ICTP workshop

On commutative algebra and algebraic geometry in prime characteristics

Monday, 8 May, 2023 : 9 am-12.30 pm

Chair: Bernd Ulrich

9.00-9.50 Steven Dale Cutkosky, University of Missouri

Title: Analytic spread of filtrations

Abstract: The definition of the analytic spread $\ell(I)$ of an ideal I in a local ring R extends to define the analytic spread $\ell(\mathcal{I})$ of a filtration \mathcal{I} in R. We will show that many of the classical theorems about analytic spread for ideals continue to be true for filtrations or at least for divisorial filtrations. The filtration of integral closures of powers of an ideal in a local ring and the filtration of symbolic powers of an ideal in a regular local ring are examples of divisorial filtrations. We will also discuss some examples of filtrations which do not have such good behavior. Much of this work is with Parangama Sarkar.

10.00-10.30 Alessandro De Stefani, Università di Genova

Title: Blowup algebras of determinantal ideals in prime characteristic

Abstract: In this talk we will present the notion of F-pure filtration, a concept which allows to obtain F-purity and strong F-regularity of certain blowup algebras in prime characteristic. Our main focus will be on filtrations given by symbolic and ordinary powers of different types of determinantal ideals. Among other facts, we will show that the symbolic Rees algebra and symbolic associated graded ring of an ideal of minors of a generic matrix are strongly F-regular. This talk will be based on joint work with Jonathan Montaño and Luis Núñez-Betancourt. If time permits, we will also present some more recent joint work with the previous two collaborators, Lisa Seccia and Matteo Varbaro.

11.00-11.50 Keiichi Watanabe, Nihon University and Meiji University

Title: Gorenstein Property of normalized tangent cones of integrally closed ideals of 2dimensional normal local rings

Abstract: Let I be an integrally closed m-primary ideal of a normal 2-dimensional local ring. Let $\bar{G}(I)$ be the normalized tangent cone of I. Namely, $\bar{G}(I) = \bigoplus_{n \ge 0} \overline{I^n}/\overline{i^{n+1}}$, where $\overline{I^n}$ is the integral closure of I^n . It turns our that the Gorenstein property of $\bar{G}(I)$ has very mysterious very geometric characterization. For example, if $I = H^0(X, O_X(-Z))$ is an elliptic ideal $(\overline{I^{n+1}} = Q\overline{I^n} \text{ for all } n \ge 2 \text{ and } \overline{I^2} \neq QI$ for any minimal reduction of I), $\bar{G}(I)$ is Gorenstein if and only if $Z^2 + K_X Z = 0$ and we can show that such $I = I_Z$ exists at most finite number in an elliptic singularity and bounded by the geometric genus of the local ring.

12.00-12.30 Ayesha Qureshi, Sabanci University

Title: Algebraic and homological properties of Polyominoes

Abstract: Polyominoes are, roughly speaking, plane figures obtained by joining squares of equal size (cells) edge to edge. Let K be a field and X be a $m \times n$ matrix of indeterminates.

Given a polyomino \mathcal{P} , we associate a quadratic binomial ideal known as the polyomino ideal $I_{\mathcal{P}}$. The polyomino ideals are generated by quite general sets of 2-minors of X, and they include some other well-known classes of binomial ideals such as the ideals generated by 2-minors of ladders and the join-meet ideals of distributive lattices. Let S be the polynomial ring over K in the variables x_a with $a \in V(\mathcal{P})$, where $V(\mathcal{P})$ is the vertex set of \mathcal{P} . We denote by $K[\mathcal{P}]$ the quotient ring $S/I_{\mathcal{P}}$. We will investigate the algebraic and homological properties of $K[\mathcal{P}]$ for given different shapes of \mathcal{P} . We will also present a conjecture on the reduced Hilbert series of $K[\mathcal{P}]$ in terms of particular arrangements of non-attacking rooks that can be placed on the polyomino.

Monday, 8 May, 2023 : 2.00 pm-4.45 pm Chair : Keiichi Watanabe

2.00-2.30 V. Trivedi, Tata Institute of Fundamental Research, Mumbai, India

Title: Hilbert-Kunz density function and its applications to some Hilbert-Kunz multiplicity conjectures

Abstract: In this talk we introduce a compactly supported and real valued continuous function called Hilbert-Kunz (HK) density function. We briefly describe its properties and its applications to study characteristic p-invariants like HK multiplicity and F-thresholds. Further we discuss in more detail some long standing conjectures of Watanabe-Yoshida and Yoshida on the HK multiplicities of quadric hypersurfaces. Here, using the classification of ACM bundles on the smooth quadric via matrix factorizations, we describe the HK density functions of the quadrics. The structure of the function explains the difficulties in nailing down the computations of HK multiplicities of even such simple class of rings.

2.40-3.10 Clare D'Cruz, Chennai Mathematical Institute, India

Title: Symbolic Rees algebras and set-theoretic complete intersections

Abstract: In this paper we extend a result of Cowsik on set-theoretic complete intersection and a result Huneke, Morales and Goto and Nishida about Noetherian symbolic Rees algebras of ideals. (This is joint work with Mousumi Mandal and J. K. Verma)

3.45-4.45 Craig Huneke, University of Virginia, Charlottesville, USA

Title: 50 years of mathematics: the work of Melvin Hochster

Abstract: This talk will highlight some of the many contributions of Melvin Hochster, not including tight closure. About half of the talk will be on his work from 1969-1985, and the last half will be discussing his solution with Ananyan of Stillman's Conjecture, which appeared in 2019.

Tuesday, 9 May, 2023 : 9.00 am -12.30 pm Chair: L. T. Hoa 9.00-9.50 Aldo Conca, Università di Genova, Italy

Title: Ideals associated to subspace arrangements

textbfAbstract: Let $L = L_1, \ldots, L_n$ be a collection of linear subspaces, a subspace arrangement, in the *d*-dimensional projective space. Each linear space L_i is the zero locus of a homogeneous linear system, i.e. the variety associated to an ideal I_i generated by linear polynomials. To L we may associate two ideals: the intersection I and the product J of the ideals I_i . They both define the union of the L_i as an algebraic variety. In the talk I will report of some recent results about the resolution and regularity of these ideals. Joint work with Manolis Tsakiris (Chinese Academy of Sciences).

10.00-10.30 Vivek Mukundan, Indian Institute of Technology Delhi, India

Title: Reduced type in one dimensional analytic algebras

Abstract: Let R be a one dimensional analytic k algebra which is a domain and k is algebraically closed. In a recent work with C. Huneke and S. Maitra, we introduced the notion of reduced type s(R) and also used it to solve certain cases of the Berger's conjecture. Along with S. Maitra, we are interested in studying this invariant, s(R) in more detail and compare it to the known invariant type(R). We explain the method used to reduce it to study of reduced type of the the corresponding valuation semigroup. The enables us to use the tools provided by numerical semigroup theory to study extremal cases of s(R). We also study the bounds for s(R) in the case of far flung Gorenstein ideals, introduced recently by Herzog, Kumashiro and Stamate.

11.00-11.50 V. Srinivas, Tata Institute of Fundamental Research, Mumbai, India

Title: Finite presentation for the tame fundamental group in positive characteristic

Abstract: This is a report on joint work with H. Esnault and M. Schusterman. Recall that the étale fundamental group of a variety over an algebraically closed field of characteristic 0 is known to be a finitely presented profinite group; this is proved by first reducing to varieties over the complex numbers, and then comparing with the topological fundamental group. In positive characteristics, even if we restrict to smooth varieties, finite generation fails in general forétale fundamental groups of non-proper varieties (eg, for the affine line). For a smooth variety with a smooth, projective compactification with a SNC boundary divisor, we show that the tame fundamental group is a finitely presented profinite group. In particular, this holds for the etale fundamental groups of smooth projective varieties.

12.00-12.30 Tony Puthenpurakal, IIT Bombay, Mumbai, India

Title: On a generalization of two results of Happel to Commutative rings

Abstract: In this paper we extend two results of Happel to commutative rings. Let (A, \mathfrak{m}) be a commutative Noetherian local ring. Let $D_f^b(mod A)$ be the bounded derived category of complexes of finitely generated modules over A with finite length cohomology. We show

 $D_f^b(mod \ A)$ has Auslander-Reiten(AR)-triangles if and only if A is regular. Let $K_f^b(proj \ A)$ be the homotopy category of finite complexes of finitely generated free A-modules with finite length cohomology. We show that if A is complete and if A is Gorenstein then $K_f^b(proj \ A)$ has AR triangles. Conversely we show that if $K_f^b(proj \ A)$ has AR triangles and if A is Cohen-Macaulay or if dim A = 1 then A is Gorenstein.

Tuesday, 9 May, 2023 : 2 pm-4.45 pm

Chair: V. Srinivas

2.00-2.30 Matteo Varbaro, Università degli Studi di Genova, Genova, Italy

Title: Singularities and Hilbert functions

Abstract: We will discuss how singularities affect the Hilbert function of a standard graded algebra R over a field. As it is well known, if R is Cohen-Macaulay then its h-vector $h = (h_0, h_1, \ldots, h_s)$ has nonnegative entries. If R has high depth can we say something on the nonnegativity of the entries of h? Without any extra assumptions, it is not difficult to produce examples with $h_2 < 0$ and depth(R) = dim(R) - 1 (and even R Buchsbaum). In contrast, if R is an isolated singularity and has depth r, then $h_i \ge 0$ for all $i \le r$ whenever the base field has characteristic 0. In general, it is more appropriate to consider the Serre's condition (S_r) rather than the depth: we will see which conditions on R, both in characteristic 0 and in positive characteristic, force the nonnegativity of the first r entries of the h-vector assuming the (S_r) condition, and raise some related questions. The material of this talk is based on a joint work with Hailong Dao and Linquan Ma.

2.40-3.10 Neil Epstein, George Mason University, USA

Title: Geometry of the tight closure operation

Abstract: Conventional wisdom says that tight closure is an inherently ungeometric operation, due to its failure to commute with localization at arbitrary multiplicative sets, and that any geometric information comes from other objects associated with tight closure, such as test ideals. Despite this, I can show how one can make the tight closure of any sheaf of ideals or any sheaf of finitely generated modules, over a Noetherian scheme over \mathbb{F}_p . We will see what properties a closure (or preclosure) operation must have in order to make such a construction work. I introduce a new kind of singularity (semi-F-regularity) and show the relevance of the as yet open question: Does tight closure commute with localization at single elements?

3.45-4.45 **B. Ulrich**, Purdue University, USA

Title: The mathematical contributions of Craig Huneke: a sampler

Abstract: The talk will highlight some of Craig Huneke's work on asymptotic properties of ideals, linkage, Rees rings, local cohomology, and symbolic powers. Given the tremendous breadth and depth of Craig's work, this can only be a small and somewhat biased sampler.

Wednesday, 10 May, 2023 : 9 am - 12.30pm Chair : Claudia Polini

9.00-9.50

Marc Chardin, CNRS & Sorbonne Université, Paris, France

Title: Residual intersections

Abstract: This lecture will present residual intersections, starting from first works of Huneke and Ulrich, after Artin and Nagata article that introduced this theme in a commutative algebra setting. Several advances in the last decades provided answers to a couple of open questions and shed some light on these objects. I will present some of these progresses and try to give a view on today's knowledge in this field.

10.00-10.30 Pham Hung Quy, FPT University, Hanoi, Vietnam

Title: Normal and Tight Hilbert Polynomials

Abstract: We compute the normal/tight Hilbert polynomials and study the first normal/tight Hilbert coefficients. An explicit formula is given for tight Buchsbaum rings. This is based on joint works with Linquan Ma and with Saipriya Dubey and Jugal Verma.

11.00-11.50 Mitsuyasu Hashimoto, Osaka Metropolitan University, Japan

Title: Frobenius maps on the rings of invariants

Abstract: Let k be a perfect field of characteristic p > 0, G an affine algebraic k-group scheme, V a finite-dimensional G-module, and S = Sym V be the symmetric algebra. This talk is on the ring-theoretic properties and numerical invariants of the ring of invariants $A = (\text{Sym } V)^G$ defined using the iteration of the Frobenius map F_A . We review some results on F-regularity and F-rationality. We generalize the result of Peter Symonds and the speaker on the description of the Frobenius limit $\text{FL}([A]) = \lim_{e \to \infty} [{}^eA]/p^{de}$ (where $d = \dim A$, and the limit is taken in certain normed space with the classes [M] of indecomposable modules Mas the \mathbb{R} -basis) for a small action of a finite (constant) group G to the (not necessarily étale) general finite group schemes (joint with Fumiya Kobayashi), where we say that the action of G on V is small if there exists some Zariski closed subset Z of V such that $\operatorname{codim}(Z, V) \geq 2$ and the stabilizer G_v is trivial for $v \in V \setminus Z$. The Frobenius limit possesses the information of the generalized F-signatures and Hilbert-Kunz multiplicity. We also give an example of a finite constant group G and its representation V such that the action is small but A does not have finite F-representation type (joint with Anurag Singh).

12.00-12.30 K. Sato, Kyushu University, Japan

Title: General hyperplane section of log canonical threefolds in positive characteristic **Abstract**: In this talk, we prove that if a 3-dimensional quasi-projective variety X over an algebraically closed field of characteristic p > 3 has only log canonical singularities, then so does a general hyperplane section H of X. In the course of the proof, we give a sufficient condition for log canonical surface singularities over a field to be geometrically log canonical.

Wednesday, 10 May, 2023 : 2.00 pm - 4.45 pm Chair: Dale Cutkosky

2.00-2.30 Luis Nunez-Betancourt ,CIMAT, Guanajuato, Mexico

Title: Nash blowups in prime characteristic

Abstract: The Nash blowup is a natural modification of algebraic varieties that replaces singular points by limits of certain vector spaces associated to the variety at non-singular points. For several decades, it has been studied whether it is possible to resolve singularities of algebraic varieties by iterating Nash blowups. This problem has mostly been treated in characteristic zero due to an example given by Nobile. In this talk, we will discuss a new approach in prime characteristic using differential operators, and an application to resolution of singularities of toric varieties.

2.40-3.10 A. V. Jayanthan , Indian Institute of Technology Madras, Chennai, India

Title: On the resurgence and asymptotic resurgence of homogeneous ideals

Abstract: Let K be a field and $R = K[x_1, \ldots, x_n]$. We obtain an improved upper bound for asymptotic resurgence of squarefree monomial ideals in R. We study the effect on the resurgence when sum, product and intersection of ideals are taken. We obtain sharp upper and lower bounds for the resurgence and asymptotic resurgence of cover ideals of finite simple graphs in terms of associated combinatorial invariants. We also explicitly compute the resurgence and asymptotic resurgence of cover ideals of several classes of graphs. We characterize a graph being bipartite in terms of the resurgence and asymptotic resurgence of edge and cover ideals. We also compute explicitly the resurgence and asymptotic resurgence of edge ideals of some classes of graphs.

3.45-4.45 **David Eisenbud**, University of California, Berkeley, USA

Title: Infinite Free Resolutions

Abstract: Free Resolutions, from Hilbert to now Free resolutions extend the idea of specifying an abelian group by generators and relations. They were used by David Hilbert to help count the invariants of a group action, and in the last 60 years their properties have been intensively explored in contexts including group cohomology, combinatorics and projective geometry.

I'll survey some of the motivation, history and successes of this exploration, focussing on some types of results that have been illuminating in the context of finite resolutions but not yet for infinite resolutions, including some modern conjectures from my current work with Hai Long Dao. 9.00-9.50 Claudia Polini, University of Notre Dame, USA

Title: The 3 Differents

Abstract: In joint work with B. Ulrich we compute the Kaehler different and the Dedekind different for several classes of ideals. Our techniques include residual] intersections and linkage theory. In particular we obtain interesting formulas for determinantal ideals.

10.00-10.30 Giulio Caviglia , Purdue University, West Lafayette, USA

Title: Bounds on the number of generators of prime ideals

Abstract: Let S be a polynomial ring over any field k, and let $P \subseteq S$ be a non-degenerate homogeneous prime ideal of height h. When k is algebraically closed, a classical result attributed to Castelnuovo establishes an upper bound on the number of linearly independent quadrics contained in P which only depends on h. We significantly extend this result by proving that the number of minimal generators of P in any degree j can be bounded above by an explicit function that only depends on j and h. In addition to providing a bound for generators in any degree j, not just for quadrics, our techniques allow us to drop the assumption that k is algebraically closed. By means of standard techniques, we also obtain analogous upper bounds on higher graded Betti numbers of any radical ideal. This is a joint work with Alessandro De Stefani.

11.00-11.50 Hai long Dao ,University of Kansas, Lawrence, USA

Title: Categorical approaches to local singularities

Abstract: Let R be a local ring. It has long been observed that one can measure how singular R is by looking at simple properties of certain categories of R-modules. Such results are typically very easy to understand, but non-trivial to prove. For instance, R is regular if and only if all dim R-syzygies are free. As another famous example, ADE singularities can be characterized by being normal, Gorenstein and has finite Cohen-Macaulay type (there exist only finitely many indecomposable maximal Cohen-Macaulay modules up to isomorphism). In this talk, I will survey the history of this fascinating subject and some recent new results, including novel characterizations of Arf rings and the class of Ulrich-split singularities. To obtain these results, sometimes we need to develop new techniques which might be of interest in their own rights. One such technique is using sub-functors of Ext arising from exact categories. Most of the new work are joint with Souvik Dey and Monalisa Dutta.

12.00-12.30 D. T. Cuong , Institute of Mathematics, Hanoi, Vietnam

Title: Remarks on Macaulayfications of Noetherian schemes

Abstract: Macaulayfications are weak variants of resolutions of singularities in which Cohen-Macaulay property replaces regularity. While resolutions of singularities have not been constructed for almost Noetherian schemes over positive characteristics, Macaulayfications exist for a large class of Noetherian schemes over any characteristic due to the works of Faltings, Brodmann, Kawasaki, Cesnavicius and others. In this talk, I first review briefly a history of this problem, then I present a necessary condition for existence of Macaulayfication. As an application, a relation between the existence of Macaulayfications and strong Macaulayfications is presented.

Thursday, 11 May, 2.00 pm-4.45 pm Chair: Marc Chardin

2.00-2.30 A. Caminata ,Università di Genova, Italy

Title: F-signature of some hypersurfaces

Abstract: We extend the theory of p-fractals introduced by Monsky and Teixeira to the setting of F-signature. We use this to compute the F-signature of some hypersurfaces of interest such as Fermat hypersurfaces. In particular, for the Fermat cubic in four variables we prove that the F-signature is strictly less than 1/8. This provides a negative answer to a question by Watanabe and Yoshida. This is a report on an ongoing project with F. Zerman.

2.40-3.10 Arindam Banerjee, Indian Institute of Technology Kharagpur, India

Title: Hilbert-Samuel coefficients of the Frobenius power of an ideal

Abstract: We provide suitable conditions under which the asymptotic limit of the Hilbert-Samuel coefficients of the Frobenius powers of an *m*-primary ideal exists in a Noetherian local ring (R, m) with prime characteristic p > 0. This, in turn, gives an expression of the Hilbert-Kunz multiplicity of powers of the ideal. We also prove that for a face ring R of a simplicial complex and an ideal J generated by pure powers of the variables, the generalized Hilbert-Kunz function $l(R/((J^{[q]})^k))$ is a polynomial for all q, k and also give an expression of the generalized Hilbert-Kunz multiplicity of powers of J in terms of Hilbert-Samuel multiplicity of J. We conclude by giving a counter-example to a conjecture proposed by I. Smirnov which connects the stability of an ideal with the asymptotic limit of the first Hilbert coefficient of the Frobenius power of the ideal.

3.45-4.45 Irena Swanson, Purdue University, West Lafayette, USA

Title: Tight closure

Abstract: This talk is an overview of the work of Mel Hochster and Craig Huneke on tight closure and its far-reaching consequences.

Friday, 12 May, 9.00 am-12.30 pm

Chair: Hai long Dao

9.00-9.50 L.T. Hoa ,Institute of Mathematics Hanoi, Vietnam Academy of Sciences and Technology

Title: Castelnuovo-Mumford regularity of powers of an ideal

Abstract: This is a joint work in progress with Nguyen Dang Hop and Ngo Viet Trung. D. Cutkosky - J. Herzog - N. V. Trung; V. Kodiyalam and N. V. Trung - Wang proved that there are non-negative integers p(I), e(I) such that

$$\operatorname{reg}(I^n) = p(I)n + e(I) \quad \forall n \gg 0,$$

where I is a graded ideal in a standard graded ring. In this talk, we study the problem which sequences of non-negative integers arise as the functions $\operatorname{reg}(I^{n+1}/I^n)$, $\operatorname{reg}(R/I^n)$, $\operatorname{reg}(I^n)$ for an ideal I generated by forms of degree d in a standard graded algebra R. These functions are asymptotically linear with slope p(I) = d. If $\dim(R/I) = 0$, we give a complete description of the functions $\operatorname{reg}(I^{n+1}/I^n)$, $\operatorname{reg}(R/I^n)$, and show that $\operatorname{reg}(I^n)$ can be any numerical function $f(n) \geq dn$ that weakly decreases first and then becomes linear with slope d. The latter result gives a negative answer to a question of Eisenbud and Ulrich. If $\dim(R/I) \geq 1$, we show that $\operatorname{reg}(I^{n+1}/I^n)$ can be any numerical asymptotically linear function $f(n) \geq dn - 1$ with slope d and $\operatorname{reg}(I^n)$ can be any numerical asymptotically linear function $f(n) \geq dn - 1$ with slope d that is weakly increasing. We also prove that the function of the saturation degree of I^n is asymptotically linear for any graded ideal I.

When R is a polynomial ring, the problem is more difficult. In this case we assume that I can be generated in different degrees. If dim R/I = 0, we give some restrictions on the *defect* sequence $e_n = \operatorname{reg}(I^n) - p(I)n$ and show that any non-increasing non-negative numerical function can be realized as a defect sequence of a monomial ideal I. In the case of higher dimensions, we show that there is a monomial ideal I whose defect sequence (considered as a numerical function) has exactly a given number of local maxima.

10.00-10.30 Manoj Kummni ,Chennai Mathematical Institute, Chennai, India

Title: F-rationality of Rees algebras

Abstract: In this talk, I will describe some results related to F-rationality of Rees algebras. Much of it is motivated by results and questions of Hara-Watanabe-Yoshida (J. Algebra, 2002) and analogous results for Hyry for rational singularities in characteristic zero (Manuscripta Math. 1999). This is based on joint work with M. Koley and with N. Kotal.

11.00-11.50 Ian Aberbach, University of Missouri, Columbia, Missouri, USA

Title: On the equality of test ideals

Abstract: This is joint work with Craig Huneke and Thomas Polstra. Let (R, m) be an excellent F-finite local ring of positive prime characteristic. It is an open problem whether or not weak F-regularity is equivalent to strong F-regularity. We show that these two notions are equivalent in the case that the anti-canonical cover of R is Noetherian on the punctured spectrum.