On the thermodynamics of 4d Kerr-Newman black holes

Kiril Hristov

Sofia University and INRNE, BAS

ICTP workshop on
String Theory, Holography, and Black Holes
24 October 2023, Trieste

Based on..

- BPS thermodynamics and Euclidean saddles - 2207.12437
- Left- and right-moving sectors - 2304.07320, 2310.13437

Motivation

- The laws of black holes thermodynamics formulated in the 70's, *[Bekenstein, Bardeen, Carter, Hawking, Gibbons, ...]* - in *analogy* to the laws of statistical physics
- Major string theory advances in the 90's - succesful match with microscopic entropy via BPS state counting, SCFT_2 *index* (Cardy formula) *[Strominger, Vafa, Maldacena, Witten, ...]*
- Quantum entropy via string corrections: OSV formula and conjecture, *[Ooguri, Strominger, Vafa'04, ...]*

$$I^{\text{OSV}}(\varphi^i) = -S_{\text{BH}} - \varphi^i\, Q_i\ , \qquad S_{\text{BH}} = \frac{A(\text{hor})}{4G_N} + ...$$

- Last 2 decades: advances driven by holography, asymptotically AdS black holes and (n)AdS horizons. Not much progress in the general thermal 4d asymptotically flat case

4d Kerr-Newman black holes

- 4d Einstein-Maxwell theory, ($G_N = 1$):

$$I_{\mathsf{EM}} = \frac{1}{16\pi} \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-g} \left(R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) . \tag{1}$$

- General black hole solution (no magnetic charge for brevity):

$$\begin{aligned} \mathrm{d}s^2 = & - \frac{\Delta(r)}{\rho^2} \left(\mathrm{d}t - a \sin^2\theta \, \mathrm{d}\phi \right)^2 + \frac{\rho^2}{\Delta(r)} \mathrm{d}r^2 \\ & + \rho^2 \, \mathrm{d}\theta^2 + \frac{\sin^2\theta}{\rho^2} \left(a \, \mathrm{d}t - (r^2 + a^2) \, \mathrm{d}\phi \right)^2 , \end{aligned} \tag{2}$$

$$\Delta(r) = r^2 - 2Mr + a^2 + Q^2 , \quad \rho^2 = r^2 + a^2 \cos^2\theta . \tag{3}$$

- Conserved asymptotic charges: mass M, electric charge Q, angular momentum $J = aM$.

Extremality

- Event horizons, roots of $\Delta(r)$:

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}\ . \tag{4}$$

- Hawking temperature of the two horizons:

$$T_{\pm}^{\mathsf{H}} = \frac{1}{2\pi} \frac{r_{\pm} - M}{r_{\pm}^2 + a^2}\ . \tag{5}$$

- Extremality: vanishing temperature, $T_{\pm}^{\mathsf{H}} = 0$, such that $r_+ = r_-$,

$$M_{\mathsf{extr}} = \sqrt{a^2 + Q^2}\ . \tag{6}$$

- Regular thermal black holes: real positive temperature of the outer horizon, $T_{+}^{\mathsf{H}} > 0$,

$$M > M_{\mathsf{extr}}\ . \tag{7}$$

- $M < M_{\mathsf{extr}}$: complex $r_{\pm}$, a *naked* singularity in Lorentzian signature

Supersymmetry

- Einstein-Maxwell theory $=$ bosonic part of minimal $\mathcal{N}=2$ ungauged supergravity. Poincaré superalgebra:

$$\{\mathcal{Q}^A_\alpha, \mathcal{Q}^B_\beta\} = M\,\delta_{\alpha\beta}\,\delta^{AB} + Q\,(i\gamma^0)_{\alpha\beta}\,\epsilon^{AB}\ . \tag{8}$$

- Full supersymmetry: $M=Q=0$, Minkowski$_4$

- Half supersymmetry (1/2 BPS):

$$M_{\text{BPS}} = |Q|\ . \tag{9}$$

- The Lorentz group is not on the r.h.s. of (8): $J\sim a$ completely arbitrary, and

$$M_{\text{BPS}} \leq M_{\text{extr}}\ . \tag{10}$$

- $M_{\text{BPS}} = M_{\text{extr}} = |Q|$ if and only if $J=0$: static extremal BPS black holes

Smooth Euclidean saddles and NH geometry

- The BPS limit, $M = |Q|$ and $J \neq 0$, still produces *regular* spacetimes, but only in Euclidean signature!

$$t \to i\,\tau \;, \qquad a \to i\,\alpha \;. \tag{11}$$

- Need to periodically identify τ:

$$\tau \sim \tau + \beta_+ \;, \qquad \beta_+ = \frac{1}{T_+} = 2\pi\, Q\, \frac{(M + 2\alpha)}{\alpha} \;. \tag{12}$$

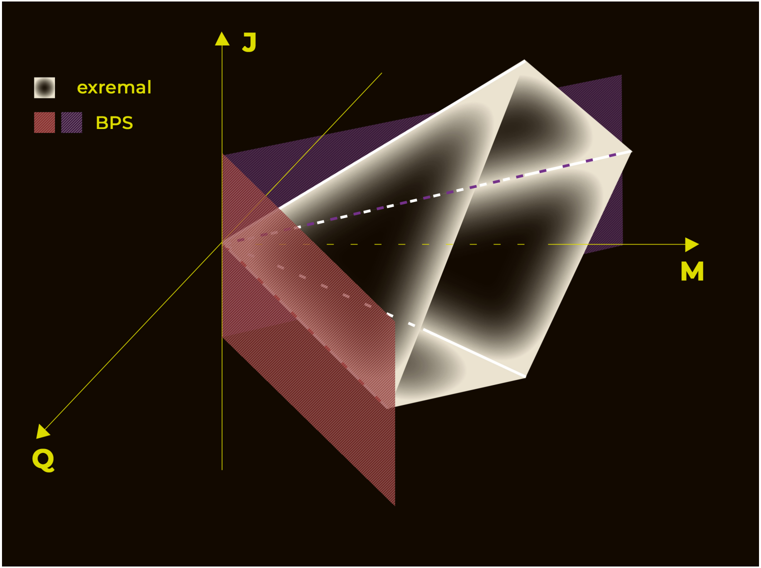
- Cigar shaped geometry: near the horizon the space smoothly caps off,

$$\mathrm{d}s^2_{\text{NH}} = \mathbb{R}^2 \times_w S^2_{\text{sq}} \;. \tag{13}$$

- For extremal BPS black holes, $\alpha \to 0$: infinite throat of (Euclidean) AdS_2 and $\beta_+ \to \infty$,

$$\mathrm{d}s^2_{\text{extr. NH}} = \mathbb{H}^2 \times S^2 \;. \tag{14}$$

The phase space



Inner and outer thermodynamics

- Formally, the laws of thermodynamics hold at each horizon, *separately*! *[Curir'79, Cvetic-Larsen'97,Wu'04, ...]*

- Corresponding chemical potentials and entropy:

$$\beta_\pm = 2\pi \frac{r_\pm^2 + a^2}{r_\pm - M} \ , \quad \Omega_\pm = \frac{a}{r_\pm^2 + a^2} \ , \quad \Phi_\pm = \frac{Q\, r_\pm}{r_\pm^2 + a^2} \ , \tag{15}$$

$$S_\pm = \pi(r_\pm^2 + a^2) \ . \tag{16}$$

- First law of thermodynamics and quantum statistical relation:

$$\beta_\pm \, \delta M = \delta S_\pm + \beta_\pm \Omega_\pm \, \delta J + \beta_\pm \Phi_\pm \, \delta Q \ , \tag{17}$$

$$I_\pm(\beta_\pm, \Omega_\pm, \Phi_\pm) = \beta_\pm M - S_\pm - \beta_\pm \Omega_\pm J - \beta_\pm \Phi_\pm Q \ . \tag{18}$$

Technical/conceptual problems

- Extremality requires $T_\pm = 0$: $\beta_\pm \to \infty$ and $I_\pm$ ill-defined?
  First go to the BPS surface, understand the BPS thermodynamics and only then go to extremality. Works for asymptotically AdS black holes, *[Cassani, Papini'19]*

- Why care about r_-, it is behind r_+?
  For regular thermal Lorentzian black holes $r_- < r_+$,
  in the BPS limit $r_\pm = M \pm i\, a$, no ordering available

- Go beyond supersymmetry: (log of) microscopic thermal partition function at leading order should agree with $I_\pm$?
  Easy to express $\beta_\pm, \Omega_\pm, \Phi_\pm$ in terms of M, Q, J via $r_\pm$, but other way around involves quartic and higher roots $\to$ no analytic expression for $I_\pm(\beta_\pm, \Omega_\pm, \Phi_\pm)$, only numeric evaluation possible

Solution: introduce new variables

- Left- and right-moving chemical potentials and entropy:

$$\begin{aligned} \beta_{l,r} &:= \frac{1}{2}\left(\beta_+ \pm \beta_-\right), & \omega_{l,r} &:= \frac{1}{2}\left(\beta_+\Omega_+ \pm \beta_-\Omega_-\right), \\ \varphi_{l,r} &:= \frac{1}{2}\left(\beta_+\Phi_+ \pm \beta_-\Phi_-\right), & S_{l,r} &:= \frac{1}{2}\left(S_+ \pm S_-\right), \end{aligned} \tag{19}$$

- Left- and right-moving on-shell action

$$I_{l,r} := \frac{1}{2}\left(I_+ \pm I_-\right), \quad \Rightarrow \quad I_\pm = I_l \pm I_r \,. \tag{20}$$

- Equivalent first law and quantum statistical relation:

$$\beta_{l,r}\,\delta M = \delta S_{l,r} + \omega^\alpha_{l,r}\,\delta J_\alpha + \varphi^i_{l,r}\,\delta Q_i \,, \tag{21}$$

$$I_{l,r}(\beta_{l,r}, \omega_{l,r}, \varphi_{l,r}) = \beta_{l,r} M - S_{l,r} - \omega^\alpha_{l,r} J_\alpha - \varphi^i_{l,r} Q_i \,. \tag{22}$$

Simple expressions

- Remarkably, $\omega_l = 0$ identically $\to$ the left-moving sector is *static*.
- Easy to invert M, Q, J as a function of the new variables.
  Left-moving sector:

$$\beta_l = 4\pi\, M \ , \quad \varphi_l = 2\pi\, Q \ , \quad \Rightarrow \quad M = \frac{\beta_l}{4\pi} \ , \quad Q = \frac{\varphi_l}{2\pi} \ , \qquad (23)$$

Right-moving sector:

$$\beta_r = 2\pi \, \frac{2M^2 - Q^2}{\sqrt{M^2 - a^2 - Q^2}} = \frac{2M^2 - Q^2}{a}\omega_r = \frac{2M^2 - Q^2}{MQ}\varphi_r \ , \quad (24)$$

$$\Rightarrow \qquad M = \sqrt{\frac{\beta_r^2 - 4\varphi_r^2 + \beta_r\sqrt{\beta_r^2 + 8\varphi_r^2}}{8\pi^2 + 2\omega_r^2}} \ ,$$

$$Q = \frac{-\beta_r + \sqrt{\beta_r^2 + 8\varphi_r^2}}{4\varphi_r}\, M \ , \quad a = \frac{3\beta_r - \sqrt{\beta_r^2 + 8\varphi_r^2}}{4(4\pi^2 + \omega_r^2)}\, \omega_r \ , \qquad (25)$$

Explicitness problem solved ✓

- We find

$$
\begin{aligned}
I_l(\beta_l,\varphi_l) =& \frac{1}{8\pi}\left(\beta_l^2 - 2\varphi_l^2\right) , \\
I_r(\beta_r,\omega_r,\varphi_r) =& \ \frac{1}{16}\ \left(3\beta_r - \sqrt{\beta_r^2+8\varphi_r^2}\right)^{3/2}\ \sqrt{\frac{\beta_r+\sqrt{\beta_r^2+8\varphi_r^2}}{4\pi^2+\omega_r^2}} \ .
\end{aligned}
\tag{26}
$$

- Conjugate variables:

$$
\frac{\partial I_{l,r}}{\partial \beta_{l,r}} = M \ , \qquad \frac{\partial I_r}{\partial \omega_r} = -J \ , \qquad \frac{\partial I_{l,r}}{\partial \varphi_{l,r}} = -Q \ , \tag{27}
$$

Good BPS limit ✓

- Take the limit $M = Q$, leading to:

$$\beta_l = 2\,\varphi_l\ , \quad \omega_r = \pm 2\pi i\ , \quad \beta_r = \varphi_r\ , \tag{28}$$

  and vanishing right-moving on-shell action

$$I_l^{\text{BPS}} = \frac{\varphi_l^2}{4\pi}\ , \qquad\qquad I_r^{\text{BPS}} = 0\ . \tag{29}$$

- I_l^{BPS} independent of $\beta_{l,r} \to$ well-defined *extremal* limit.
- Agreement with OSV formula for extremal BPS black holes:

$$I^{\text{OSV}} = \frac{\varphi_{\text{BPS}}^2}{4\pi}\ . \tag{30}$$

Wider applicability of the new variables

- Left- and right-moving variables similar to holomorphic and anti-holomorphic variables for thermal BTZ/CFT_2
- The construction of $I_{l,r}$ works for any black hole solution with 2 horizons $\rightarrow$ similar structure in matter-coupled 4d/5d supergravity
- The BPS limit sets $I_r = 0$, in agreement with susy fixed point formula (OSV and generalizations) entirely expressed by holomorphic supergravity data. (Works also for *almost* BPS limits in matter-coupled 4d/5d supergravity)
- Simplest higher derivative corrections preserve the analytic structure: 4der. F-terms only correct I_l, not I_r

(A)dS asymptotics

- How to properly generalize the construction in the presence of more than 2 horizons $\to$ (A)dS black holes
- Focus on AdS$_4$ black holes with four different horizons: $r_{(i)}$, $i = 1, ..., 4$ with corresponding on-shell actions $I_{(i)}$.
  Left- and right-moving variables exist for each pair of horizons, but calculations harder
- Evidence that all horizons play a role and need to be included to find simplifications. Purely topological sum over all horizons:

$$I_{(1)} + I_{(2)} + I_{(3)} + I_{(4)} = 2\pi\,\kappa\,L^2\,|1 - \mathfrak{g}|\;, \qquad (31)$$

  with L is the AdS$_4$ scale, $\mathfrak{g}$ the genus of the horizon topology, $\kappa = \pm 1$ for spherical and hyperbolic curvature.
  Holographic meaning?

Many open questions

- Is there a clear rule to find $I_{l,r}$ for any black hole?
  Thermal on-shell action in terms of NUTs and bolts?
- Technical simplification useful for dealing with higher derivative corrections from string theory?
- Fundamental meaning of the new variables?
- What is the quantum phase space of black holes?
- Having explicit expressions allows us to begin addressing some of these questions!

Thank you!