\hat{H} - i\beta\mu\hat{Q}}\right) \qquad \hat{H} = \hat{H}_0 + m^2\varphi^2$$

$$\hat{H} = V_{d-1}\int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\, \omega_{\vec{p}}\left(\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \hat{b}^\dagger_{\vec{p}}\hat{b}_{\vec{p}} + 1\right)$$

$$\hat{Q} = V_{d-1}\int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\left(\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} - \hat{b}^\dagger_{\vec{p}}\hat{b}_{\vec{p}}\right)$$

$$\omega_{\vec{k}} = \sqrt{m^2 + \vec{p}^{\,2}}$$

The result is: $(d = 2L+1)$

$$\ln Z_L = V_{d-1}\int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\, \ln Z_0(m,\mu)$$

where $Z_0(m,\mu)$ is the partition function of two harmonic oscillators in the presence of imaginary chemical potential:

$$\hat{H} = \frac{\hat{p}_1^2}{2} + \frac{\hat{p}_2^2}{2} + \tfrac{1}{2} m^2 (\hat{x}_1^2 + \hat{x}_2^2) - i\mu (\hat{p}_1 \hat{x}_2 - \hat{p}_2 \hat{x}_1)$$

$$\Rightarrow \quad Z_0 = \mathrm{Tr}\, e^{-\beta \hat{H}} = \frac{e^{-\beta\omega}}{(1-e^{-\beta\omega - i\beta\mu})(1-e^{-\beta\omega + i\beta\mu})} = \frac{z\bar{z}}{(1-z)(1-\bar{z})}$$

This is the partition function of a relativistic bose gas in $d = 2L+1$, $L = 0, 1, 2 \ldots$

* We can take care of the divergence from the zero-point energies of the oscillators and we get a term of the form $\# m^d$.

* Note that the frequency of the harmonic oscillator becomes the mass parameter in higher dimensions.

We find: $\boxed{V_{d-1} = \ell^{d-1}}$

$$\ln Z_{L}(\beta; m, \mu) = \frac{2\ell^{2L}}{(4\pi)^L (L-1)!} \int_m^\infty d\omega\, \omega\, (\omega^2 - m^2)^{L-1} \ln Z_0(\beta; m, \mu)$$

$$= \frac{\ell^2}{2\pi} \int_m^\infty dm'\, m' \ln Z_{L-1}(\beta; m', \mu)$$

$$\alpha^2 = \frac{\ell^2}{4\pi\beta^2}$$

$$= (-2\alpha^2)^L \int_0^{|z|} \frac{dr_1}{r_1} \ln r_1 \int_0^{r_1} \frac{dr_2}{r_2} \ln r_2 \cdots \int_0^{r_{L-1}} \frac{dr_L}{r_L} \ln r_L \ln Z_0(z_L, \bar{z}_L)$$

$$z_L = e^{-\beta m_L - i\mu\beta}$$

We have written $\ln Z_L$ as an **iterated integral.**

This representation facilitates the proof of some **previously unnoticed differential equation identities.**

Define:

$$\hat{D} = \frac{1}{b^2}\frac{\partial}{\partial m^2} = \frac{1}{2\ln|z|}\left(z\partial_z + \bar{z}\partial_{\bar{z}}\right)$$

$$\hat{L} = \frac{1}{b}\, i\frac{\partial}{\partial \mu} = z\partial_z - \bar{z}\partial_{\bar{z}}$$

Then:

$$\boxed{\begin{aligned} \hat{D}\cdot \ln Z_L &= -\frac{1}{b}\langle \varphi^2 \rangle_L \\ \hat{L}\cdot \ln Z_L &= \langle \hat{Q} \rangle_L \end{aligned}}$$

For example:

$$\langle \hat{\varphi}^2 \rangle_0 \equiv \frac{1}{2}\langle \hat{x}_1^2 + \hat{x}_2^2 \rangle = -\frac{b}{2\ln|z|}\left(1 + \frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}}\right)$$

$$\langle \hat{Q} \rangle_0 = \frac{\bar{z}}{1-\bar{z}} - \frac{z}{1-z}$$

We then obtain:

$$\boxed{\begin{aligned} \hat{D}\cdot \langle \hat{\varphi}^2 \rangle_L &= -\lambda^2 \langle \hat{\varphi}^2 \rangle_{L-1} \\ \hat{D}\cdot \langle \hat{Q} \rangle_L &= -\lambda^2 \langle \hat{Q} \rangle_{L-1} \end{aligned}}$$

First order evolution equations

$\Downarrow$

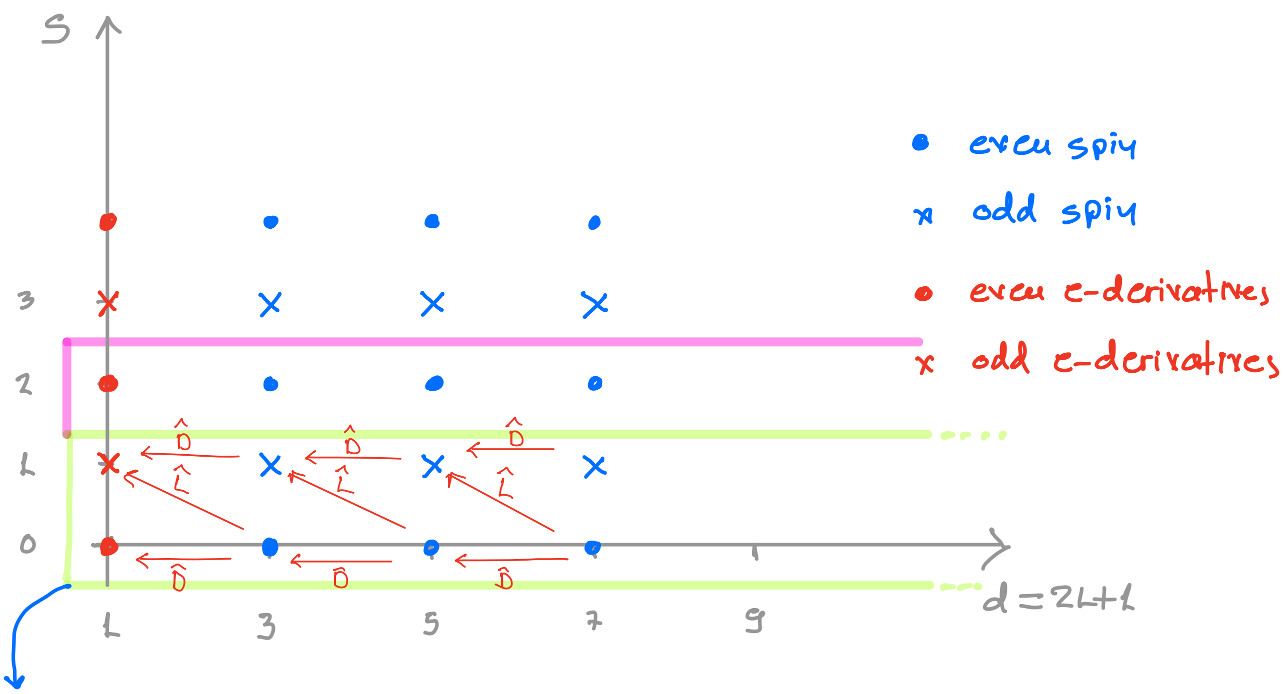
Dyson/Magnon series $\rightsquigarrow$ resummation!

There is also a second-order equation:

$$z\bar{z}\,\partial_{\bar z}\partial_z = \frac{1}{4\beta^2}\left(\frac{\partial^2}{\partial m^2}+\frac{\partial^2}{\partial \mu^2}\right) = \frac{1}{4\beta^2}\,\partial^2$$

$$\boxed{\begin{aligned} \partial^2\langle\hat{\varphi}^2\rangle_L &= -4(\alpha\beta)^2 L\,\langle\hat{\varphi}^2\rangle_{L-1} \\ \partial^2\langle\hat{Q}\rangle_L &= -4(\alpha\beta)^2 L\,\langle\hat{Q}\rangle_{L-1} \end{aligned}}$$

RECAP



The partition functions yield the spin 0, 1, 2 operators

* Notice the second order differential equation:

$$\boxed{\left[\frac{1}{L}\,z\bar{z}\,\partial_{\bar z}\partial_z - \frac{1}{2\ln|z|}\left(z\partial_z+\bar{z}\partial_{\bar z}\right)\right]\ln Z_L = 0}$$

$\Longrightarrow$ What are $\langle \varphi^2 \rangle_L$ and $\langle \hat{Q} \rangle_L$?

After some calculations we find:

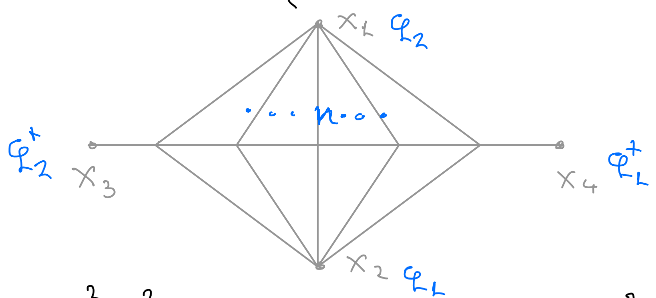
$$\boxed{\langle \hat{Q} \rangle_L = \alpha^{2L} \sum_{n=0}^{L} \frac{(-1)^n (2L-n)!\,(2\log|z|)^n}{n!\,(L-n)!} \left[\mathrm{Li}_{2L-n}(z) - \mathrm{Li}_{2L-n}(\bar{z})\right]}$$

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{n^k}$$

This is intimately related to **fishnet graphs in $d=4$.**

$$\mathcal{L} = \mathrm{Tr}\left(\partial_\mu \varphi_i \partial_\mu \varphi_i^\dagger - (4\pi g)^2 \varphi_1 \varphi_2 \varphi_1^\dagger \varphi_2^\dagger\right) \quad // \quad N_c \times N_c \text{ matrices}$$

$$\mathrm{Tr}\langle \varphi_2^n(x_1)\, \varphi_1(x_2)\, \varphi_2^{\dagger n}(x_3)\, \varphi_1^\dagger(x_4) \rangle = \frac{1}{x_{12}^{2n}\, x_{34}^2}\, \Phi_n(z,\bar{z})$$



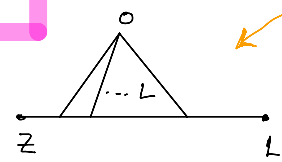
$$\left.\begin{array}{l} x_1 \to 0 \\ x_2 \to \infty \\ x_3 \to z \\ x_4 \to 1 \end{array}\right\} \text{CFT}$$

$$u = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2} = \frac{z\bar{z}}{(1-z)(1-\bar{z})} \qquad v = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} = \frac{u}{z\bar{z}}$$

$$\Phi_n(z,\bar{z}) = \frac{(1-z)(1-\bar{z})}{z-\bar{z}} \sum_{n=L}^{2L} \frac{n!\,[-\log|z|]^{2L-n}}{L!\,(n-L)!\,(2L-n)!} \left[\mathrm{Li}_n(z) - \mathrm{Li}_n(\bar{z})\right]$$

With the appropriate normalizations, we can show that:

$$\boxed{\langle Q \rangle_L = \alpha^{2L} L! \, (z-\bar{z}) \cdot}$$



The corresponding results for $\ln Z_L \sim \langle \varphi^2 \rangle_{L+1}$ are a surprise: We find.

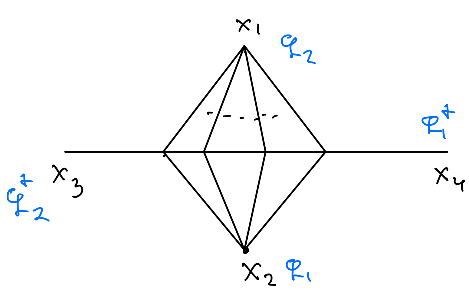
$$\ln Z_L = \alpha^{2L} \sum_{n=0}^{L} \frac{(-1)^n (2L-n)! \, (2\ln|z|)^n}{n! \, (L-n)!} \left[\mathrm{Li}_{2L+1-n}(z) + \mathrm{Li}_{2L+1-n}(\bar{z}) \right]$$

$$+ (-\alpha^2)^L \frac{L!}{2(2L+1)!} (2\ln|z|)^{2L+1}$$

This result corresponds to 4pt graphs

of a singular 2d fishnet model:

$$\mathcal{L} = \mathrm{Tr}\left[\varphi_1^+ (-\partial^2)^{\omega} \varphi_1 + \varphi_2^+ (-\partial^2)^{1-\omega} \varphi_2 + 4\pi \xi^2 \varphi_1 \varphi_2 \varphi_1^+ \varphi_2^+ \right]$$



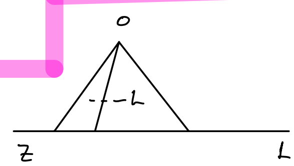
$\longrightarrow$ **in the limit $\omega \to 1$**

These ladder are proportional to:

$$\sum_{m=-\infty}^{+\infty} \int d\nu \, \frac{(z\bar{z})^{i\nu} \left(z/\bar{z}\right)^{m/2}}{\left(m^2/4 + \nu^2\right)^{L+1}}$$

.... some calculations ....

we find:

$$\ln Z_L = \alpha^{2L} L!$$ 

We expressed the thermal one-point functions of spin-0, 1 (and 2) operators in a charged massive free scalar theory in terms of 4pt fishnet graphs

Where do these formulae come from?

Consider a **charged massless** scalar field in the cylinder $S^1_\beta \times \mathbb{R}^{d-1}$

with twisted boundary conditions

$$\varphi(\tau+\beta, \vec{x}) = e^{-i\beta\mu}\varphi(\tau, \vec{x})$$

Calculate the 2pt function as:

$$\langle \varphi^\dagger(\tau, \vec{x})\varphi(0,\vec{0})\rangle = g_d(r, \cos\theta; 0, \mu) = \sum_{n=-\infty}^{+\infty} \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \frac{e^{-i(\omega_n-\mu)\tau - i\vec{p}\cdot\vec{x}}}{(\omega_n-\mu)^2 + \vec{p}^{\,2}}$$

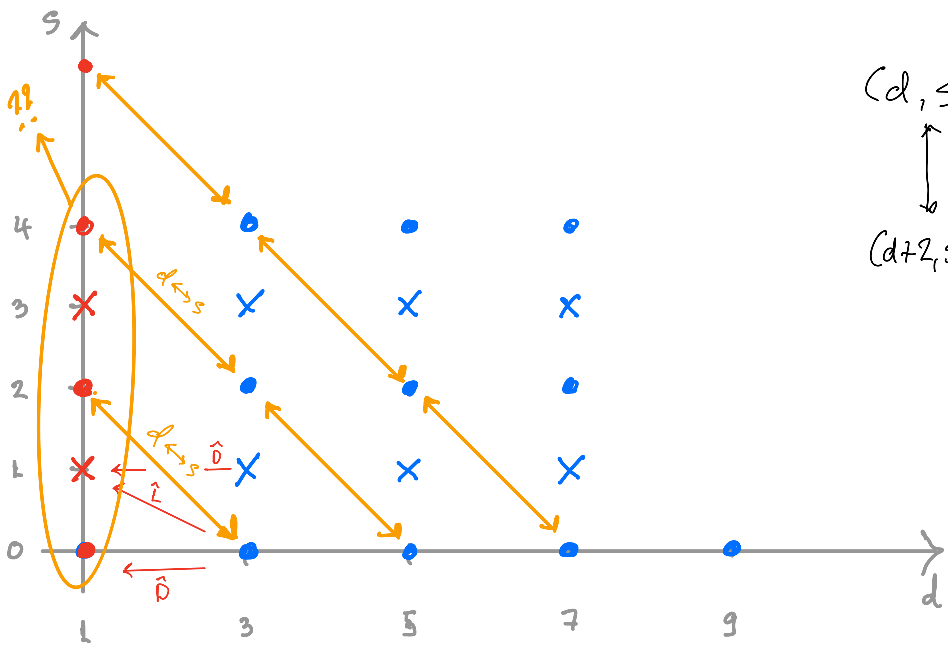
$$= \frac{\Gamma(d/2-1)}{4\pi^{d/2}}\left[\frac{1}{r^{d-2}} + \sum_{s=0,2,4}^{\infty} r^s C_s^{d/2}(\cos\theta)\, a_s^{(+)}(\mu) + \sum_{s=1,3,5}^{\infty} r^s C_s^{d/2}(\cos\theta)\, a_s^{(-)}(\mu)\right]$$

$$a_s^{(+)}(\mu) = Li_{d-2+s}(e^{-i\beta\mu}) + Li_{d-2+s}(e^{i\beta\mu}) = 2\, Ci_{d-2+s}(\mu)$$

$$a_s^{(-)}(\mu) = Li_{d-2+s}(e^{-i\beta\mu}) - Li_{d-2+s}(e^{i\beta\mu})$$

$$Li_s(\mu) = \sum_{k=1}^{\infty} \frac{e^{-ik\mu}}{k^s} = \sum_{k=1}^{\infty} \frac{\cos k\mu}{k^s} - i \sum_{k=1}^{\infty} \frac{\sin k\mu}{k^s} = Ci_s(\mu) - i\, Si_{\mu}(\mu)$$

Emerging pattern



- $d \longleftrightarrow s$ duality

$$(d, s) \longleftrightarrow (d-2, s+2)$$
$$\updownarrow$$
$$(d+2, s-2)$$

Conjecture

"Thermal" one-point functions of free theories make a lattice space of 4pt fishnet graphs

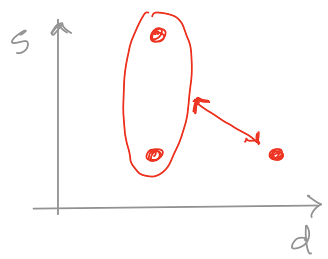
Fishnet graphs are asigned a spin $-S$

$d=2L+1$: $L \longleftrightarrow S$: Loop order $\longleftrightarrow$ spin

The two types of fishnet graphs in $d=2$, $d=4$ build the full lattice space through recursive relations

For $m\neq 0$ the relationships become more complicated. From the known results for $m\neq 0, \mu=0$ we can verify:

$$\boxed{a(d,s+2) = \frac{1}{(s+1)(s+2)}\left[\frac{1}{d/2-1}\, a(d+2,s) + \frac{m^2}{(d/2+s)(d/2-1+s)}\, a(d,s)\right]}$$



Important consequence:

The higher-spin "thermal" one-point functions correspond to linear combinations of multiloop fishnet graphs.

Further questions and remarks

- Why 4pt fishnet graphs correspond to "thermal" one-point functions?

- What is the physical meaning of the recursive relations and the differential equations?

RECAP II

$$\langle \varphi(x)\varphi(0)\rangle = \frac{1}{r^{\Delta_\varphi}} \sum_{Q_s} r^{\Delta_{Q_s}} C_s^{d/2}(\cos\theta)\, a_{Q_s}(z,\bar z)$$

Conformal invariance at finite size/temperature

Fishnet graphs at all loops

→ The "thermal" two-point function resembles an all-loop resummation of fishnet graphs.

→ The recursive relations appear to be a consequence of the free e.o.m.

e.g. $\partial^2\varphi = m^2\varphi \Rightarrow \partial_0^2\varphi = -\vec\partial^2\varphi + m^2\varphi$

$\rightarrow \quad \langle \varphi\,\partial_0^2\varphi\rangle \simeq -\langle\varphi\,\vec\partial^2\varphi\rangle + m^2\varphi^2$

$\rightsquigarrow \quad \langle T_{00}\rangle_d \simeq \langle\varphi^2\rangle_{d+2} + m^2\langle\varphi^2\rangle_d$

can reduce everything to scalar one-pt functions

$\longrightarrow$ A geometric interpretation?

The "thermal" one-point functions for $d=6$ are

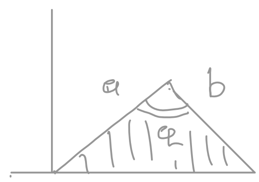
$$\langle |x|^2 \rangle_6 = -\frac{6}{2\omega|z|}\left(1+\frac{z}{1-z}+\frac{\bar z}{1-\bar z}\right)$$

$$\langle Q \rangle_6 = \frac{\bar z}{1-z} - \frac{z}{1-\bar z}$$

$$z = \frac{b}{a} e^{i\varphi}$$
$$\bar z = \frac{b}{a} e^{-i\varphi}$$

$2ab\sin\varphi$

$a^2 - b^2$



In $d=3$ $\langle Q \rangle_3$ = Bloch-Wigner function

gives volume of an ideal tetrahedron in hyperbolic space.

$$\hat D \langle \hat Q \rangle_3 = \langle \hat Q \rangle_1 \quad \cdots$$
$$\hat D \langle \hat Q \rangle_5 = \langle \hat Q \rangle_3 \quad \cdots$$

hyper-hyperbolic volumes?

to be continued...