String theory and chaos

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Chaos meets black holes and holography

- Chaos + high-energy physics: until ~ 10 years ago an unlikely marriage, now a mainstream topic

- Holography connects chaos, black hole information problem and stringy corrections to black holes

Susskind: Black holes are the fastest scramblers – the information on anything falling in quickly gets mixed up with all the degrees of freedom inside

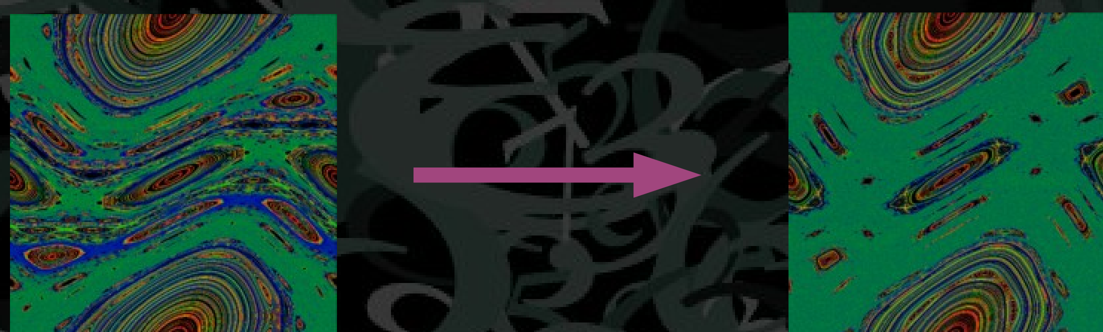
Leonard Susskind

Outline

- Quantum chaos
- Chaos bound, black holes and holography
- Away from black holes I: chaos in the string S-matrix [Savić & Čubrović 2311.xxxxx]
- Away from black holes II: chaos in matrix models [Čubrović 2203.10697, Marković & Čubrović 2202.09443 + some fresh results]

What is chaos?

- Everybody knows: nonintegrability, exponential sensitivity to initial conditions
- Integrability ⇒ one integral of motion per degree of freedom ⇒ N independent 1D oscillators
- Non-integrability ⇒ insufficient symmetry, aperiodic motion, higher-dimensional phase space



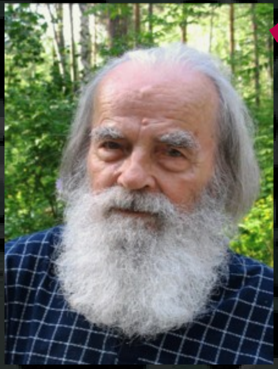
- Positive (classical) Lyapunov exponent λ_c:

$$\lambda_c = \lim_{t \to \infty} \lim_{\delta X(0) \to 0} \frac{1}{t} \log \delta X(t)$$

What is quantum chaos?

- Evolution operator is linear ⇒ no direct analogue of classical chaos, no Lyapunov exponent

Anyone who uses words "quantum" and "chaos" in the same sentence should be hung on a tree in the park behind the Niels Bohr institute!



Boris A. Chirikov, Usp. Fiz. Nauk 71, 112, (1973).

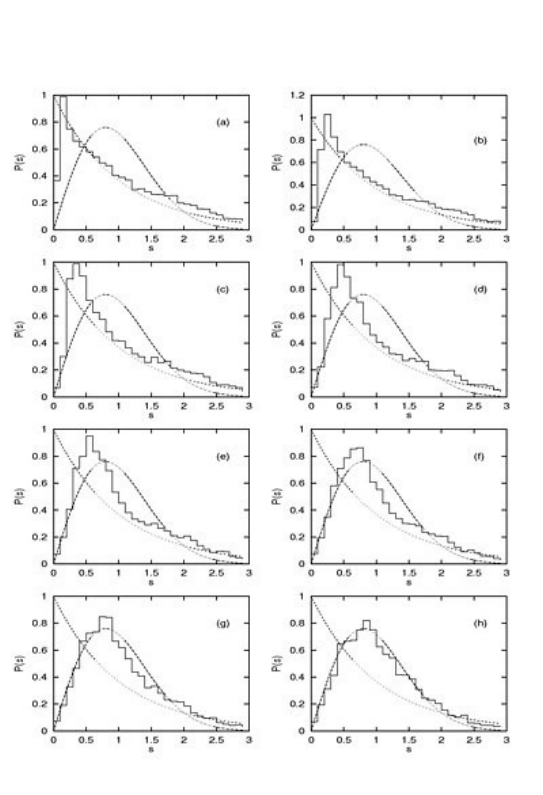
Quantum chaos and level statistics

- Evolution operator is linear ⇒ no direct analogue of classical chaos, no Lyapunov exponent
- N independent 1D oscillators vs. coupled dynamics → independent vs. correlated energy levels

Integrable quantum system: independent levels, Poisson distribution of energy level spacings $P(s) = e^{-s}$

Nonintegrable quantum system: level repulsion, Wigner-Dyson distribution

$$P(s) = c_\beta s^\beta e^{-a_\beta s^2}$$



Quantum chaos and OTOC

- Loschmidt echo: evolve the system for time t with the Hamiltonian H, perturb the Hamiltonian as $H+\Delta H$, then evolve backward in time (i.e. for time $-t$) and calculate the overlap of initial and final state
- A variation: act by operator A at $\tau=0$, observe the evolution of operator B until time $\tau=t$, then evolve backwards and compute the overlap
- This is captured by out-of-time ordered correlator (OTOC)

$$C(t)\equiv\langle|[A(t),B(0)]|^2\rangle=2\langle A^\dagger(t)A(t)B^\dagger(0)B(0)\rangle-2\langle A^\dagger(t)B^\dagger(0)A(t)B(0)\rangle=2(\mathrm{TOC}-\mathrm{OTOC})$$

- Putting $A=X, B=P$ we get the closest possible meaningful quantum generalization of the Lyapunov exponent

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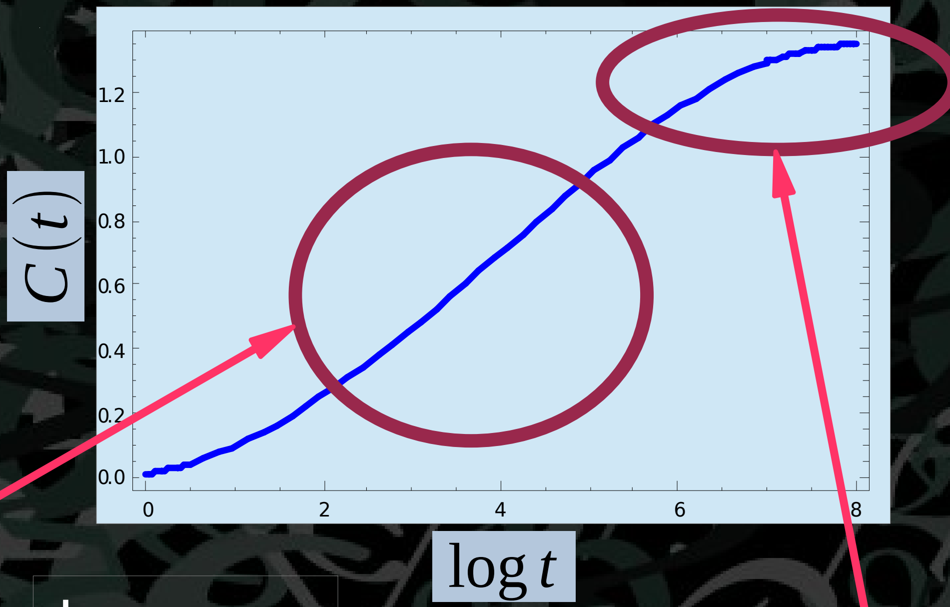


$t \ll t_s : L e^{2\lambda t}$

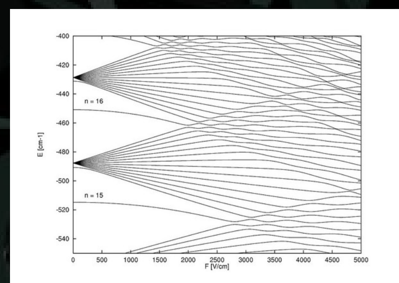
Lyapunov exponent

$t \gg t_s : \sim \mathrm{const.}$

Ruelle resonance

The confusing terminology of quantum chaos

Level statistics



Random matrix theory in the large system limit – Gaussian ensembles

Fully determined solely by the Hamiltonian

Very hard for many-body systems

Hard to relate to holography

Perturbation growth:

$$\langle A^{\dagger}(0)B^{\dagger}(t)A(0)B(t)\rangle \sim \exp(\lambda t)$$

Out-of-time-ordered correlators

Related (?) to classical Lyapunov exponents for $A=P,\ B=X$

Similar to Loschmidt echo but there are subtleties with order of limits etc

Non-equilibrium property, depends on the choice of A, B

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Scrambling and chaos

- Scrambling – a new (~10 years ago) concept of chaos, dressed in quantum information: even in a pure state detailed information about the system is effectively hidden
- Rigorously: a system of size M is scrambled if the entanglement entropy of any subsystem of size $m<M/2$ is maximal
- Interpretation: we need to study at least half of the system to retrieve info on even a small part of it
- Scrambling time: time needed for a small perturbation (adding a few qubits) to distribute over the system

Scrambling and chaos

- Dimensional analysis for scrambling time at inverse temperature β:

$$\frac{M}{2} \sim D(\beta) t_*^{d/2}, \quad D(\beta) \text{ - diffusion coefficient, } d \text{ - space dimension}$$

$$t_* \sim M^{2/d} D(\beta)^{-2/d}$$

- At infinite dimension $d \to \infty$ ⇔ mean-field limit ⇔ large N limit (number of colors, $M \sim N^2$):

$$t_* \sim \log M \times \lim_{d \to \infty} D(\beta)^{-2/d} \sim \log N \times \lim_{d \to \infty} D(\beta)^{-2/d}$$

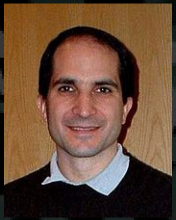
- The large-N limit of $D(\beta)$ is nontrivial

Black holes are fast scramblers

- Celebrated derivation of the scrambling time in large-N field theories from the analytic continuation of OTOC: Maldacena-Shenker-Stanford (MSS) bound [1503.01409]

$$t_* \geq \frac{1}{2\pi T} \log N^2 \Rightarrow \lambda \leq 2\pi T$$

- Maximum quantum Lyapunov exponent is reached in large-N QFTs at strong coupling ⇔ QFTs dual to classical AdS black holes



Juan Maldacena

Steve Shenker

Douglas Stanford

Black holes are fast scramblers

- The same large-N limit and the chaos bound can easily be found from semiclassical black hole horizons (Susskind 0808.2096)
- Consider the near-horizon Rindler space:

$$ds^2 = -dt^2 + dx_i dx^i + dz^2 = -\rho^2 d\omega^2 + dx_i dx^i + d\rho^2$$

$$t = \rho \sinh\omega, \quad z = \rho \cosh\omega$$

- Wave equation for a scalar perturbation (or charge density) on the horizon yields:

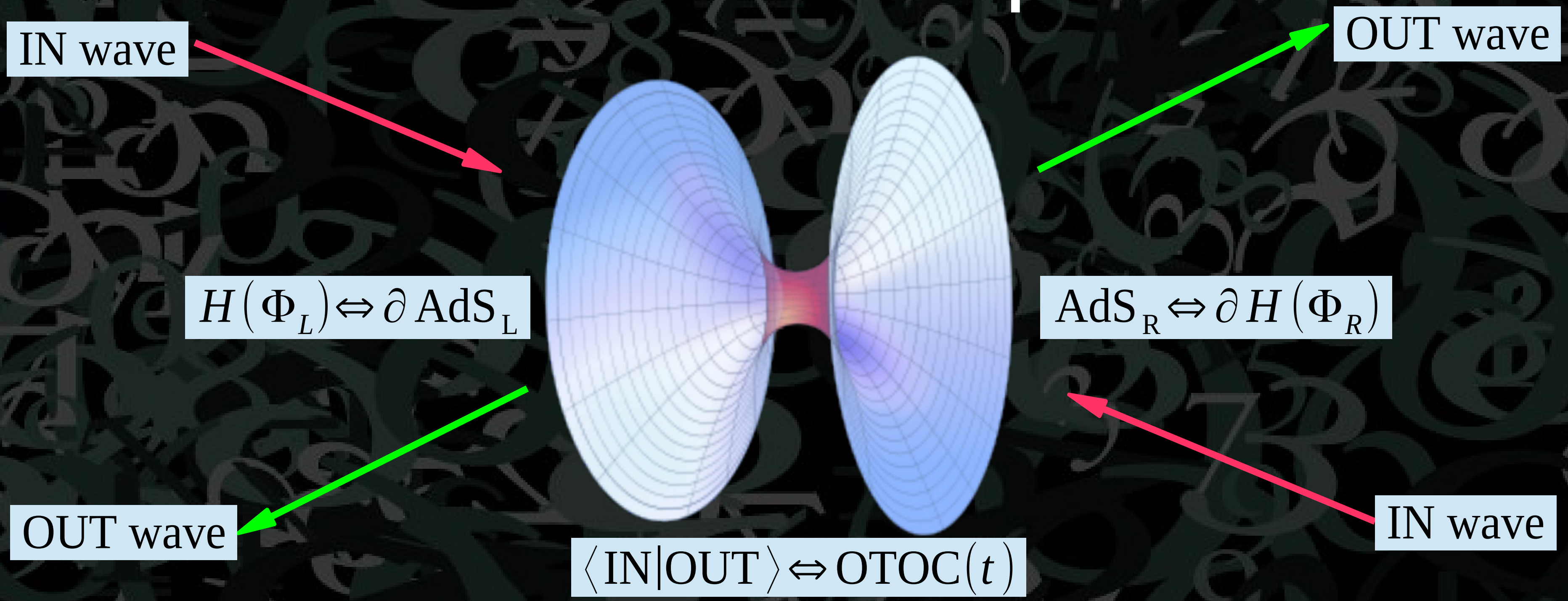
$$\Phi \sim \exp(-\omega)(x^2+1)^{-(d-1)/2} \Rightarrow t_* \sim (d-1) r_h$$

- Going to asymptotic time $t = (\beta/2\pi)\omega$:

$$t_* \sim \frac{1}{2\pi T} \log r_h^{d-1} \sim \frac{1}{2\pi T} \log S \Rightarrow \lambda = 2\pi T$$

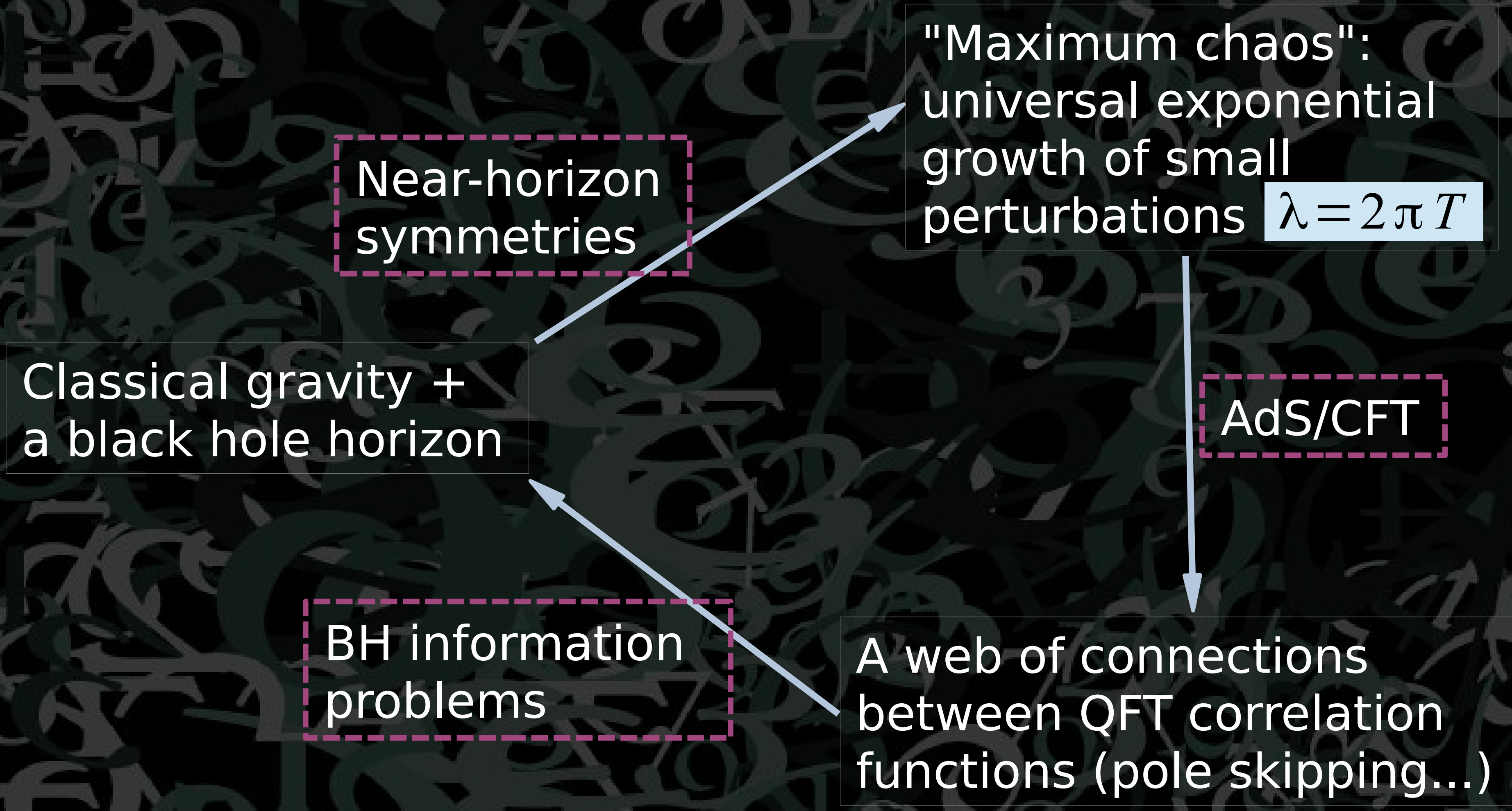
Leonard Susskind

Stanford-Shenker protocol



- Maximally extended AdS-Schwarzschild black hole
- Four-wave scattering ~ out-of-time-ordered correlator (OTOC)
- Left and right AdS with left and right dual CFT – thermofield double (a formal way of doing QFT at finite temperature)
- Fast scrambling at BH horizon ⇒ MSS bound $\lambda_{\mathrm{max}} = 2\pi T$

Fast scrambling, black holes and all that



Fast scrambling, black holes and all that

"Maximum chaos": universal exponential growth of small perturbations $\lambda = 2\pi T$

Near-horizon symmetries

~~Classical gravity + a black hole horizon~~

AdS/CFT

BH information problems

A web of connections between QFT correlation functions (pole skipping...)

Fast scrambling, black holes and all that

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Highly excited strings and black holes

- Highly excited string (HES): occupation number $N \gg 1$
- Horowitz&Polchinski 1990s: BH/string complementarity

$$M_{\mathrm{BH}} = \frac{r_s^{D-2}}{G_N}, \quad M_{\mathrm{string}} = \frac{N}{\alpha'}$$

when string becomes a black hole: $M_{\mathrm{string}} \sim M_{\mathrm{BH}},\ l_s = \sqrt{\alpha'} \sim r_s$

- Newton's constant: $G_N = \alpha' g^2$
- Altogether at the transition we get:

$$N g^4 \sim (\alpha')^{D-3} \quad \Rightarrow \quad N_c \sim 1/g^4$$

Highly excited strings and black holes

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- The bottom line: the transition to black hole can be observed at the tree level (small g) at the cost of going to large occupation numbers N
- Can we see how the string approaches the fast scrambling regime?
- Sensitive dependence of amplitudes on initial conditions found by Gross&Rosenhaus 2103.15301, Bianchi, Firrotta, Sonnenschein & Weissman 2303.17233 – does that mean chaos?

Highly excited string scattering

Nikola Savić

- The idea: look at the S-matrix structure of the string-string scattering amplitude, when the strings are highly excited
- It is known how to build a highly excited string (HES) in an analytically controled way: DDF formalism (Di Vecchia, Del Guidice & Fubini)
- Start from the tachyon ($N=0$ state) and add to it $J \gg 1$ photons ($N=1$ states) to get a HES with $N \gg 1$:

$$|\mathrm{HES}\rangle \propto \xi^{i_1 \ldots i_J} P\left(\partial X, \partial^2 X \ldots \partial^N X\right)$$

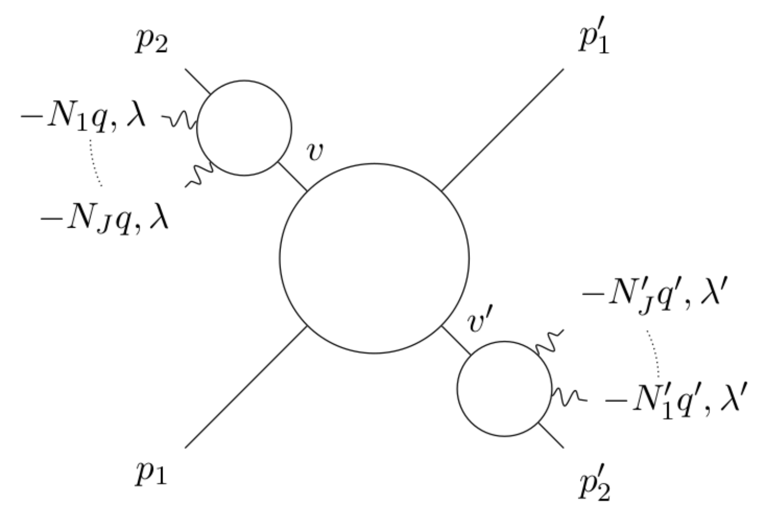
- Analytically doable within the DDF formalsim with the usual stock of tricks (OPE expansions etc)

Highly excited open string amplitudes

- The setup: HES + tachyon → HES' + tachyon'
- Scattering amplitude at tree-level is found analytically:

$$A = A_{st} + A_{tu} + A_{us} + A_{ts} + A_{su} + A_{ut}$$

$$A_{st} = \frac{1}{\text{Vol SL}(2,R)} \int DX\, e^{-S_P} \int \prod_i dw_i V_t(w_i, p_i) \int \prod_{a=1}^{J} dz_a V_p(z_a, -N_a q, \zeta) \int \prod_{b=1}^{J'} d z_b' V_p(z_b', -N_b' q, \zeta)$$



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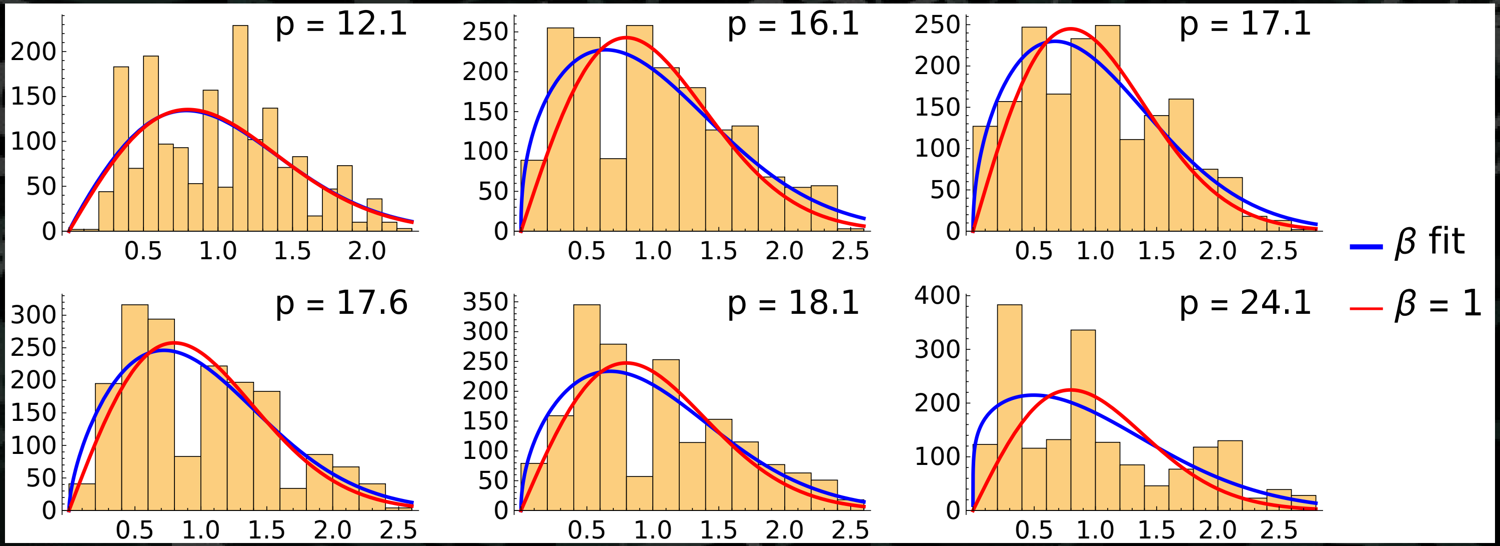
- Worldsheet integrals yield expressions of the form:

$$A_{st} = \sum_{i_a} \sum_{j_b} \sum_{k_a} \sum_{l_b} \prod_{i_a} C(k_{i_a}) \prod_{j_b} D(l_{j_b}) B(-1 - s/2 + k, -1 - t/2 + l)$$

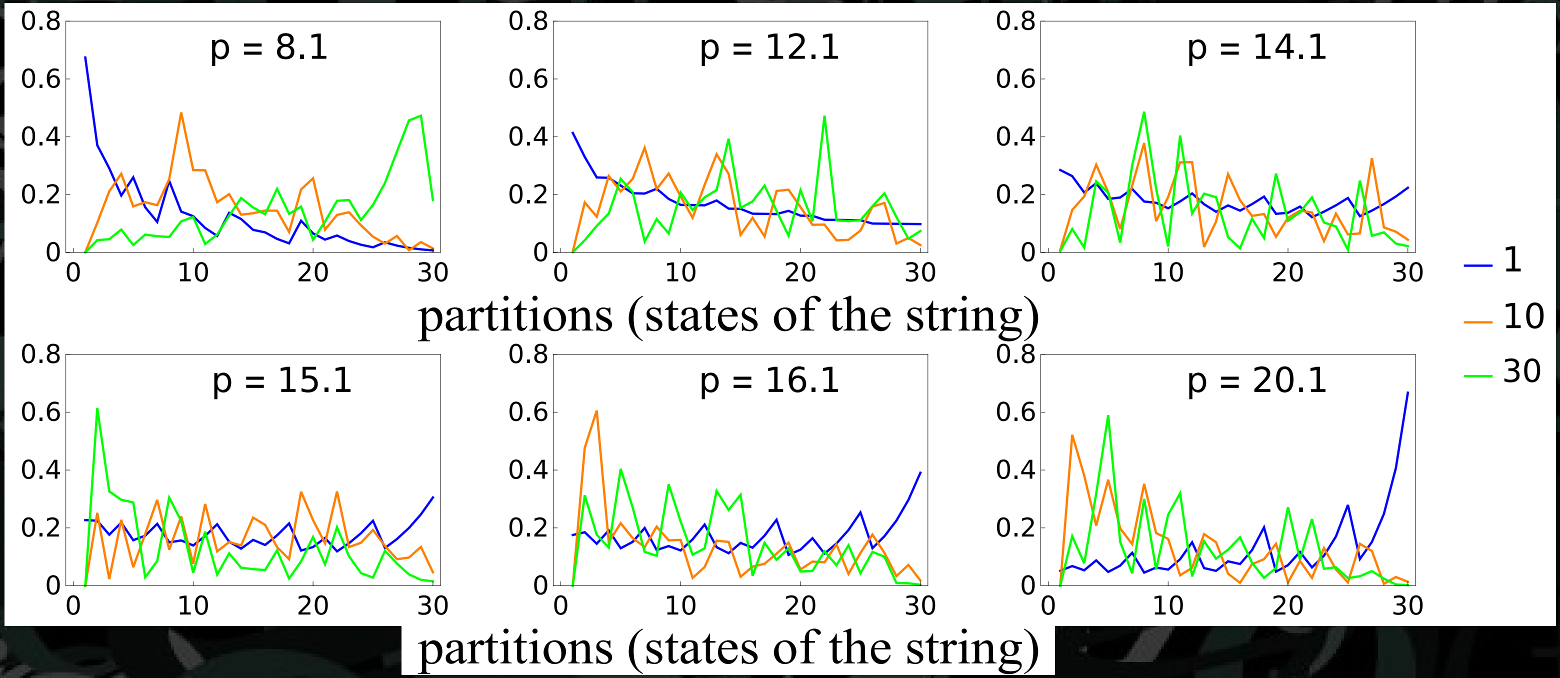
- The indices i_a, j_b, k_a, l_b go over all the permutations of the photon insertions ⇒ the number of terms grows superexponentially (roughly with $N!$)

Mixed dynamics of the S-matrix



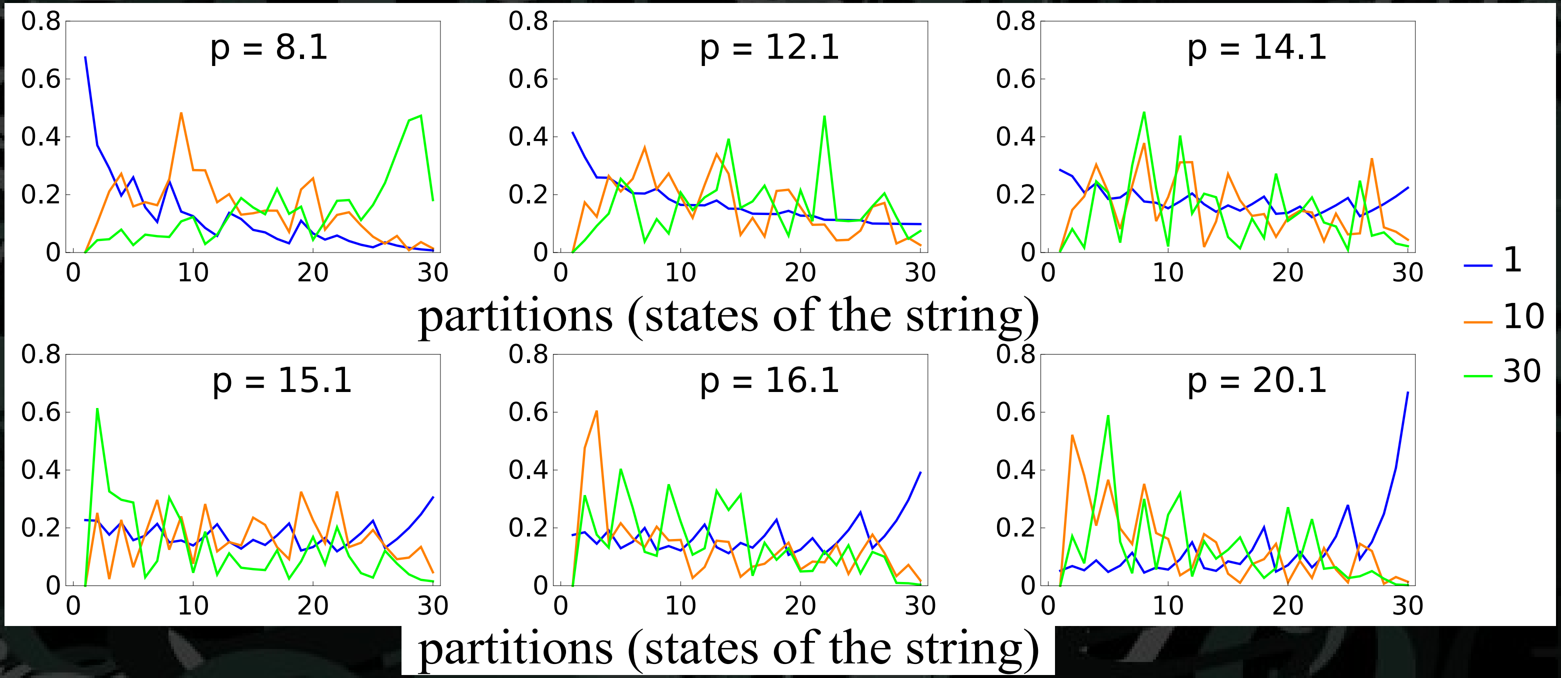
- Textbook test of quantum chaotic scattering: differences between the phases of the S-matrix eigenvalues vs. Random matrix theory (Gaussian orthogonal ensemble)
- Decent fit but there are clear deviations, in particular the excess of near-zero spacings (islands of regular dynamics)

Crossover from short to long partitions



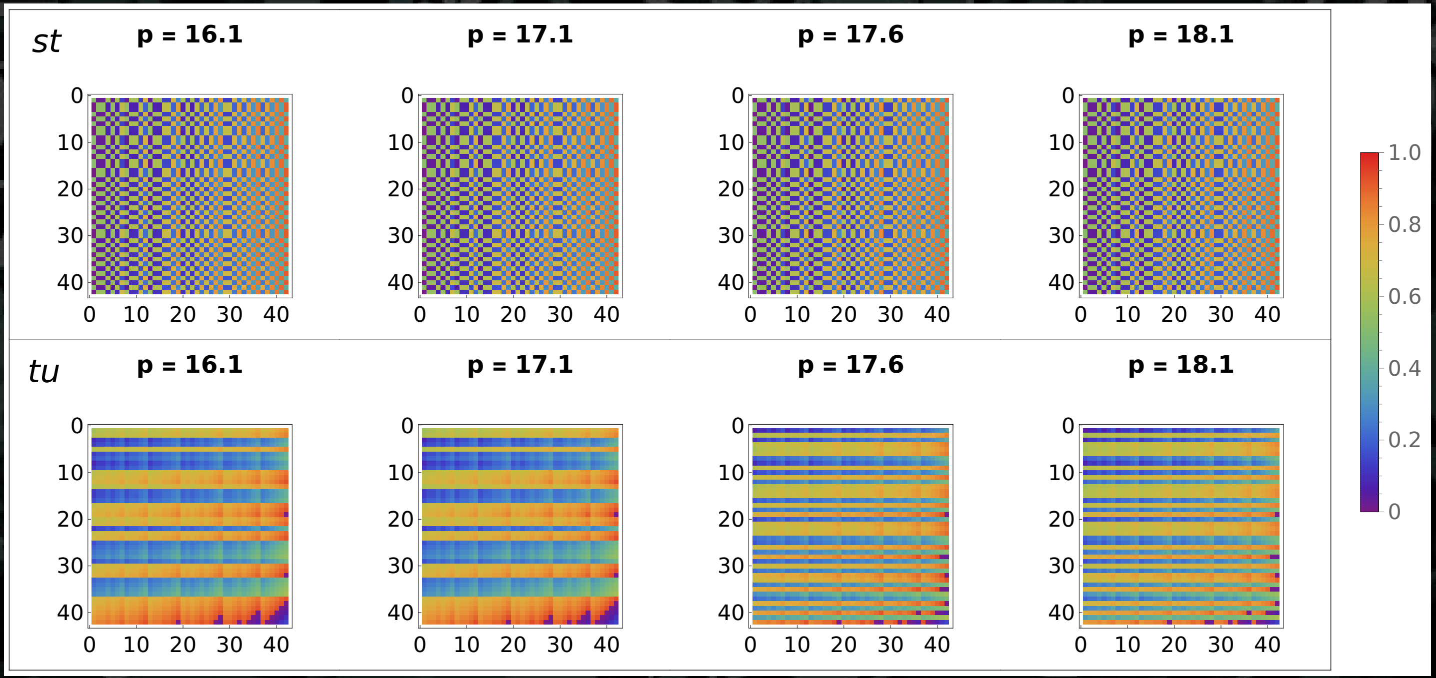
- Eigenvectors of the S-matrix ordered from the largest eigenvalue (n=1, blue) toward smaller eigenvalues (here n=10, red and n=30, green)
- The leading eigenvector (blue) contributes most to the scattering

Crossover from short to long partitions



- Small momenta: the leading eigenvector consists mainly of short partitions, like (0, 0, 0.... N, 0, ... 0)
- Large momenta: the leading eigenvector consists mainly of long partitions, like (1, 1, 1, ... 1)

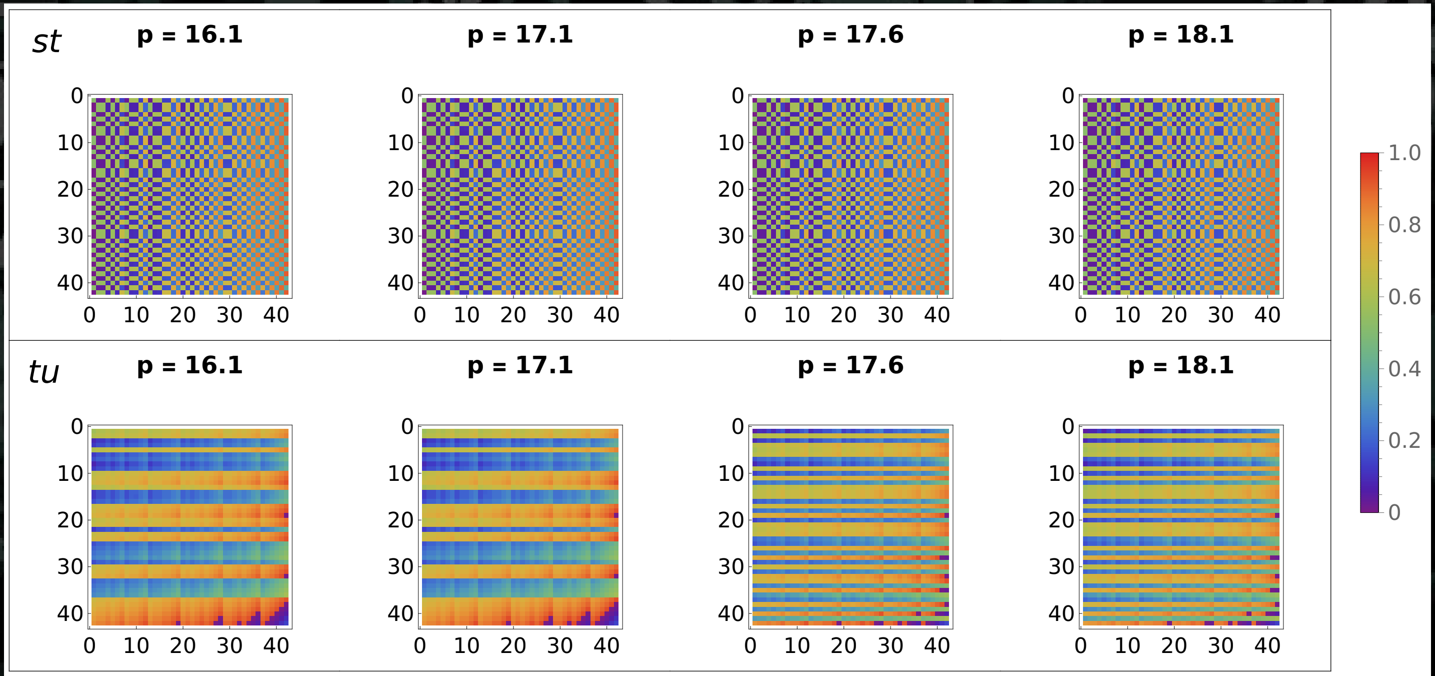
Persistent states



Phases of the S-matrix in the permutation basis (color code): random structure predicted by Wigner-Dyson but in the tu channel we see nearly-invariant states

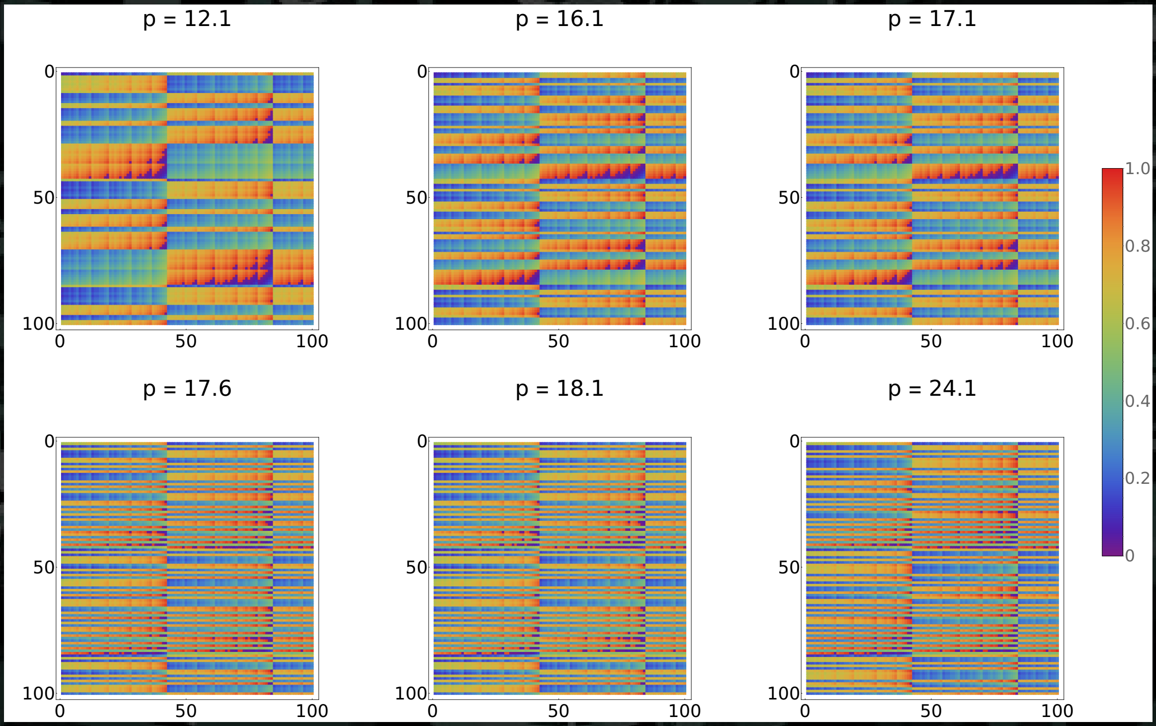
- Persists for large N – into the black hole regime!

Persistent states



- Origin of the crossover: competition between the chaotic states (majority) and a few states that almost do not change upon scattering

Persistent states



- Same happens in the tu channel for closed strings. Amplitudes computed either through KLT relations or directly (brute-force numerics)

Near-fixed points of the random walk model

- Some analytical insight comes from the $N \to \infty$ probabilistic analysis of the S-matrix elements:

$$S_{\vec{n}_1 \vec{n}_2} = \sum_{i_a} \sum_{j_b} \sum_{k_a} \sum_{l_b} (\ldots) = \prod_{i=1}^{|\vec{n}_1|} \frac{C_{\mathrm{st}}}{1-s^i} \prod_{j=1}^{|\vec{n}_2|} \frac{D_{\mathrm{st}}}{1-t^j} + (\text{all other channels})$$

- For the $N \to \infty$ model there are always states with eigenvalues $|s-1| < \mathrm{const.}/N^2$ i.e. states that remain almost unchanged

ToDo's and conclusions (so far)

- Unlike individual amplitudes in 2103.15301 and 2303.17233 the ***S-matrix has strong and persistent (with growing N) deviations*** from RMT statistics and strong chaos ⇒ important to study the whole S-matrix
- By definition we look at the asymptotic states at $t \to \infty$ just like in the Shenker-Stanford protocol but the individual string fluctuations have much richer dynamics than the MSS scaling

ToDo's and conclusions (so far)

- Immediate task: understand the physical meaning of persistent states, relate to the well-known random walk models of highly-excited strings (Kazakov&Migdal 1985, Sagerstam 1989)

- Redo for curved background – is that the missing link to chaos? From the S-matrix formula it seems not but one should check...

- Relation to quantum scars?

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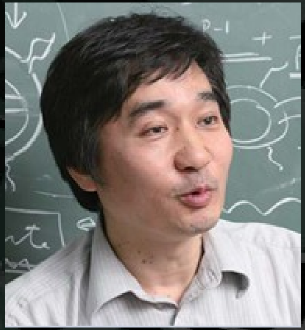
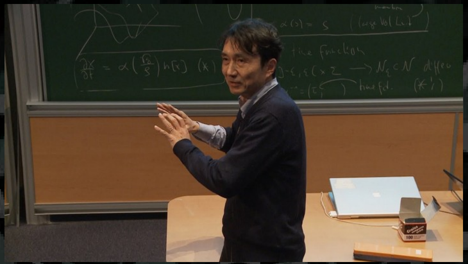
Type IIB matrix model (IKKT model)

- The matrix formulation of type IIB string theory – Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) model
- Perfect testing ground for string dynamics:

  - rich dynamics, including brane configurations (full nonpertrubative string theory?)

  - 0-dimensional ⇒ no derivatives ⇒ simple path integrals



N. Ishibashi	H. Kawai	Y. Kitazawa	A. Tsuchiya

Path integral of the IKKT model

- Discretization of the Schild action for type IIB string theory in 0 dimensions:

$$S = \frac{1}{4}[X^\mu, X^\nu]^2 + \frac{1}{2}\bar{\psi}_\alpha \Gamma_\mu [X^\mu, \psi_\alpha] + \beta$$

$$\mu = 1 \ldots 10, \quad \alpha = 1 \ldots 16$$

X^μ – bosonic coordinates – NxN Hermitian matrices

ψ_α – Majorana-Weyl spinors – NxN Hermitian matrices

- Lorentzian signature: always real but not positive definite because of the time component → sign problem

$$Z_L = \sum_N \int D[A_\mu] \int D[\psi_\alpha] \int D[\bar{\psi}_\alpha] \exp\left(iS_L[A_\mu, \Psi_\alpha, \bar{\psi}_\alpha]\right)$$

Dp-brane solutions

- Remember: IIB string theory has Dp brane excitations with p odd: $p=-1$ – D-instantons, $p=1$ – strings, etc.
- D-instantons – points in spacetime as elementary degrees of freedom; any configuration is a collection of N instantons
- Single Dp brane solution of the matrix model (IKKT 1997, Aoki, IKK, Tada & Tsuchiya 1999):

$$A_\mu = \left(q_1, k_1, q_2, k_2 \ldots q_{(p+1)/2}, k_{(p+1)/2}, 0, \ldots 0 \right), \qquad [q_\mu, k_\nu] = i \frac{L^2}{2\pi N^{2/(p+1)}}$$

- Hermitian random matrices q_i, k_i with compactification radius L_i and eigenvalues bounded as $0 \le \alpha_j^{p_i}, \alpha_j^{p_i} \le L_i$
  $j = 1 \ldots N$

Dynamics in type IIB string regime

- Consider the matrix X_0 as the "time operator" and its eigenvalues τ_i as discrete time increments:

$$0<\tau_1<\tau_2<\ldots\tau_N$$

- Off-diagonal terms are exponentially small (Tsuchiya et al, 1108.1540, 1311.5579)
- Therefore one might identify the eigenvalues with time instants...
- But that leaves strong fluctuations. Easier to work with "coarse-grained" time – suming over n neighboring eigenvalues:

$$t_I=\frac{1}{n}\sum_{j=0}^{n}\tau_{I+j}, \qquad I=0\ldots N-n$$

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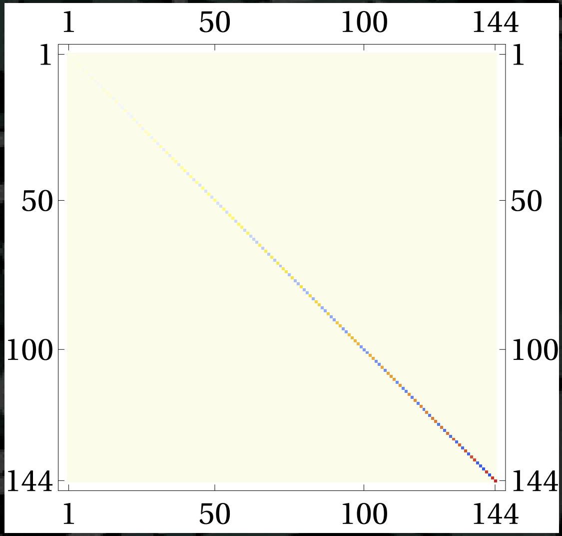
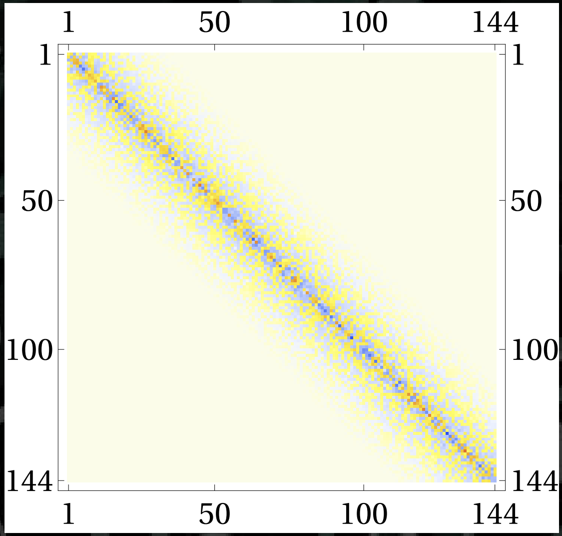
- Time instants ~ $n \times n$ blocks



X_0



X_1

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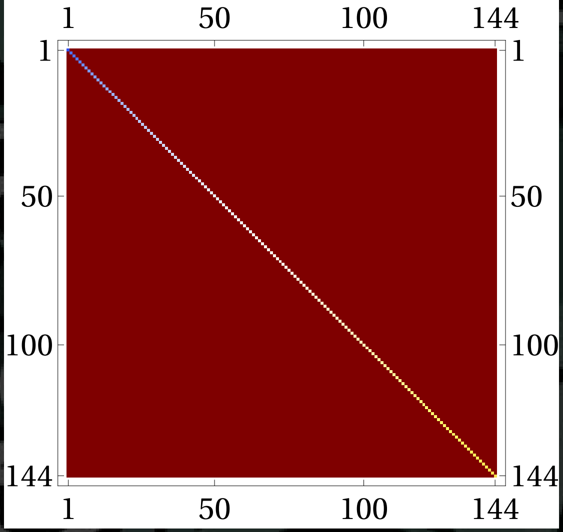
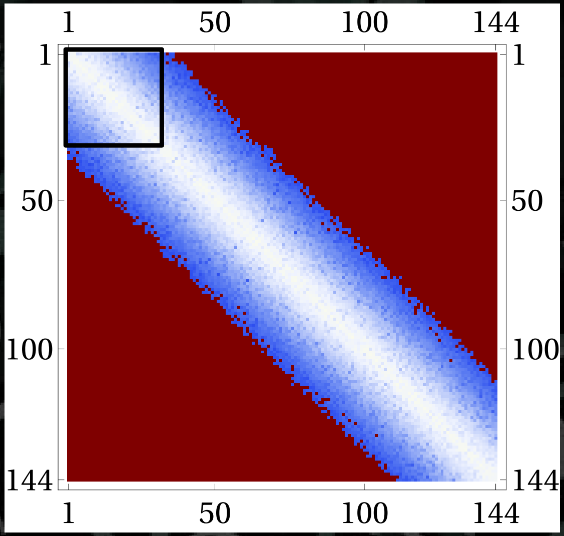


$\log X_0$



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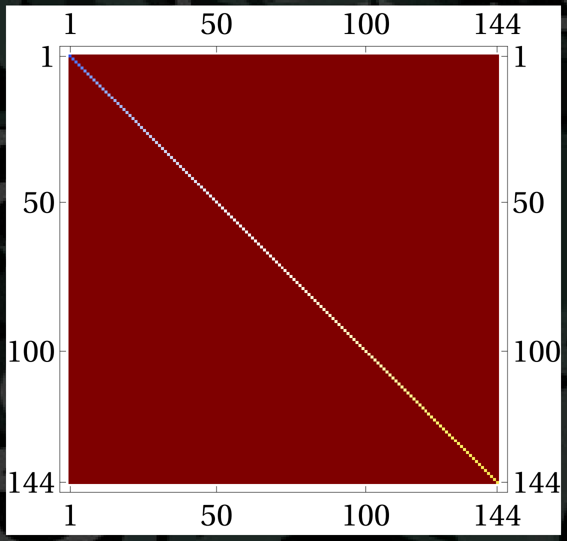

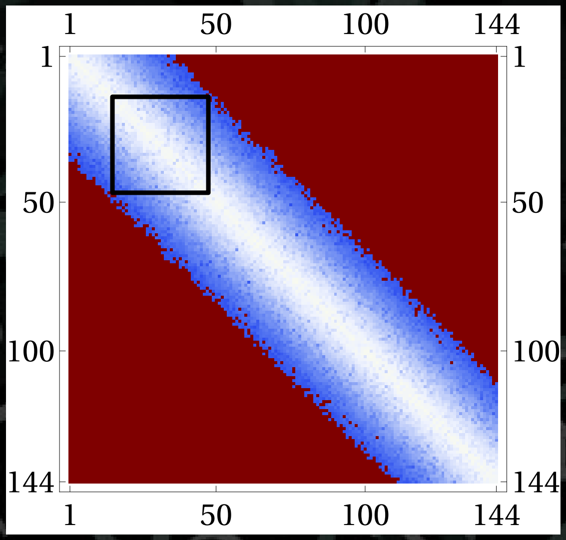

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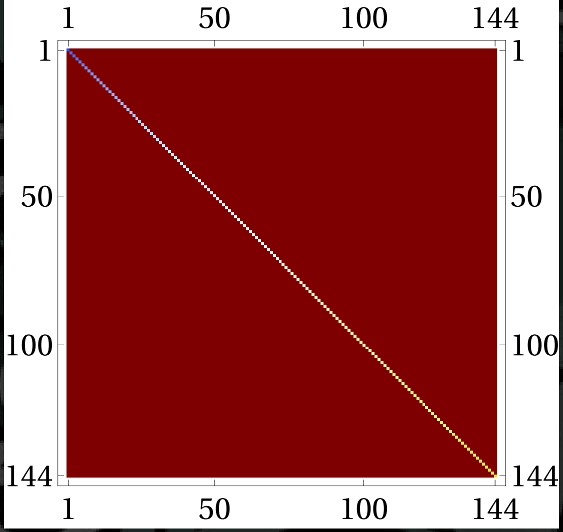
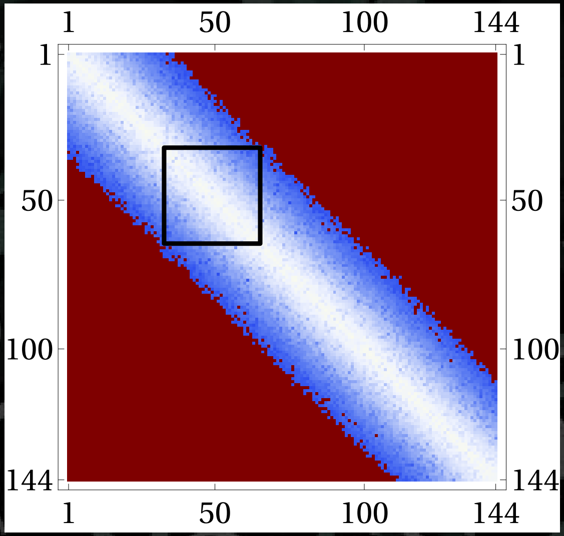


$\log X_0$



$\log X_1$

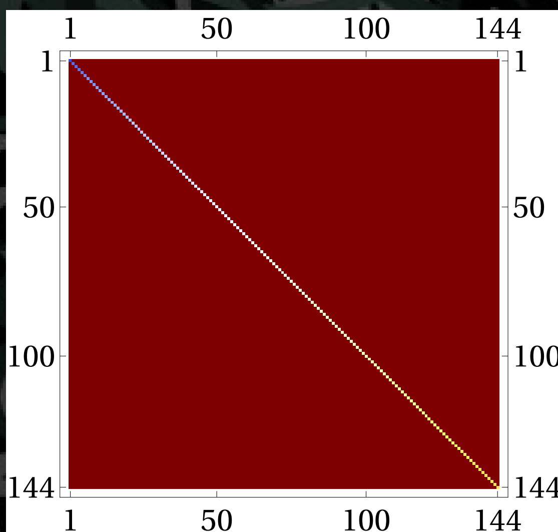
Dynamics in type IIB string regime

- Consider the matrix X_0 as the "time operator" and its eigenvalues τ_i as discrete time increments:

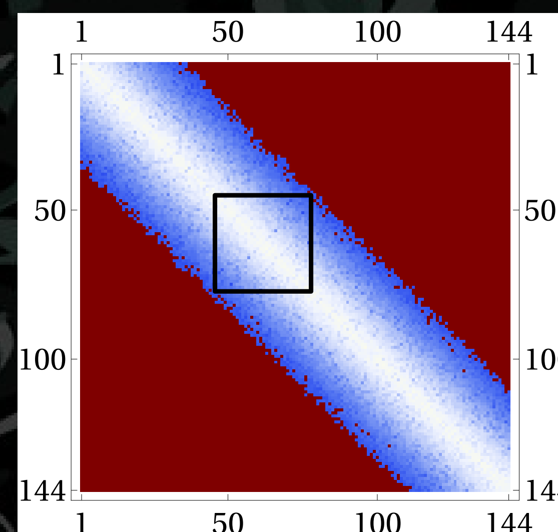
$$0 < \tau_1 < \tau_2 < \ldots \tau_N$$

$$t_I = \frac{1}{n} \sum_{j=0}^{n} \tau_{I+j}, \qquad I = 0 \ldots N-n$$

- Time instants ~ $n \times n$ blocks



$\log X_0$



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Dynamics in type IIB string regime

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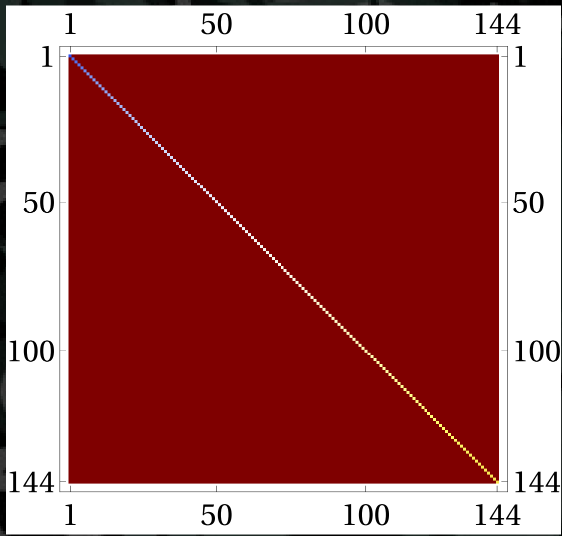
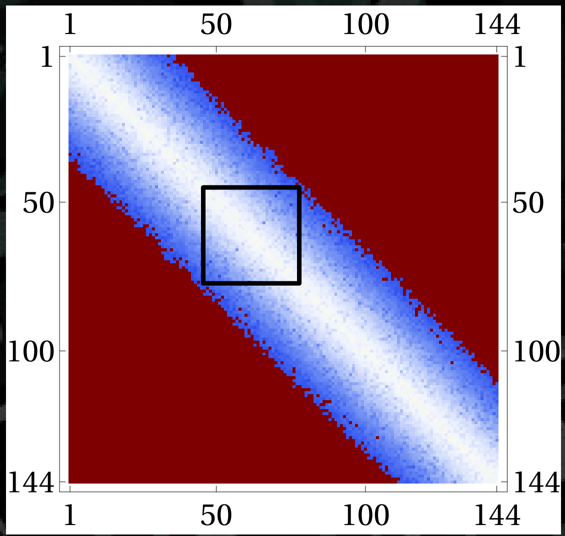
$$0<\tau_1<\tau_2<\ldots\tau_N$$

$$t_I=\frac{1}{n}\sum_{j=0}^{n}\tau_{I+j}, \qquad I=0\ldots N-n$$

- Off-diagonal elements decay exponentially fast (~eigenstate thermalization)



$\log X_0$



$\log X_1$

Time-disordered correlators in type IIB matrix model

- Time-ordered correlator (TOC): $\langle X^\dagger(t)\, X(t)\, X^\dagger(0)\, X(0)\rangle$
- Out-of-time ordered correlator (OTOC): $\langle X^\dagger(t)\, X^\dagger(0)\, X(t)\, X(0)\rangle$
- The usual definition of TOC and OTOC applied to matrices (coordinates $X \equiv X_1,\ \ Y \equiv X_2$:

$$C(t_I) \equiv \langle |[\widetilde{X}_I, \widetilde{Y}_0]|^2 \rangle = 2\left(\mathrm{TOC} - \mathrm{OTOC}\right)$$

$$\mathrm{TOC} = \langle \widetilde{X}_I^\dagger \widetilde{X}_I \widetilde{Y}_0^\dagger \widetilde{Y}_0 \rangle = \frac{1}{Z_L} \int D[X_\mu]\, \widetilde{X}_I^\dagger \widetilde{X}_I \widetilde{Y}_0^\dagger \widetilde{Y}_0\, e^{i S_L}$$

$$\mathrm{OTOC} = \langle \widetilde{X}_I^\dagger \widetilde{Y}_0^\dagger \widetilde{X}_I \widetilde{Y}_0 \rangle = \frac{1}{Z_L} \int D[X_\mu]\, \widetilde{X}_I^\dagger \widetilde{Y}_0^\dagger \widetilde{X}_I \widetilde{Y}_0\, e^{i S_L}$$

- Crucial analytical trick: separation into diagonal elements $X_{ii} \equiv q_i,\, Y_{ii} \equiv p_i$ and off-diagonal elements $x_{ij},\, y_{ij} \ll q_i,\, p_i$

(O)TOC in type IIB regime

- Crucial analytical trick: separation into diagonal elements $X_{ii} \equiv q_i, Y_{ii} \equiv p_i$ and off-diagonal elements $x_{ij}, y_{ij} \ll q_i, p_i$
- Schematically:

$$\mathrm{TOC} = \sum_{i=1}^{n} |q_{I+i}|^2 |p_i|^2 + \sum_{i,j=1}^{n} |q_{I+i}|^2 |y_{ij}|^2 + \sum_{i,j=1}^{n} |p_i|^2 |x_{I+i,I+j}|^2 + \dots$$

$$\mathrm{OTOC} = \sum_{i=1}^{n} |q_{I+i}|^2 |p_i|^2 +$$
$$+ \sum_{i,j=1}^{n} \left(q_{I+i}^* p_{I+j} - \mathrm{c.c.}\right) \left(\tilde{x} \cdot \tilde{y}\right)_{I+i,i} + \sum_{i,j=1}^{n} \left(p_i^* q_i + \mathrm{c.c.}\right) \left(\tilde{x} \cdot \tilde{y}\right)_{I+i,i} + \dots$$

$$C(t_I) = 2 \sum_{i,j=1}^{n} |q_{I+i}|^2 |y_{ij}|^2 + 2 \sum_{i,j=1}^{n} |p_i|^2 |x_{I+i,I+j}|^2 + \mathrm{subleading}$$

$$\tilde{X} = \mathrm{diag}(q_1 \dots q_{N-n}) + \tilde{x}, \quad \tilde{Y} = \mathrm{diag}(p_1 \dots p_{N-n}) + \tilde{y}$$

(O)TOC in type IIB regime

- Crucial analytical trick: separation into diagonal elements $X_{ii} \equiv q_i, Y_{ii} \equiv p_i$ and off-diagonal elements $x_{ij}, y_{ij} \ll q_i, p_i$
- Schematically:

$$\text{TOC} = \sum_{i=1}^{n} |q_{I+i}|^2 |p_i|^2 + \sum_{i,\, j=1}^{n} |q_{I+i}|^2 |y_{ij}|^2 + \sum_{i,\, j=1}^{n} |p_i|^2 |x_{I+i,\, I+j}|^2 + \ldots$$

$$\text{OTOC} = \sum_{i=1}^{n} |q_{I+i}|^2 |p_i|^2 + \\ + \sum_{i,\, j=1}^{n} \left(q_{I+i}^{*} p_{I+j} - \text{c.c.}\right)\left(\tilde{x}\cdot\tilde{y}\right)_{I+i,\, i} + \sum_{i,\, j=1}^{n} \left(p_i^{*} q_i + \text{c.c.}\right)\left(\tilde{x}\cdot\tilde{y}\right)_{I+i,\, i} + \ldots$$

$$C(t_I) = 2\sum_{i,\, j=1}^{n} |q_{I+i}|^2 |y_{ij}|^2 + 2\sum_{i,\, j=1}^{n} |p_i|^2 |x_{I+i,\, I+j}|^2 + \text{subleading}$$

From numerics and statistical arguments $\langle |q_{I+i}|^2 \rangle, \langle |p_{I+i}|^2 \rangle \sim \exp(r t_I)$

(O)TOC in type IIB regime

- Crucial analytical trick: separation into diagonal elements $X_{ii} \equiv q_i, Y_{ii} \equiv p_i$ and off-diagonal elements $x_{ij}, y_{ij} \ll q_i, p_i$
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$$C(t_I) = 2\sum_{i,j=1}^{n} |q_{I+i}|^2 |y_{ij}|^2 + 2\sum_{i,j=1}^{n} |p_i|^2 |x_{I+i,I+j}|^2 + \text{subleading}$$

Irrelevant for chaos – overall change of scale $\sim \exp(2 r t_I)$

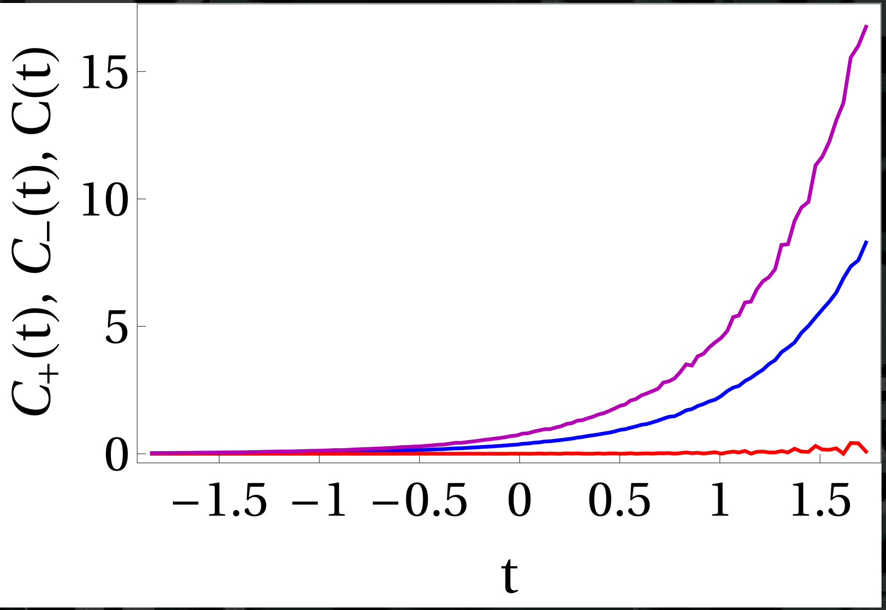
No equilibrium state and no maximal chaos

$$C(t_I) = 2\sum_{i,j=1}^{n} |q_{I+i}|^2 |y_{ij}|^2 + 2\sum_{i,j=1}^{n} |p_i|^2 |x_{I+i,I+j}|^2 + \text{subleading}$$

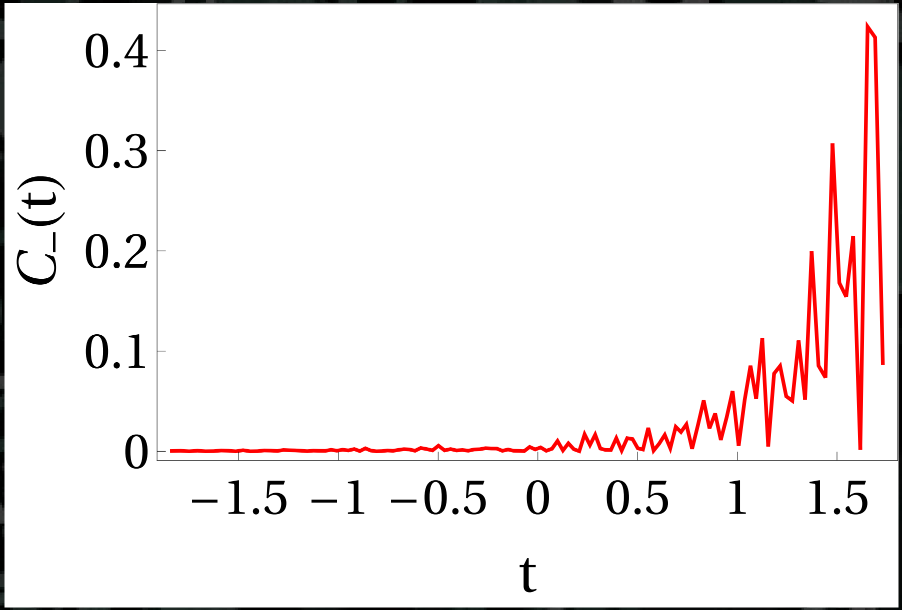
From numerics and statistical arguments $\langle |q_{I+i}|^2 \rangle, \langle |p_{I+i}|^2 \rangle \sim \exp(r t_I)$

- Tempting to claim r as the Lyapunov exponent but...
- This is completely wrong! The exponentially growing term ***comes from TOC, not OTOC!***
- Non-stationary TOC: no equilibrium solution, the geometry is non-stationary

Non-maximal chaos from Monte Carlo numerics



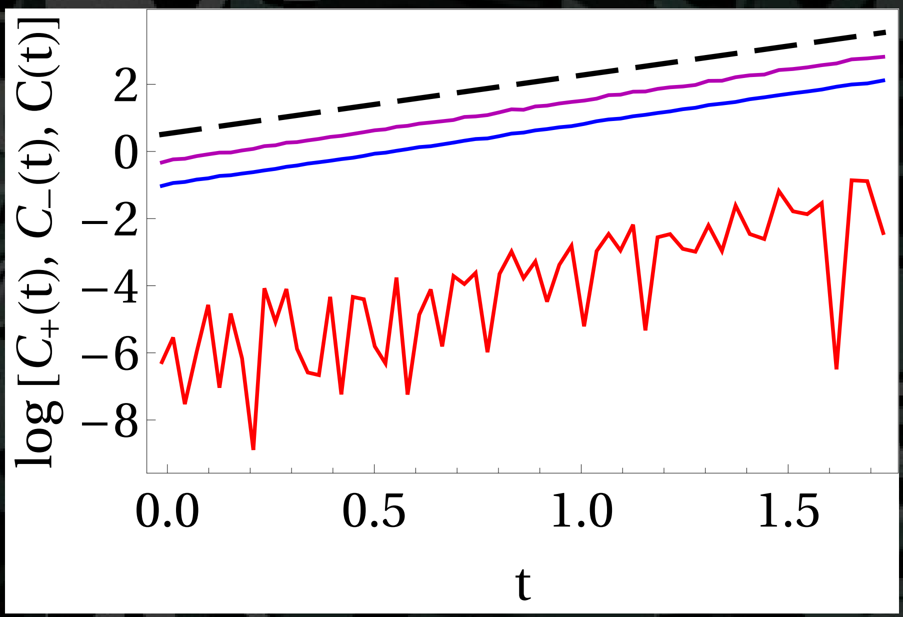
Regular exponential growth of TOC ($C_+(t)$, blue), absence of exponential growth of OTOC ($C_-(t)$, red) and their (doubled) difference ($C(t)$, violet)



Zoom-in onto the slow (sub-exponential) growth of OTOC, the signature of weak chaos

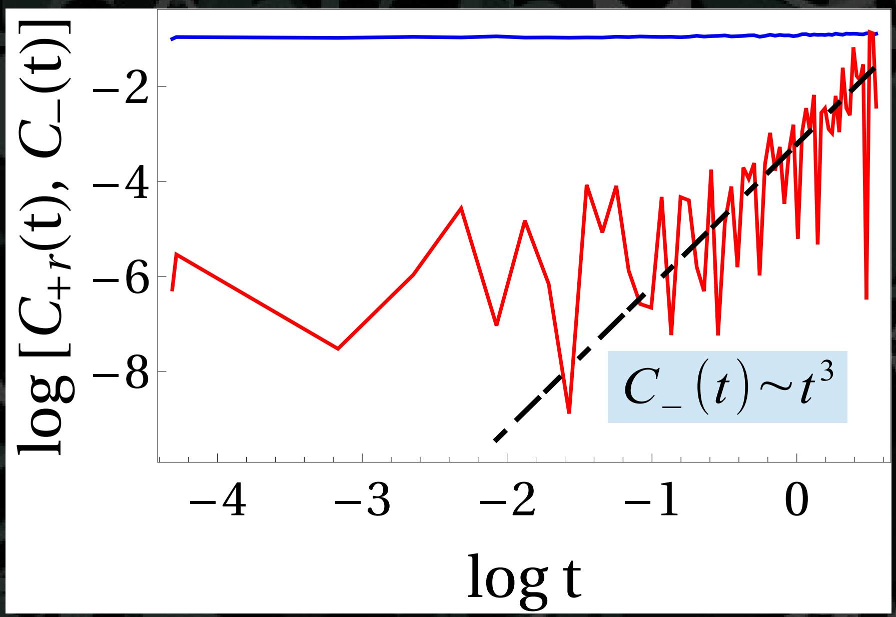
Non-maximal chaos from Monte Carlo numerics

$C_+(t), C(t) \sim \exp(r\,t)$



Log-linear plot confirming the exponential trend (black dashed line – exponential fit)

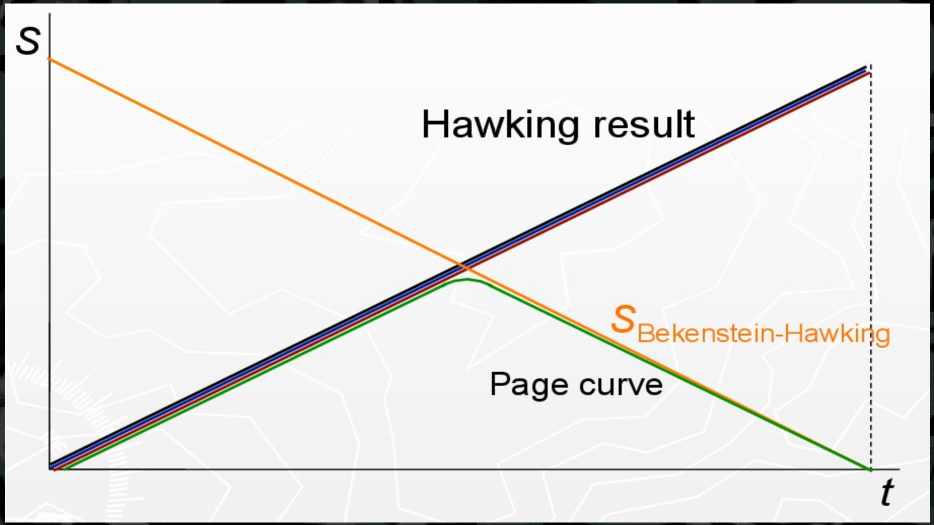
$C_{+r}(t) \equiv e^{-rt} C_{+r}(t) \sim \text{const.}$



Log-log plot of OTOC and rescaled TOC (by the exponential growth function) – power-law growth of OTOC appears

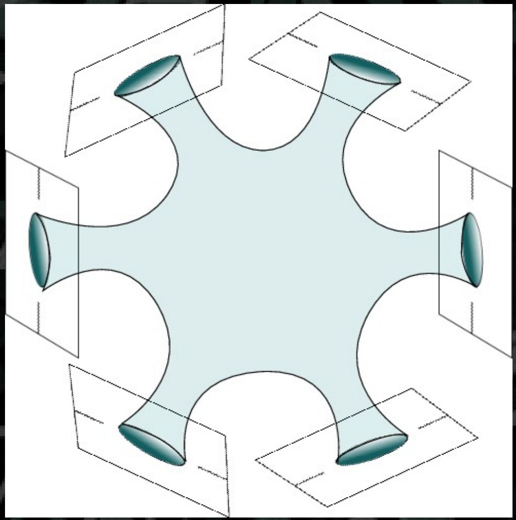
Connection to replica wormholes, factorization and all that

Proposed resolution of the black hole information problems through replica wormholes and entanglement islands: Almheiri, Hartman, Maldacena, Shaghoulian, Tajdini 1911.12333; Penington, Shenker, Stanford and Yang 1911.11977



Page curve of an evaporating black hole

⇦



Sum over saddles, including wormholes

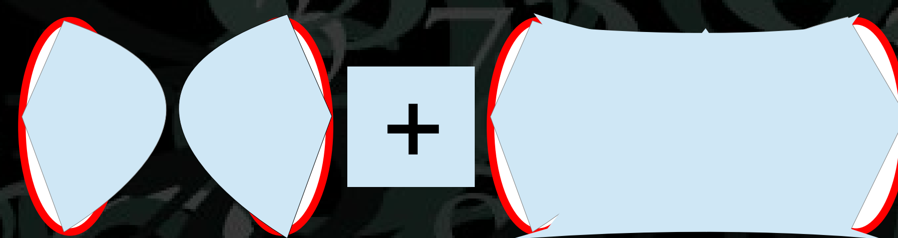
The factorization puzzle

- Remember AdS/CFT: $Z_{\text{gravity}} = Z_{\text{CFT}}$

gravity partition function = CFT partiton function

- Wormholes ruin the factorization:

$Z_1 =$ 

$Z_2 =$ 

$$Z_2 \neq Z_1^2 \quad ?!$$

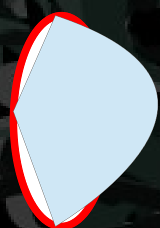
- We can live with $Z_{1g}^2 \neq Z_{2g}$ but we do expect $Z_{1\text{CFT}}^2 = Z_{2\text{CFT}}$

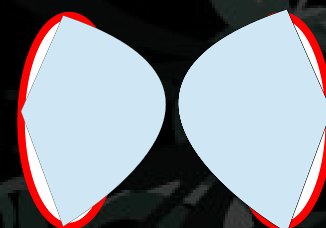
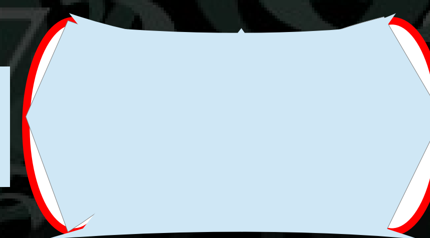
Factorization and averaging

- Remember AdS/CFT: $Z_{\text{gravity}} = Z_{\text{CFT}}$

gravity partition function = CFT partiton function

- Wormholes ruin the factorization:

$\langle Z_1 \rangle =$ 

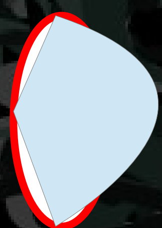
$\langle Z_2 \rangle =$  $+$ 

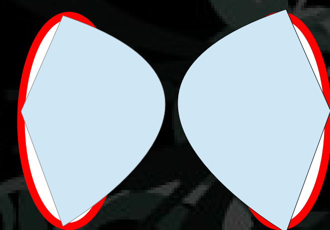
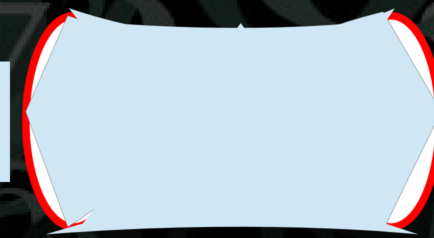
$$\langle Z_2 \rangle \neq \langle Z_1^2 \rangle$$

- We can easily have $\langle Z_{1\text{CFT}}^2 \rangle \neq \langle Z_{2\text{CFT}} \rangle$

Where does averaging come from?

- Averaging over what?
- Is the average fundamental (over quenched disorder) or emergent (coarse-graining or time binning)?

$\langle Z_1 \rangle =$ 

$\langle Z_2 \rangle =$  $+$ 

$$\langle Z_2 \rangle \neq \langle Z_1^2 \rangle$$

- Many ideas: 2008.08570, 2103.16754, 2105.02129, 2105.08270, 2107.13130, 2110.06221, 2111.07863, 2111.11705, 2202.01372, 2203.09537, 2211.09398 ...

Averaging over the fluctuations

- Divide the fields (matrices) into slow (quenched, semi-classical) and fast degrees of freedom:

$A_\mu \rightarrow A_\mu + a_\mu$ $\quad Z_E = \int D[a_\mu] \int D[A_\mu] \exp\left(-S_{\text{IKKT}}[A_\mu + a_\mu]\right)$

- Annealed partition function:

$S_{\text{IKKT}}[A_\mu + a_\mu]$ $\xrightarrow{\text{elliminate } a_\mu}$ $\int D[a_\mu] e^{-S_{\text{IKKT}}[A_\mu + a_\mu]} = e^{-W(A_\mu)}$

- Big issue: Does the replica partition function factorize?

$$\langle Z^n \rangle \ ?\approx? \ \langle Z \rangle^n + \text{small corrections}$$

- The plan: compute Z, Z^2, Z^4 vs. $\langle Z \rangle, \langle Z^2 \rangle, \langle Z^4 \rangle$

Collective field formalism

- The trick: collective fields – used for SYK and similar models (Sachdev et al 2017, Saad-Shenker-Stanford-Yao 2103.16754)

$$\langle Z\rangle = \int Da_\mu \int D\lambda_i \int Dg \exp\left[-a_\mu^\dagger P^2 a_\mu - \frac{4}{L^{2N-2}}\left(\mathrm{Tr}\, g - \mathrm{Tr}\, a_\mu^\dagger a_\mu\right)\right]\delta\left(g - a_\mu^\dagger a_\mu\right)\wp(\lambda_i)$$

$$\langle Z\rangle = \int Da_\mu \int D\lambda_i \int Dg \int Ds \exp\left[-a_\mu^\dagger P^2 a_\mu - \frac{4}{L^{2N-2}}\left(\mathrm{Tr}\, g - \mathrm{Tr}\, a_\mu^\dagger a_\mu\right) - i\, s\left(g - a_\mu^\dagger a_\mu\right)\right]\wp(\lambda_i)$$

$$\langle Z\rangle = \int Dg \int Ds \exp\left[-\frac{1}{2}\log\det s - i\, s\, g - \frac{4}{L^{2N-2}}\mathrm{Tr}\, g\right]$$

- Solution: $s = \frac{2i}{L^{2N-2}} I\,, \quad g = \frac{L^{2N-2}}{4} I$

- Effective action: $S_{\mathrm{eff}}^{(1)} \equiv -\log\langle Z\rangle = \left(N^2 - N\right)\log L + N\log\sqrt{2}$

Four replicas

- Replicas L, R, L', R'
- Two- and four-field combinations for bosons:

$g_{AB'} \equiv a_A^\dagger a_{B'}$ $G_{AAB'B'} \equiv a_A^\dagger a_A a_{B'}^\dagger a_{B'}$ $A, B \in \{L, R\}, \quad A', B' \in \{L', R'\}$

- Two-field combinations for fermions: $\gamma_{AB} \equiv \frac{1}{N} a_A^\dagger a_B, \gamma_{AB'} \equiv \frac{1}{N} a_A^\dagger a_B{}'$
- Effective action:

$$S_{\text{eff}}^{(4)} = \frac{1}{2} \log \det s_{AB} + \frac{1}{2} \log \det s_{A'B'} + \frac{1}{2} \log \det S_{AAB'B'} - i S_{AAB'B'} G_{AAB'B'} + \frac{8}{L^{2N-4}} \mathrm{Tr}\, g_{AA} \mathrm{Tr}\, g_{BB} - \frac{4}{L^{2N-4}} \mathrm{Tr}\, G_{AAB'B'}$$

- Hubbard-Stratonovich fields $S_{AAB'B'}, \quad s_{AB'}, \quad \sigma_{AB}, \quad \sigma_{AB'}$
- Wormhole couplings $\sigma_{LR}, \quad \sigma_{L'R'}$
- Half-wormhole couplings $s_{LR'}, \quad S_{LLL'L'}, \quad S_{LLR'R'}$

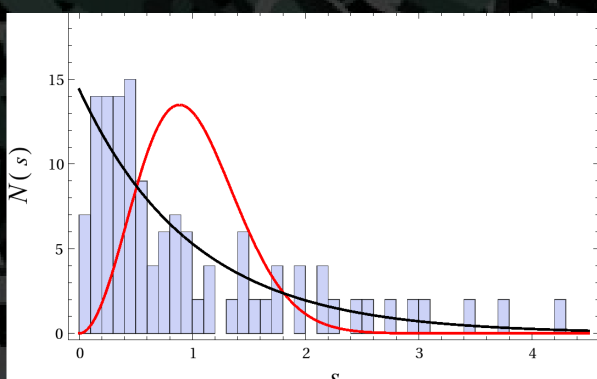
Four replicas - solutions

- Trivial solution: $\langle Z^4 \rangle \sim \langle Z \rangle^4$
- Wormhole: $\langle Z^4 \rangle \neq \langle Z \rangle^4$
- Half-wormhole: $\langle Z^4 \rangle \sim \langle Z \rangle^4$
- Wormhole + half wormhole: $\langle Z^4 \rangle \sim \langle Z \rangle^4$
- Full expressions for solutions and partition functions + fermionic contributions can be found in 2203.10697

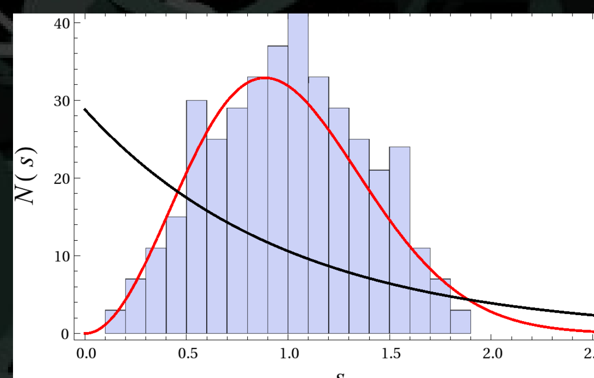
Factorizing solutions have chaotic level statistics

- WH saddle: self-averaging, regular
- HWH saddle: factorizing, chaotic
- WH+HWH saddle: self-averaging, factorizing, chaotic

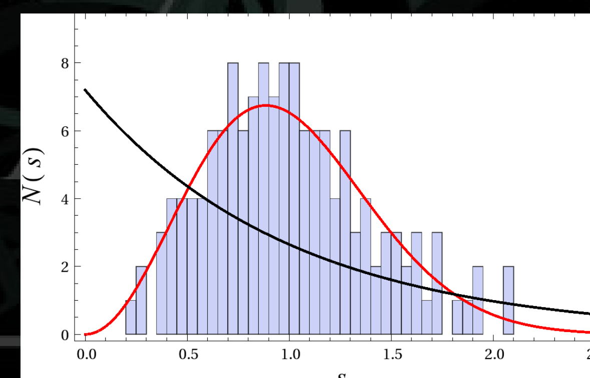
Dragan Marković



WH - regular



HWH – chaotic

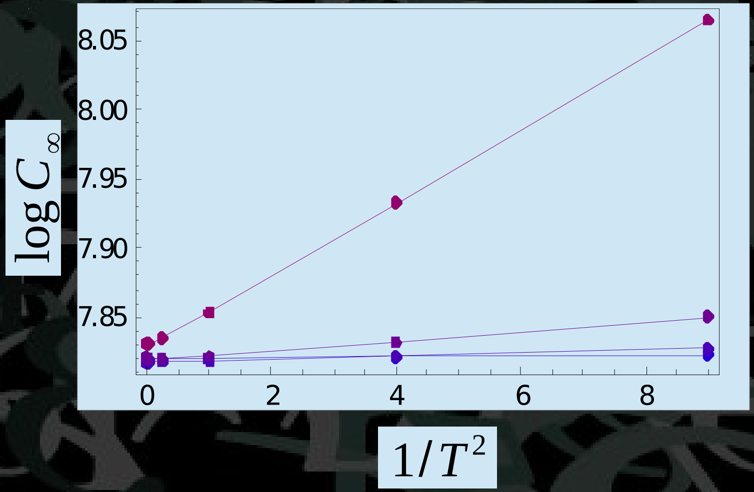


WH+HWH - chaotic

Black – Poisson, Red – Gaussian Unitary Ensemble

Strong chaos in the BMN model

- In the IIA matrix quantum mechanics likewise power-law behavior of OTOC. In 2202.09443 we even find the same in vanilla random-matrix models.
- Universality is subtler, in the OTOC plateau region: the plateau scales as a universal power-law (Bessel funs etc)



$$\log C_\infty = c_0 + c_1/T^2$$

Red: Wigner-Dyson $N(s) \sim s^2 \exp(-\pi s^2)$

Black: Poisson $N(s) \sim \exp(-s)$

- In line with the concept of weak quantum chaos formulated in Kukuljan, Grozdanov & Prosen 1701.09147

What's the memo?

- MSS universality and maximal chaos really hinges on the existence of a ***classical horizon with infinite redhsift*** (so the eikonal approximation becomes exact) which only sees the constant factorized TOC and exponential OTOC with MSS Lyapunov exponent at $t \rightarrow \infty$
- Once we are deep in the stringy regime there is no (sharp) horizon so we see the ***finite-time dynamics*** of OTOC with non-universal and weak chaos from microscopic brane fluctuations. This is something like-prethermalization regime where TOC is also non-stationary

What's the memo?

- On the other side the level statistics see chaos because it's a stationary quantity, resolving individual levels and thus characterizing long timescales longer then the Ehrenfest time
- Factorization and chaos go hand-in-hand as expected from the "effective disorder" proposal – chaotic level statistics provides a coarse-graining mechanism
- Interesting but difficult: explicit connection to black holes and the holographic dual. Easier to do in the IIA (BFSS) matrix model. Some ideas maybe in 2310.11617?