On a generalization of two results of Happel to commutative rings

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Happel's results

Let Γ be a finite dimensional algebra over a field k.
Happel proved that its bounded derived category $D^b(\text{mod}(\Gamma))$ has Auslander-Reiten (AR) triangles if and only if Γ has finite global dimension.
He also proved that $K^b(\text{proj}\,\Gamma)$ the homotopy category of bounded complexes of finitely generated projective Γ-modules has right AR triangles if and only if Γ is a Gorenstein algebra.

Auslander's philosophy

Auslander philosophy is that concepts in the study of representation theory of Artin algebras have natural analogues in study of maximal Cohen-Macaulay modules over Cohen-Macaulay local rings (which are free on the punctured spectrum). We study natural analogues of Happel's results in the context of commutative Noetherian rings.

Our Generalization

Let $(A, \mathfrak{m})$ be a commutative Noetherian local ring of dimension d. Let $D^b_f(\text{mod}(A))$ be the bounded derived category of complexes of finitely generated modules over A with finite length cohomology. Then it can be shown that $D^b_f(\text{mod}(A))$ is a Hom-finite Krull-Schmidt triangulated category. Our generalization of first of Happel's results is the following:

Theorem 1

Let $(A, \mathfrak{m})$ be a commutative Noetherian local ring. The following conditions are equivalent:

1. *A is regular.*
2. *$D^b_f(\text{mod}(A))$ has AR-triangles.*

Our proof is not similar to Happel's.

Let $K_f^b(\operatorname{proj} A)$ be the subcategory of $K^b(\operatorname{proj} A)$ with finite length cohomology. By Happel's result if $(A, \mathfrak{m})$ is a zero-dimensional commutative Gorenstein ring then $K^b(\operatorname{proj} A)$ has AR-triangles. For higher dimensional Gorenstein rings we prove the following extension of one direction of Happel's result.

Theorem 2

Let $(A, \mathfrak{m})$ be a complete Gorenstein local ring. Then $K_f^b(\operatorname{proj} A)$ has AR-triangles.

We believe the converse to Theorem 2 is true. More precisely

Conjecture 3

Let $(A, \mathfrak{m})$ be a complete Noetherian local ring. If $K^b_f(\operatorname{proj} A)$ has AR-triangles then A is Gorenstein.

We prove Conjecture 3 under the following cases:

Theorem 4

Let $(A, \mathfrak{m})$ be a complete Noetherian local ring. Assume $K^b_f(\operatorname{proj} A)$ has AR-triangles. Then

1. *if A is Cohen-Macaulay then A is Gorenstein.*
2. *if $\dim A \leq 1$ then A is Gorenstein.*

AR-triangles

Let $\mathcal{C}$ be a Krull-Schmidt triangulated category with shift functor $\sum$.

A triangle $N \xrightarrow{f} E \xrightarrow{g} M \xrightarrow{h} \sum N$ in $\mathcal{C}$ is called a *right AR-triangle* (ending at M) if

(RAR1) M, N are indecomposable.

(RAR2) $h \neq 0$.

(RAR3) If D is indecomposable then for every non-isomorphism $t\colon D \to M$ we have $h \circ t = 0$.

Dually, a triangle $\sum^{-1} M \xrightarrow{w} N \xrightarrow{f} E \xrightarrow{g} M$ in $\mathcal{C}$ is called a *left AR-triangle* (starting at N) if

(LAR1) M, N are indecomposable.

(LAR2) $w \neq 0$.

(LAR3) If D is indecomposable then for every non-isomorphism $t\colon N \to D$ we have $t \circ w = 0$.

We say $\mathcal{C}$ has AR-triangles if for any indecomposable $M \in \mathcal{C}$ there exists a right AR-triangle ending at M and a left AR-triangle starting at M.

Serre-functors

Let $(A, \mathfrak{m})$ be a Noetherian local ring and let E be the injective hull of $k = A/\mathfrak{m}$. Set $(-)^\vee = \operatorname{Hom}_A(-, E)$. Let $\mathcal{C}$ be a Hom-finite A-triangulated Krull-Schmidt category. By a right Serre-functor on $\mathcal{C}$ we mean an additive functor $F\colon \mathcal{C} \to \mathcal{C}$ such that we have isomorphism

$$\eta_{C,D}\colon \operatorname{Hom}_{\mathcal{C}}(C, D) \to \operatorname{Hom}_{\mathcal{C}}(D, F(C))^\vee,$$

for any $C, D \in \mathcal{C}$ which are natural in C and D. If F is an equivalence then we call F to be a Serre functor. We will use the following result due to Reiten and Van den Bergh

Theorem 5

Let $\mathcal{C}$ be a Hom-finite A-linear triangulated Krull-Schmidt category. Then the following are equivalent

1. *$\mathcal{C}$ has AR-triangles.*
2. *$\mathcal{C}$ has a Serre-functor.*

Krull-Schmidt property of $D^b_f(A)$

Lemma 6

Let $(A, \mathfrak{m})$ be a Noetherian local ring. Then $D^b_f(A)$ is a Hom-finite, triangulated Krull-Schmidt category.

Proof.

Clearly $D^b_f(A)$ is a triangulated category.
The category $D^b(A)$ has split idempotents. Using this fact it is easy to show that $D^b_f(A)$ has split idempotents.
Set $\mathcal{K} = \mathcal{K}^{-,b}(\operatorname{proj} A)$. Let $\mathbf{X}_\bullet, \mathbf{Y}_\bullet$ be bounded complexes in $D^b_f(A)$. Let $\mathcal{F}$ be a minimal projective resolution of $\mathbf{X}_\bullet$. Then

$$\operatorname{Hom}_{D^b(A)}(\mathbf{X}_\bullet, \mathbf{Y}_\bullet) \cong \operatorname{Hom}_{\mathcal{K}}(\mathcal{F}, \mathbf{Y}_\bullet).$$

Using this fact it can be easily shown that $\operatorname{Hom}_{D^b(A)}(\mathbf{X}_\bullet, \mathbf{Y}_\bullet)$ has finite length. Thus $D^b_f(A)$ is a Hom-finite category

Krull-Schmidt property of $D^b_f(A)$ (continued)

Let $\mathbf{X}_\bullet \in D^b_f(A)$. Let $c = h(\mathbf{X}_\bullet) = \sum_i H^i(\mathbf{X}_\bullet)$. Then it is clear that $\mathbf{X}_\bullet$ cannot be a direct sum of $c+1$ non-zero complexes in $D^b_f(A)$. Thus $\mathbf{X}_\bullet$ is a finite direct sum of indecomposable objects in $D^b_f(A)$.

As $D^b_f(A)$ is Hom-finite with split-idempotents it follows that the endomorphism rings of indecomposable objects are local.

It follows that $D^b_f(A)$ is a Hom-finite Krull-Schmidt triangulated category.

Proof of First result (only forward implication)

Assume $(A, \mathfrak{m})$ is regular. Let E be the injective hull of $A/\mathfrak{m}$. Set

$$\mathcal{D}(-) = \mathcal{H}om_A(-, A) \quad \text{and} \quad \mathcal{E}(-) = \mathcal{H}om_A(-, E).$$

Recall the equivalence from $D^{-,fg}(A)$ to $K^{-,b}(\operatorname{proj} A)$ ($K^{+,b}(\operatorname{Inj} A)$)is given by the projective (injective) resolution functor $\mathbf{p}$ ($\mathbf{i}$). Consider the functor F which is the following composite of A-linear functors:

$$K_f^b(\operatorname{proj} A) \xrightarrow{\mathcal{D}} K_f^b(\operatorname{proj} A) \xrightarrow{\mathcal{E}} K_f^b(\operatorname{Inj} A) \xrightarrow{\mathbf{p}} K_f^b(\operatorname{proj} A).$$

Theorem 7

1. *The functor F is dense.*
2. *F is a Serre-functor, i.e., for $\mathbf{X}_\bullet, \mathbf{Y}_\bullet \in K_f^b(\operatorname{proj} A)$ we have a natural isomorphism*

   $\eta_{\mathbf{X}_\bullet, \mathbf{Y}_\bullet} \colon \operatorname{Hom}_{\mathcal{K}}(\mathbf{X}_\bullet, \mathbf{Y}_\bullet) \to \operatorname{Hom}_{\mathcal{K}}(\mathbf{Y}_\bullet, F(\mathbf{X}_\bullet))^\vee.$

Proof of second theorem

Let $(A, \mathfrak{m})$ be a complete Gorenstein local ring of dimension d. Let $\mathcal{S}$ be the category of finite length A-modules which also have finite projective dimension. As A is Gorenstein each element M in $\mathcal{S}$ also has finite injective dimension. If $M \in \mathcal{S}$ then $M^\vee = \operatorname{Hom}_A(M, E) \in \mathcal{S}$.
For each $M \in \mathcal{S}$ fix a minimal projective resolution $\mathbf{P}_\bullet^M$. Set $\mathbf{I}_\bullet^M = \mathcal{H}om_A(\mathbf{P}_\bullet^{M^\vee}, E)$ which is a minimal injective resolution of M. Set

$$\mathcal{F}_f = \operatorname{thick}(\{\mathbf{P}_\bullet^M \mid M \in \mathcal{S}\}) \quad \text{in } K^b(\operatorname{proj} A)\text{and}$$

$$\mathcal{I}_f = \operatorname{thick}(\{\mathbf{I}_\bullet^M \mid M \in \mathcal{S}\}) \quad \text{in } K^{b,fg}(\operatorname{Inj} A).$$

It is easily verified that $\mathcal{F}_f \subseteq K_f^b(\operatorname{proj} A)$ and $\mathcal{I}_f \subseteq K_f^b(E)$.

Proof of second theorem(contd)

Lemma 8

Consider the three equivalences $\mathcal{D}\colon K^b(\operatorname{proj} A) \to K^b(\operatorname{proj} A)^{op}$, $\mathcal{E}\colon K^b(\operatorname{proj} A) \to K^b(E)^{op}$ *and* $\mathbf{p}\colon K^{-,fg}(\operatorname{Inj} A) \to K^{-,fg}(\operatorname{proj} A)$. *Then*

1. $\mathcal{D}$ *induces an equivalence* $\mathcal{D}_r\colon \mathcal{F}_f \to \mathcal{F}_f^{op}$.
2. $\mathcal{E}$ *induces an equivalence* $\mathcal{E}_r\colon \mathcal{F}_f \to \mathcal{I}_f^{op}$.
3. $\mathbf{p}$ *induces an equivalence* $\mathbf{p}_r\colon \mathcal{I}_f \to \mathcal{F}_f$.

Consider $G\colon \mathcal{F}_f \to \mathcal{F}_f$ which is the composite of triangle equivalences

$$\mathcal{F}_f \xrightarrow{\mathcal{D}_r} \mathcal{F}_f^{op} \xrightarrow{\mathcal{E}_r^{op}} \mathcal{I}_f \xrightarrow{\mathbf{p}_r} \mathcal{F}_f.$$

Proof of second theorem(contd)

Remark

By A.Neeman's classification of thick subcategories of $K^b(\text{proj}\, A)$ it follows that $K^b_f(\text{proj}\, A)$ does not have proper thick subcategories. So $\mathcal{F}_f = K^b_f(\text{proj}\, A)$.

Proof.

Just as in proof of previous Theorem we have for $\mathbf{X}_\bullet, \mathbf{Y}_\bullet \in K^b_f(\text{proj}\, A)$ an isomorphism

$$\eta_{\mathbf{X}_\bullet, \mathbf{Y}_\bullet} \colon \text{Hom}_{\mathcal{F}_f}(\mathbf{X}_\bullet, \mathbf{Y}_\bullet) \to \text{Hom}_{\mathcal{F}}(\mathbf{Y}_\bullet, G(\mathbf{X}_\bullet))^\vee.$$

It is easily verified that $\eta_{\mathbf{X}_\bullet, \mathbf{Y}_\bullet}$ is natural in $\mathbf{X}_\bullet\, \mathbf{Y}_\bullet$. So G is a right Serre functor. We have shown G is an equivalence. In particular G is dense. So G is a Serre-functor. Thus, $K^b_f(\text{proj}\, A)$ has AR-triangles. □

Proof of Theorem 1 ((ii) $\implies$ (i))

Theorem 9

Let $(A, \mathfrak{m})$ be a Noetherian local ring. If $D_f^b(A)$ has AR-triangles then A is regular local.

Note $D_f^b(A) \cong K_f^{-,b}(\operatorname{proj} A)$. Let $\mathbf{X}_\bullet$ be a minimal projective resolution of k. Then clearly $\mathbf{X}_\bullet$ is indecomposable in $K_f^{-,b}(\operatorname{proj} A)$. We will use a result independently proved by Gulliksen and Schoeller, i.e., one can use the Tate process to yield a minimal resolution of k. The previous theorem follows from the following result:

Lemma 10

Let $\mathbf{X}_\bullet$ be a minimal resolution of k. If A is not regular then there does not exist a right AR-triangle in $K_f^{-,b}(\operatorname{proj} A)$ ending at $\mathbf{X}_\bullet$.

Tate process

We describe Tate process for creating algebra resolution of A/I where I is an ideal in A. An associative algebra $\mathbf{X}_\bullet$ over A is called an *non-positive DG-algebra* over A if the following hypotheses are satisfied:

1. $\mathbf{X}_\bullet$ is non-positively graded $\mathbf{X}_\bullet = \bigoplus_{n \leq 0} \mathbf{X}_\bullet^n$ with each $\mathbf{X}_\bullet^i$ a finitely generated A-module and $\mathbf{X}_\bullet^i \mathbf{X}_\bullet^j \subseteq \mathbf{X}_\bullet^{i+j}$ for all $i, j \leq 0$.
2. $\mathbf{X}_\bullet$ has a unit element $1 \in \mathbf{X}_\bullet^0$ such that $\mathbf{X}_\bullet^0 = A1$.
3. $\mathbf{X}_\bullet$ is strictly skew-commutative; (for homogeneous element $x \in \mathbf{X}_\bullet^i$ set $|x| = i$) For homogeneous elements x, y we have
   1. $x.y = (-1)^{|x||y|} yx$.
   2. $x^2 = 0$ if $|x|$ is odd.
4. There exists a skew derivation $d \colon \mathbf{X}_\bullet \to \mathbf{X}_\bullet$ such that
   1. $d(\mathbf{X}_\bullet^n) \subseteq \mathbf{X}_\bullet^{n+1}$ for all $n \leq 0$.
   2. $d^2 = 0$.
   3. For x, y homogeneous,

$$d(xy) = d(x)y + (-1)^{|x|} x d(y).$$

Tate process continued

Next we recall Tate's process of killing cycles. Let $\mathbf{X}_\bullet$ be a non-positive DG-algebra. Let $\rho < 0$ be a negative integer. Let $t \in Z^{\rho+1}(\mathbf{X}_\bullet)$ be a cycle of degree $\rho + 1$.

If ρ is odd one can adjoin exterior variable to $\mathbf{X}_\bullet$. If ρ is even we add divided power variable to $\mathbf{X}_\bullet$. In the DG-algebra $\mathbf{X}_\bullet < T >$ the cycle t is killed.

By Tate process we can give construct a DG algebra resolution of A/I for any ideal I.

Problem Say I is an $\mathfrak{m}$-primary ideal and let $\mathbf{X}_\bullet$ be the Koszul complex of I. Note $H^*(\mathbf{X}_\bullet)$ has finite length. If t is a cycle in degree -1 and if $\mathbf{Y}_\bullet = X < T >$ with $dT = t$ then note $H^*(\mathbf{Y}_\bullet)$ need not have finite length. However note that $H^i(\mathbf{Y}_\bullet)$ has finite length for all $i \in \mathbb{Z}$.

Good filtrations of DG-algebras

Let $\mathbf{X}_\bullet$ be a non-positive DG-algebra over A. We assume $\mathbf{X}_\bullet^i$ is a finitely generated free A-module for all $i \leq 0$ and $\ell(H^*(\mathbf{X}_\bullet)^n) < \infty$ for all $n \in \mathbb{Z}$. By a good filtration $\mathcal{F} = \{\mathbf{F}_\bullet(i)\}_{i \geq 0}$ of $\mathbf{X}_\bullet$ we mean

1. $\mathbf{F}_\bullet(i)$ is a sub-complex of $\mathbf{X}_\bullet$ with $\mathbf{F}_\bullet(i)^n$ a direct summand of $\mathbf{X}_\bullet^n$ for all $n \leq 0$.
2. $\mathbf{F}_\bullet(i) \subseteq \mathbf{F}_\bullet(i+1)$ for all $i \geq 0$ and $\bigcup_{i \geq 0} \mathbf{F}_\bullet(i) = \mathbf{X}_\bullet$.
3. $\mathbf{F}_\bullet(i)^n$ is a direct summand of $\mathbf{F}_\bullet(i+1)^n$ for all $n \leq 0$.
4. There exists c depending on $\mathbf{X}_\bullet$ and $\mathcal{F}$ such that
$$\mathbf{F}_\bullet(i)^n \mathbf{F}_\bullet(j)^m \subseteq \mathbf{F}_\bullet(i+j+c)^{n+m}.$$
5. $\ell(H^*(\mathbf{F}_\bullet(i))) < \infty$ for all $i \geq 1$.

We say $\mathcal{F}$ is a *proper* good filtration of $\mathbf{X}_\bullet$ if $\mathbf{F}_\bullet(i) \neq \mathbf{X}_\bullet$ for all i.

A Lemma on Good filtrations

Theorem 11

Let $\rho < 0$ be such that $H^i(\mathbf{X}_\bullet) = 0$ for $0 > i \geq \rho + 2$. Suppose $H^{\rho+1}(\mathbf{X}_\bullet) \neq 0$ and let $t \in Z^{\rho+1}(\mathbf{X}_\bullet)$ be a cycle of degree $\rho + 1$ such that its residue class in $H^{\rho+1}(\mathbf{X}_\bullet)$ is non-zero. Let $\mathbf{Z}_\bullet = \mathbf{X}_\bullet < T >$ with $d(T) = t$. Then $\mathbf{Z}_\bullet$ also has a good filtration. Furthermore

1. *If the filtration on $\mathbf{X}_\bullet$ is proper then the constructed filtration on $\mathbf{Z}_\bullet$ is proper*
2. *If ρ is even then regardless of filtration on $\mathbf{X}_\bullet$ the constructed filtration on $\mathbf{Z}_\bullet$ is proper.*

Two Propositions

Let $(A, \mathfrak{m})$ be local Noetherian ring. First we show

Proposition 12

Let $\mathbf{U}_\bullet \in K_f^{-,b}(\operatorname{proj} A)$ *be a minimal complex, i.e.,* $\partial(\mathbf{U}_\bullet) \subseteq \mathfrak{m}\mathbf{U}_\bullet$*. Let* $\mathbf{V}_\bullet \subseteq \mathbf{U}_\bullet$ *be a sub-complex such that* $\mathbf{V}_\bullet^n$ *is a direct summand of* $\mathbf{U}_\bullet^n$ *for all* $n \in \mathbb{Z}$*. If* $\mathbf{V}_\bullet \neq \mathbf{U}_\bullet$ *then the inclusion* $i \colon \mathbf{V}_\bullet \to \mathbf{U}_\bullet$ *is not a retraction in* $K_f^{-,b}(\operatorname{proj} A)$*.*

Next we prove the following result:

Proposition 13

Let $\mathbf{U}_\bullet \in K^{-,b}(\operatorname{proj} A)$*. Let* $\mathbf{V}_\bullet \subseteq \mathbf{U}_\bullet$ *be a sub-complex such that* $\mathbf{V}_\bullet^n$ *is a direct summand of* $\mathbf{U}_\bullet^n$ *for all* $n \in \mathbb{Z}$*. Let* $\mathbf{W}_\bullet \in K^{-,b}(\operatorname{proj} A)$ *be such that* $H^n(\mathbf{W}_\bullet) = 0$ *for* $n \leq m$*. Suppose* $\mathbf{V}_\bullet^n = \mathbf{U}_\bullet^n$ *for all* $n \geq m$*. Let* $g \colon \mathbf{U}_\bullet \to \mathbf{W}_\bullet$ *be a chain map such that* g *restricted to* $\mathbf{V}_\bullet$ *is null-homotopic. Then* g *is null-homotopic.*

Proof of Lemma 10

Suppose if possible there exists a right AR-triangle ending at $\mathbf{X}_\bullet$. Say we have an AR-triangle

$$\alpha\colon \mathbf{U}_\bullet \to \mathbf{K}_\bullet \to \mathbf{X}_\bullet \xrightarrow{g} \mathbf{U}_\bullet[1]$$

in $K_f^{-,b}(\operatorname{proj} A)$. Assume $H^i(\mathbf{U}_\bullet[1]) = 0$ for $i \geq -m$. We consider two cases:

Case (1): A is a complete intersection. Let $\mathbf{Y}_{\bullet 1}$ be the Koszul complex on a minimal set of generators of $\mathfrak{m}$. Let $\mathbf{Y}_{\bullet 2} = \mathbf{Y}_{\bullet 1} < T_1, \dots T_r >$ be the DG-complex obtained by killing cycles in degree minus one of $\mathbf{Y}_{\bullet 1}$. Then Tate shows that $\mathbf{Y}_{\bullet 2}$ is a minimal resolution of k.

Let $\mathcal{F} = \{\mathbf{F}_\bullet(i)\}_{i \geq 0}$ be a proper good filtration of $\mathbf{Y}_{\bullet 2}$. We may assume that $\mathbf{F}_\bullet(i)^j = \mathbf{Y}_{\bullet 2}^j$ for $i \geq -m$ for all $i \geq i_0$. We have an inclusion $\phi_{i_0}\colon \mathbf{F}_\bullet(i_0) \to \mathbf{X}_\bullet = \mathbf{Y}_{\bullet 2}$. Then ϕ_{i_0} is not a retraction, by our earlier proposition. So $g \circ \phi_{i_0} = 0$ in $K_f^{-,b}(\operatorname{proj} A)$ (as α is an AR-triangle in $K_f^{-,b}(\operatorname{proj} A)$).

Proof of Lemma 10 (continued)

By the second of our proposition for $i \geq i_0$ the null homotopy $g \circ \phi_i$ can be extended to a null homotopy $g \circ \phi_{i+1}$. Thus inductively we have defined homotopy on $\mathbf{F}_\bullet(i)$ for all $i \geq i_0$. But $\mathbf{X}_\bullet = \bigcup_{i \geq i_0} \mathbf{F}_\bullet(i)$. It follows that g is null-homotopic which is a contradiction.

Case (2): A is *not* a complete intersection.
Then $\mathbf{X}_\bullet = \bigcup_{i \geq 1} \mathbf{Y}_{\bullet i}$ where $\mathbf{Y}_{\bullet 1}$ is the Koszul complex on a minimal set of generators of $\mathfrak{m}$ and for $i \geq 2$ the DG-complex $\mathbf{Y}_{\bullet i}$ is obtained by killing cycles of $\mathbf{Y}_{\bullet i-1}$ in degree $-i+1$. We note that $\mathbf{X}_\bullet^n = \mathbf{Y}_{\bullet i}^n$ for $n \geq -i$. It is known thatthat $\mathbf{Y}_{\bullet i} \neq \mathbf{X}_\bullet$ for all i.
Let $\mathcal{F} = \{\mathbf{F}_\bullet(i)\}_{i \geq 0}$ be a good filtration of $\mathbf{Y}_{\bullet m}$. We may assume that $\mathbf{F}_\bullet(i)^j = \mathbf{Y}_{\bullet m}^j$ for $i \geq -m$ for all $i \geq i_0$. We have an inclusion $\phi_{i_0} \colon \mathbf{F}_\bullet(i_0) \to \mathbf{X}_\bullet$. Then clearly ϕ_{i_0} is not a retraction. So $g \circ \phi_{i_0} = 0$ in $K_f^{-,b}(\operatorname{proj} A)$ (as α is an AR-triangle in $K_f^{-,b}(\operatorname{proj} A)$). Then as constructed above we can extend to a homotopy from $\mathbf{Y}_{\bullet m} \to \mathbf{U}_\bullet[1]$.

Proof of Lemma 10 (continued)

Now suppose the map $g\colon \mathbf{X}_\bullet \to \mathbf{U}_\bullet[1]$ when restricted to $\mathbf{Y}_{\bullet i}$ is null-homotopic for some $i \geq m$. Then by our earlier proposition we can extend the homotopy to $\mathbf{Y}_{\bullet i+1}$. As $\mathbf{X}_\bullet = \bigcup_{i \geq m} \mathbf{Y}_{\bullet i}$ we have that we have extended the homotopy to $\mathbf{X}_\bullet$. In particular g is null-homotopic which is a contradiction.