On the Virasoro Fusion Kernel at $c = 25$

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based on: [2310.09334] w/ Sylvain Ribault .

Motivation

Conformal Bootstrap in 2d CFTs

- The basic **2d CFT data** at central charge c is a list of primary operators $\mathcal{O}_i$, along with their
  - scaling dimensions $\Delta_i = h_i + \overline{h}_i$, and spins $l_i = |h_i - \overline{h}_i|$
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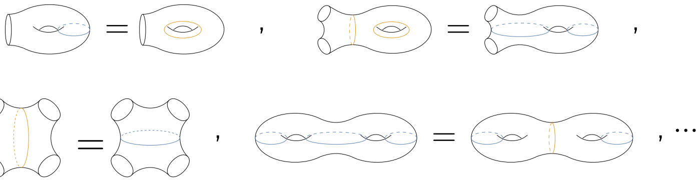

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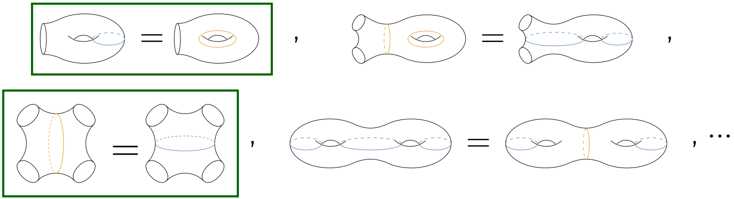

[Moore-Seiberg, '89]

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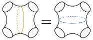
2d Conformal Bootstrap without the blocks

Let's focus on the case of the four-point functions: 

$$\sum_{h_s,\overline{h_s}} C^2_{\text{s-channel}} \left|\mathcal{V}_s(z)\right|^2 = \sum_{h_t,\overline{h_t}} C^2_{\text{t-channel}} \left|\mathcal{V}_t(1-z)\right|^2$$

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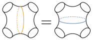
Let's focus on the case of the four-point functions: 

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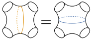
Let's focus on the case of the four-point functions: 

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One then gets

$$\boxed{\rho_{\text{t-channel}}(h, \overline{h}) = \sum\!\!\!\!\!\!\!\int_{h', \overline{h'}} \mathbf{F}_{h,h'} \mathbf{F}_{\overline{h}, \overline{h'}} \, \rho_{\text{s-channel}}(h', \overline{h'})}$$

where ρ is an appropriate *distribution*.

The Virasoro fusion kernel in 2d

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- Its form is too complicated; so far practically used in asymptotic/special limits [Collier-Maxfield-Gobeil-Perlmutter,'18; Collier-Maxfield-Maloney-**I.T.**,'19; **I.T.** '20; Numasawa-**I.T.**,'22; Chandra-Collier-Hartman-Maloney, '22, ⋯]

A *simpler* expression at $c = 25$

- In [2310.09334] we show:

$$\mathbf{F}^{(c=25)}_{p_s p_t}\left[\begin{smallmatrix} p_2 & p_3 \\ p_1 & p_4 \end{smallmatrix}\right] = \frac{4\pi^2}{i}\frac{G(\pm 2ip_t)}{G(2\pm 2ip_s)} \prod_{f\in F} \prod_{\substack{\sigma\in\mathbb{Z}_2^f| \\ \sigma_f=\eta_t(f)}} G\left(1+i\sum_{j\in f}\sigma_j p_j\right)^{-\sigma_f}$$

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where $G(x)$ is the Barnes G−function, $\widetilde{G} \equiv \frac{G(1+x)}{G(1-x)}$, and $d, \omega_\pm$ are specific (trigonometric) functions of p_i's.

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  - $\rho_{\text{t-channel}} = \not{\int}_{h',\overline{h'}}\ \mathbf{F}_{h,h'}\mathbf{F}_{\overline{h},\overline{h'}}\ \rho_{\text{s-channel}}$ implemented more easily
  - surprising connection with Painleve VI non-linear diff equation
  - $c \leftrightarrow 26-c$ duality in 2d CFTs!

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  - ⋆ surprising connection with Painleve VI non-linear diff equation
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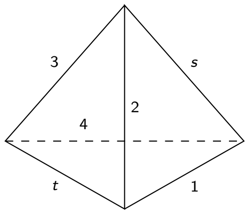
Thank you!

Extra slides

Tetrahedron notation:



Name	Notation	Value
Edges	E	$\{1, 2, 3, 4, s, t\}$
Pairs of opposite edges	P	$\{13, 24, st\}$
Faces	F	$\{12s, 34s, 23t, 14t\}$
Vertices	V	$\{14s, 12t, 34t, 23s\}$

Formulas will involve assigning signs to edges. We use the notations:

- $\sigma \in \mathbb{Z}_2^E$ is an assignment of a sign $\sigma_i \in \{+, -\}$ for any $i \in E$, and $\sigma \in \mathbb{Z}_2^f$ for a triple of signs on a face $f \in F$.
- $\sigma_E, \sigma_v, \sigma_f, \sigma_p$ for products of $6, 3, 3$ or 2 signs on all edges, a vertex, a face, or two opposite edges.
- $\sigma \in \mathbb{Z}_2^E | \sigma_V = 1$ for sign assignments whose products are 1 at each vertex. There are 8 such assignments, and they can be split in two halves according to $\sigma_E = \pm 1$.
- The indicator function $\eta_i \in \mathbb{Z}_2^F$ is $\eta_i(f) = 1$ if the edge i belongs to the face f, and $\eta_i(f) = -1$ otherwise.

Here is the set $\sigma \in \mathbb{Z}_2^E | \sigma_V = 1$:

s	t	1	2	3	4	σ_E
−	+	−	−	+	+	−
−	+	+	+	−	−	−
+	−	−	+	+	−	−
+	−	+	−	−	+	−
+	+	−	−	−	−	+
+	+	+	+	+	+	+
−	−	−	+	−	+	+
−	−	+	−	+	−	+

The functions $d, \omega_\pm$:

$$d = \det\left[\begin{pmatrix} 2 & -2\cosh(2\pi p_2) & -2\cosh(2\pi p_3) & 2\cosh(2\pi p_s) \\ -2\cosh(2\pi p_2) & 2 & 2\cosh(2\pi p_t) & -2\cosh(2\pi p_1) \\ -2\cosh(2\pi p_3) & 2\cosh(2\pi p_t) & 2 & -2\cosh(2\pi p_4) \\ 2\cosh(2\pi p_s) & -2\cosh(2\pi p_1) & -2\cosh(2\pi p_4) & 2 \end{pmatrix}\right],$$

$$e^{2\pi i \omega_\pm} = \frac{\sum_{ij\in P} 4\sinh(2\pi p_i)\sinh(2\pi p_j) \mp \sqrt{d}}{2\sum_{\substack{\sigma\in\mathbb{Z}_2^E| \\ \sigma_V=1}} \sigma_E e^{-\pi\sum_{k\in E}\sigma_k p_k}}.$$