

Singularities and Hilbert functions

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Workshop on Commutative Algebra and Algebraic Geometry
in Prime Characteristic, ICTP Trieste 2023

Basic definitions and results

Let $R = \bigoplus_{i \in \mathbb{N}} R_i$ be a standard graded algebra over a field $R_0 = K$. The *Hilbert Function* and *Hilbert Series* of R are

$$\text{HF}_R(i) = \dim_K R_i \quad \forall i \in \mathbb{N}, \quad \text{HS}_R(t) = \sum_{i \in \mathbb{N}} \text{HF}_R(i) t^i \in \mathbb{Z}[[t]].$$

If $d = \dim R$, Hilbert proved that

$$\text{HS}_R(t) = \frac{h_R(t)}{(1-t)^d}$$

where $h_R(t) = h_0 + h_1 t + h_2 t^2 + \dots + h_s t^s \in \mathbb{Z}[t]$ is the *h-polynomial* of R . ($h_0 = 1, h_1, h_2, \dots, h_s$) the *h-vector* of R .

Remark

While $\text{HS}_R(t) \in \mathbb{N}[[t]]$, $h_R(t)$ may *not* belong to $\mathbb{N}[t]$. If $d = \dim R = 0$, then $h_R(t) = \text{HS}(t) \in \mathbb{N}[t]$.

If $\ell \in R_1$ is an R -regular element, then $h_{R/(\ell)}(t) = h_R(t)$, so if R is Cohen-Macaulay we have $h_i \geq 0$ for all i . Without the CM assumption things change:

Example

Let $S = K[x_i, y_i : i = 1, \dots, r+1]$ and $I \subset S$ the ideal

$$I = (x_1, \dots, x_{r+1})^2 + (x_1y_1 + \dots + x_{r+1}y_{r+1}).$$

$R = S/I$ has dimension $(r+1)$, depth $R = r$ and has h -vector

$$(1, r+1, -1) \quad (h_2 < 0).$$

Such an R is even Buchsbaum (in particular R is generalized CM).

Remark

The integer $e(R) = h_R(1) = h_0 + h_1 + h_2 + \dots$, being the *multiplicity of R* , is always positive.

Let $S = K[X_1, \dots, X_n]$, where $n = \dim_K R_1$. Hence $R \cong S/I$ for some homogeneous ideal $I \subset S$ which *contains no linear forms*. Let c denote the height of I .

Remark

Since $I_1 = \{0\}$ we have:

- $h_1 = c$.
- $h_2 \geq 0 \iff \dim_K I_2 \leq \binom{c+1}{2}$.

If $I_j = \{0\}$ for all $j < k$, then $h_k \geq 0 \iff \dim_K I_k \leq \binom{c+k-1}{k}$.

Theorem (Murai-Terai)

Let R be a Stanley-Reisner ring (i.e. I is generated by squarefree monomials of S) satisfying Serre condition (S_r) . Then $h_i \geq 0$ for all $i \leq r$ and $e(R) \geq h_0 + h_1 + \dots + h_{r-1}$. Furthermore, if $h_i = 0$ for some $i \leq r$, then R is Cohen-Macaulay.

For $r \in \mathbb{N}$, we recall that R satisfies the Serre condition (S_r) if:

$$\text{depth } R_{\mathfrak{p}} \geq \min\{\dim R_{\mathfrak{p}}, r\} \quad \forall \mathfrak{p} \in \text{Spec } R.$$

Remark

If R is generalized Cohen-Macaulay then R satisfies (S_r) if and only if $\text{depth } R \geq r$. Hence, by the previous example, (S_r) alone is not enough to infer $h_i \geq 0$ for all $i \leq r$, not even assuming Buchsbaum.

We say that $R = S/I$ satisfies the condition MT_r if

$$\operatorname{reg} \operatorname{Ext}_S^{n-i}(R, S(-n)) \leq i - r \quad \forall i = 0, \dots, \dim R - 1.$$

This notion is good for several reasons:

- MT_r does not depend on S .
- MT_r is preserved by taking general hyperplane sections.
- MT_r is preserved by saturating.

Lemma (Murai-Terai, Dao-Ma-)

If R satisfies MT_r , then

- $h_i \geq 0$ for all $i \leq r$.
- $h_r + \dots + h_s \geq 0$, or equivalently $e(R) \geq h_0 + \dots + h_{r-1}$.

Furthermore, if $\text{reg } R < r$ or $h_i = 0$ for some $i \leq r$, then R is CM.

Theorem (Dao-Ma-)

Assume that $R = S/I$ has dimension d and satisfies Serre condition (S_r) , and suppose either

- K has characteristic 0 and R is Du Bois in codimension $d - 1$ (e.g. (R_{d-1})), or
- K has characteristic 0, R is Du Bois in codimension $d - 2$ (e.g. (R_{d-2})) and $d = r + 1$, or
- K has positive characteristic and R is F -pure.

Then R satisfies condition MT_r . In particular, $h_i \geq 0$ for all $i \leq r$ and $e(R) \geq h_0 + h_1 + \dots + h_{r-1}$. Furthermore, if $\text{reg } R < r$, or if $h_i = 0$ for some $i \leq r$, then R is Cohen-Macaulay.

Stanley-Reisner rings are Du Bois in characteristic 0 and F -pure in positive characteristic, so we recover Murai-Terai result. However by results of Kummini-Murai MT_r holds whenever I is a monomial ideal and S/I satisfies (S_r) condition.

Questions:

Is it true that R satisfies MT_r if R satisfies (S_r) and either

- 1 K has characteristic 0 and R_{red} is Du Bois in codimension $r - 1$,
- 2 K has characteristic 0 and R is Du Bois in codimension $r - 2$, (in particular, if R is normal and satisfies (S_3) , is $h_3 \geq 0$?),
- 3 K has positive characteristic and R_{red} is F -pure, or
- 4 K has positive characteristic and R is F -injective???

A new F -singularity

Suppose that K has positive characteristic:

Definition

We say that R is *deformation equivalent to an F -pure ring (de F -pure)* if there exist a domain A , essentially of finite type over K , and a flat finitely generated A -algebra R_A such that all fibres are standard graded, one fibre is R and one is F -pure. In other words, there exist prime ideals $\mathfrak{p}, \mathfrak{q} \in A$ such that $R_A \otimes_A \kappa(\mathfrak{p}) \cong R$ and $R_A \otimes_A \kappa(\mathfrak{q})$ is F -pure.

Remark

If R is F -pure, then it is de F -pure ($A = K$, $R_A = R$, $\mathfrak{p} = \mathfrak{q} = \{0\}$).

Proposition (Dao, Ma, ...)

If R is de F -pure and satisfies (S_r) , then it satisfies MT_r .

A new F -singularity

Recall that $R = S/I$ where $S = K[X_1, \dots, X_n]$ and $I \subset S$ is a homogeneous ideal. Suppose that $\text{in}(I)$ is a squarefree monomial ideal for some term order. One can check that R is de F -pure.

Example

If $n = 3$ and $I = (X_1^3 + X_2^3 + X_3^3 - X_1X_2X_3) \subset S$, then $R = S/I$ is de F -pure: taking $A = K[t]$, $J = (tX_1^3 + tX_2^3 + tX_3^3 - X_1X_2X_3)$ and $R_A = S[t]/J$, one has $R_A \otimes_A \kappa((t-1)) \cong R_A/(t-1) \cong S/I = R$. $R_A \otimes_A \kappa((t)) \cong R_A/(t) \cong S/(X_1X_2X_3)$ (that is F -pure). Note that, if $\text{char}(K) = 5$, R is not even F -injective.

Question

Is an F -injective standard graded K -algebra de F -pure?

THANK YOU !