Mutual information of subsystems and the Page curve of Hawking radiation

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A. Saha, S. Gangopadhyay, J.P. Saha, Euro. Phys. Journal C 82 (2022) 476
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Outline:

- Demystification of the entropic paradox: Page curve of Hawking radiation
- Resolution: the *island* proposal
- Role of mutual information of subsystems in the Page curve
- Insights from gauge/gravity duality

Page curve of radiation: an entropic paradox

Hawking's observation: [Comm. Math. Phys. 43, 199–220 (1975)]

The BH radiation appears as a thermal radiation for an observer at asymptotic infinity and the von Neumann entropy of the radiation is monotonically increasing with respect to the observer's time.

Page's observation: [Phys. Rev. Lett. 71, 3743–3746 (1993)]

The von Neumann entropy of radiation of an evaporating black hole should fall after the Page time and for eternal black holes, it should reach a constant value which is the coarse-grained entropy of the black holes.

- These two observations give us a paradoxical situation.
- The definitions of fine-grained and coarse-grained entropy help us in choosing the right path.

Page curve of radiation: an entropic paradox

Considering Total system = BH + Radiation ($R \cup R^c$ = Full system) and the state of the full system (on a Cauchy slice) is a pure state. Then one should have [Almheiri et al. Rev.Mod.Phys. 93, 035002 (2021)]

Entropic bound:

$$S(R) = S(R^c) \leq S_{\text{coarse}} \; .$$

- This is due to the fact that S_{coarse} provides a measure of the total number of degrees of freedom available to the system, it sets an upper bound on how much the system can be entangled with something else.
- The above mentioned property can also be understood as an artefact of the Bekenstein bound.
- This understanding of the situation motivates us to look for a new method of computation which shall produce the correct time-evolution of radiation.
- The Page time efficiently points out the region of paradox.

Replica wormholes and island

- **island formula:** [Almheiri et al. JHEP 12, 063 (2019), JHEP 03, 149 (2020); Pennington JHEP 09, 002 (2020)]

$$S(R) = \min \ \underset{\mathrm{I}}{\mathrm{ext}} \left\{ \frac{\mathrm{Area}(\partial I)}{4G_N} + S_{\mathrm{mat}}(I \cup R) \right\} .$$

- The island appears due to the application of replica technique in dynamical gravitational background. There two different saddle points of the gravitational path integral (partition function)

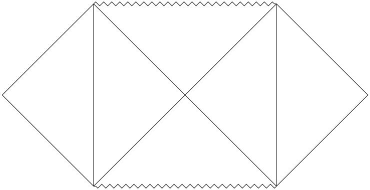
Hawking saddle $\rightarrow$ monotonically increasing $S(R)$

Replica wormhole saddle $\rightarrow$ $S(R) = S(R^c) \leq S_{\mathrm{coarse}}$.

- It has also been argued that at early times, the Hawking saddle dominates whereas in the late time, the replica wormhole saddle points dominate. [Almheiri et al. arXiv:1910.11077]

Gravitational set up: JT gravity + flat baths

- We consider the $2d$ eternal black hole solution of Jackiw-Teitelboim (JT) gravity coupled to a pair of flat baths, filled with conformal matter (free CFT) of central charge c.



Two-sided eternal BH in AdS + flat thermal baths

- The flat baths introduce transparent boundary condition for the outgoing Hawking quantas.
- Metric in Schwarzschild coordinate:

$$\boxed{ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)};\ f(r) = \frac{(r^2 - r_+^2)}{l^2}}\ .$$

Gravitational set up: Kruskal coordinate

- In Kruskal coordinates, the metric corresponding to gravitating region reads

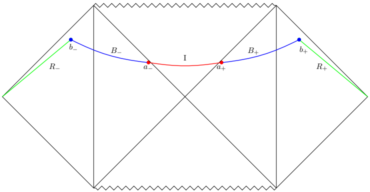
$$ds^2_{JT} = -F^2(r)dudv \ ; \ F^2(r) = -\frac{f(r)}{\kappa^2 uv} \ .$$

- We assume that the curved spacetime influences of the JT gravity vanishes at a certain hypothetical cut-off distance, namely, $r_{R(L)} = \xi$, which lies inside the AdS boundary.
- In Kruskal coordinates, the metric corresponding to non-gravitating region (flat baths) reads

$$ds^2_{Bath} = -F^2(\xi, r)dudv \ ; \ F^2(\xi, r) = -\frac{f(\xi)}{\kappa^2 uv}; \ \xi = \alpha r_+, \ \alpha \gg 1 \ .$$

- This ensures that both of the metrics (associated to bath and JT gravity) are continuously connected along $r_{R(L)} = \xi$.

After Page time scenario: in presence of island



Island region (in red) with boundaries $a_\pm = (\pm t_a, a)$

- Geometrical Part: $\boxed{\frac{\text{Area}(\partial I)}{4G_N} = 2 \times \frac{2\pi a}{4G_N}}$.
- In this set up, the matter entropy can be realized as

$$S_{\text{mat}}(I \cup R) = S_{\text{mat}}(B_+ \cup B_-) \ .$$

- The regions of $B_\pm$ can be specified as $(b_\pm \to a_\pm)$.
- We now make use of the Calabrese-Cardy formula for computing von Neumann entropy associated to two-disjoint subsystems.

After Page time scenario: in presence of island

Calabrese-Cardy formula $2d$ CFT disjoint subsystem formula:

$$S_{\rm mat}(B_+ \cup B_-) = \left(\frac{c}{3}\right) \log\left[\frac{d(a_+,a_-)d(b_+,b_-)d(a_+,b_+)d(a_-,b_-)}{d(a_+,b_-)d(a_-,b_+)}\right] .$$

- Explicit computation of the matter entropy suggests that it can be recast to the following form

$$\boxed{S_{\rm mat}(B_+ \cup B_-) = S_{\rm mat}(B_+) + S_{\rm mat}(B_-) + \mathcal{O}(e^{-\frac{2\pi t_a}{\beta}}) + \mathcal{O}(e^{-\frac{2\pi t_b}{\beta}})}.$$

late time approximation of the Island formula:

$$S(R) \approx \min\ \underset{\rm I}{\rm ext}\left\{\frac{\text{Area}(\partial I)}{4G_N} + S_{\rm mat}(B_+) + S_{\rm mat}(B_-)\right\} .$$

After Page time scenario: in presence of island

Extremization of the island parameters (t_a and a) yields $t_a - t_b = 0$, $a \approx r_+$ and upon substitution one obtains

$$\boxed{S(R) \approx 2S_{BH} + ... = S_{\text{coarse}}} \, .$$

Some crucial points:

- In the above computation, usually one just simply ignores all the contributions from the terms with time dependency $\mathcal{O}(e^{-\frac{2\pi t}{\beta}})$.
- However, if we keep those terms this will lead to a time-dependent expression of $S(R)$!
- One has to justify the approximation and also needs to incorporate the consequences it produces.

Our method:

- We note that under late time approximation the mutual information between the subsystems B_+ and B_- is approximately zero,

$$I(B_+ : B_-) = S_{\rm mat}(B_+) + S_{\rm mat}(B_-) - S_{\rm mat}(B_+ \cup B_-) \approx 0.$$

In this set up, the standard extremization of the Island parameters (t_a and a) fixes the position of the QES (end points of the Island).

Our Proposal: [A. Saha et al. EPJC 82 (2022) 476]

Just after the Page time, inclusion of the Island contribution leads to the exact saturation of the mutual correlation between B_+ and B_-,i.e. $I(B_+ : B_-) = 0$.

- We observe that $I(B_+ : B_-)$ vanishes only if the following condition is satisfied

Condition for $I(B_+ : B_-) = 0$

$$t_a - t_b = |r^*(a) - r^*(b)|; \quad r^*(r) = \left(\frac{l^2}{2r_+}\right)\log\left(\frac{r - r_+}{r + r_+}\right) .$$

Results:

Finding 't_a'

The condition of vanishing mutual information fixes t_a as

$$t_a = t_b + |r^*(a) - r^*(b)|$$

Computation of $S_{\mathbf{mat}}(I \cup R)$ [Phys.Rev.D 106 (2022) 8, 086019]

Substituting t_a in $S_{\text{mat}}(B_+ \cup B_-)$ we get

$$S_{\text{mat}}(I \cup R) = \frac{c}{3} \log \left[\left(\frac{\beta}{\pi} \right) \sqrt{(\alpha^2 - 1)(a^2 - r_+^2)} \right] .$$

- Remarkably, the obtained expression of $S_{\text{mat}}(I \cup R)$ has no time dependency.
- We now use the fact the Geometrical Part: $\frac{\text{Area}(\partial I)}{4G_N} = 2 \times \frac{2\pi a}{4G_N}$ and apply the extremization condition for 'a'.

Results:

Finding 'a'

$$\partial_a S(R) = 0 \quad \rightarrow \quad a = r_+ + \left(\frac{2cG_N l}{3}\right)^2 \frac{1}{8r_+} + \ldots \,.$$

We note that the correction term to "a" suggests that the QES has formed just outside the horizon.

Finding $S(R)$

$$S(R) = 2S_{BH} - \left(\frac{2c}{3}\right) \log\left(S_{BH}\right) + \frac{\left(\frac{c}{2}\right)^2}{2S_{BH}} + \frac{\left(\frac{c}{3}\right)^3}{32S_{BH}^2} + \ldots.$$

- Apart from the leading piece $2S_{BH}$, the expression of $S(R)$ contains universal corrections involving the Hawking entropy of the black hole.
- The saturation of the mutual information $I(B_+ : B_-)$ leads to the correct Page curve.

Results:

Simplifying the condition for $I(B_+ : B_-) = 0$

$$t_a - t_b \equiv |r^*(a) - r^*(b)| = \boxed{\left(\frac{\beta}{2\pi}\right) \log\left(S_{BH}\right)} + \left(\frac{\beta}{16\pi}\right)\left(\frac{c}{12}\right)^2 \frac{1}{S_{BH}^2} + \ldots$$

Finding out the Scrambling time in the Page curve

$$t_a - t_b = t_{scr} + \mathcal{O}\left(\frac{1}{S_{BH}^2}\right) \ .$$

- As soon as the time difference $t_a - t_b$ equals t_{scr}, the mutual information between B_+ and B_- vanishes which results in a time independent nature of $S(R)$.
- The Scrambling time is defined as the minimum time required to reterive the information after sending the information into the black hole.

Results:

Page time t_p

The fine grained entropy of radiation stops growing at

$$t_p = \left(\frac{3\beta}{\pi c}\right) S_{BH} - \left(\frac{\beta}{\pi}\right) \log\left(S_{BH}\right) + \left(\frac{3c}{8}\right) \frac{\beta}{2\pi S_{BH}} + \dots \, .$$

Observations:

- Saturation of the Mutual Information $I(B_+ : B_-)$ happens when the time difference $t_a - t_b$ equals the Scrambling time t_{scr}.
- $I(B_+ : B_-) = 0$ yields a time independent $S(R)$ and eventually leads to the correct Page curve.
- $S(R)$ contains universal corrections which are logarithmic and inverse power law in form.

Insights from gauge/gravity duality:

- In the context of holography, the mutual information is a crucial parameter which determines the phase of the entanglement wedge.
- $I(A:B) \neq 0$ means a connected (phase) entanglement wedge of $A \cup B$ and $I(A:B) = 0$ means disconnected (phase) entanglement wedge of $A \cup B$. [Takayanagi et al. Nature Phys. 14 (2018) 6]
- Disconnected EW ($I(A:B) = 0$): $\rho_{A\cup B} = \rho_A \otimes \rho_B$.

Concluding remarks for the after Page time scenario:

Just after the Page time, inclusion of the Island contribution leads to the disconnected phase of the entanglement wedge corresponding to $B_+ \cup B_-$, that is,

$$\rho_{B_+ \cup B_-} = \rho_{B_+} \otimes \rho_{B_-} \ .$$

- The island is precisely what that separates the entanglement wedge of $B_+ \cup B_-$ which in turn saturates the bound $S(R) \leq S_{\text{coarse}}$.
- This particular observation was also mentioned later in [JHEP 03 (2022) 136].


Thank You!!