Normal and Tight Hilbert Polynomials

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with Saipriya Dubey and Jugal Verma, Res. Math. Sci. 2023;
with Linquan Ma, arXiv:2301.13084)

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Outline

1. Normal/Tight Hilbert polynomials
2. Tight Buchsbaum rings and its tight Hilbert polynomials
3. Normal/Tight Hilbert coefficients

Outline

In this talk

- $(R, \mathfrak{m})$ denotes a complete local ring of dimension d.
- $H^i_{\mathfrak{m}}(R)$ denotes the i**-th local cohomology** of R.
- Let I is an $\mathfrak{m}$-primary ideal.
- Let $Q = (x_1, \ldots, x_d)$ be a **parameter ideal** of R.
- $e(I)$, $\ell(R/I)$ denote the **Hilbert multiplicity** and the **colength** of I.
- Let $\bar{I}$ denote the **integral closure** of I.
- If R has positive characteristic p, then I^* denotes the **tight closure** of I.

Normal/Tight Hilbert polynomials

It is well-known that $\ell(R/I^{n+1})$ becomes a Hilbert polynomial of degree d, i.e. for all $n \gg 0$

$$\ell(R/I^{n+1}) = P_I(n) = e_0(I)\binom{n+d}{d} - e_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d(I),$$

where $e_0(I)$ is the multiplicity of R with respect to I, and $e_i(I)$, $0 \leq i \leq d$, is the Hilbert coefficients of R with respect to I.

Normal/Tight Hilbert polynomials

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If R is reduced, by Rees we have an integer number c such that $\overline{I^n} \subseteq I^{n-c}$ for all n. Hence $\ell(R/\overline{I^{n+1}})$ becomes a polynomial of degree d, i.e. for all $n \gg 0$

$$\ell(R/\overline{I^{n+1}}) = \overline{P}_I(n) = \overline{e}_0(I)\binom{n+d}{d} - \overline{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \overline{e}_d(I),$$

where $\overline{e}_0(I) = e_0(I)$.

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where $\overline{e}_0(I) = e_0(I)$.
$\overline{P}_I(n)$ is called the **normal Hilbert polynomial** of R with respect to I. We call $\overline{e}_0(I)$ the normal multiplicity, and $\overline{e}_i(I)$, $0 \leq i \leq d$, the normal Hilbert coefficients of R with respect to I.

Characteristic p

Let $I = (a_1, \ldots, a_t)$ be an ideal of R. The **Frobenius power** $I^{[p]}$ of I is the extension of I via the Frobenius endomorphism, that is

$$I^{[p]} = (a^p \mid a \in I).$$

In general, for every $q = p^e$ we can work with the e-th Frobenius endomorphism

$$F^e : R \to R; x \mapsto x^q,$$

and the e-th Frobenius power $I^{[q]} = (a^q \mid a \in I)$.

Normal/Tight Hilbert polynomials

Let I be an ideal of R. The **tight closure** I^* of I is defined as follows

$$I^* = \{x \mid cx^q \in I^{[q]} \text{ for some } c \in R^\circ \text{ and for all } q = p^e \gg 0\},$$

where $R^\circ = R \setminus \cup_{P \in \mathrm{minAss}(R)} P$. The ideal I is called tight closed if $I = I^*$.

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where $R^\circ = R \setminus \cup_{P \in \mathrm{minAss}(R)} P$. The ideal I is called tight closed if $I = I^*$. We have $I \subseteq I^* \subseteq \overline{I}$. Thus if R is reduced, $\ell(R/(I^{n+1})^*)$ becomes a polynomial of degree d, i.e. for all $n \gg 0$

$$\ell(R/(I^{n+1})^*) = P_I^*(n) = e_0^*(I)\binom{n+d}{d} - e_1^*(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d^*(I$$

where $e_0^*(I) = e_0(I)$.
$P_I^*(n)$ is called the **tight Hilbert polynomial** of R with respect to I. We call $e_0^*(I)$ the tight multiplicity, and $e_i^*(I)$, $0 \le i \le d$, the tight Hilbert coefficients of R with respect to I.

Normal/Tight Hilbert polynomials

Suppose R is reduced, so $e_0(I) = \overline{e}_0(I) = e_0^*(I)$. Moreover

$$\ell(R/\overline{I^{n+1}}) \leq \ell(R/(I^{n+1})^*) \leq \ell(R/I^{n+1}).$$

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$$\ell(R/\overline{I^{n+1}}) \leq \ell(R/(I^{n+1})^*) \leq \ell(R/I^{n+1}).$$

Corollary

$\overline{e}_1(I) \geq e_1^*(I) \geq e_1(I)$.

(Normal) Hilbert cofficients

Theorem (Goto, 1987)

Let Q be a parameter ideal of R. Suppose $\overline{Q} = Q$. Then R is regular, and $\mu(\mathfrak{m}/Q) \leq 1$.

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Theorem (Morales-Trung-Villamayor, 1990)

Let $(R, \mathfrak{m})$ be a reduced and equidimensional local ring. Suppose $\overline{e}_1(Q) = e_1(Q)$. Then R is regular, and $\mu(\mathfrak{m}/Q) \leq 1$.

(Normal) Hilbert coefficients

It is well known that $\ell(R/Q) \geq e_0(Q)$ for all Q.

Theorem (Hayasaka-Hyry, 2010)

Let Q be a parameter ideal of R. Then $\ell(R/Q^{n+1}) \geq e_0(Q)\binom{n+d}{d}$ for all n. If $\ell(R/Q^{n+1}) = e_0(Q)\binom{n+d}{d}$ for some n, then R is Cohen-Maucaulay.

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We get a solution for the Vasconcelos non-positive conjecture of $e_1(Q)$.

Theorem (Mandal-Singh-Verma, 2011)

Let Q be a parameter ideal of R. Then $e_1(Q) \leq 0$.

(Normal) Hilbert coefficients

For the Vasconcelos vanishing conjecture of $e_1(Q)$, we have the following theorem.

Theorem (GGHOPV, 2010)

Let Q be a parameter ideal of an unmixed local ring R. If $e_1(Q) = 0$, then R is Cohen-Macaulay.

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For integral closure, we have the following result.

Theorem (Goto-Hong-Mandal, 2011)

Let $(R, \mathfrak{m})$ be a reduced and equidimensional local ring. Then $\overline{e}_1(Q) \geq 0$.

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Let $(R, \mathfrak{m})$ be a reduced and equidimensional local ring. Then $\overline{e}_1(Q) \geq 0$.

For any $\mathfrak{m}$-primary ideal, we choose Q a minimal reduction of I. Therefore $\overline{P}_I(n) = \overline{P}_Q(n)$, so $\overline{e}_1(I) \geq 0$.

Question

Theorem (Goel-Verma-Mukundan, 2020)

Let $(R, \mathfrak{m})$ be a reduced Cohen-Macaulay local ring of positive characteristic. Then R is F-rational if and only if $e_1^*(Q) = 0$ for some Q.

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Question (Huneke, 2023)

Let $(R, \mathfrak{m})$ be a reduced and equidimensional local ring of positive characteristic. Then is it true that R is F-rational if and only if $e_1^*(Q) = 0$ for some parameter ideal Q.

Buchsbaum rings

In general, we have $\ell(R/Q) \geq e(Q)$ for all parameter ideals Q.

Definition (Stuckrad-Vogel)

Let $(R, \mathfrak{m})$ be a local ring of dimension d. Then R is called Buchsbaum if and only if $\ell(R/Q) - e(Q)$ is a constant for all parameter ideals Q.

Buchsbaum rings

Theorem (Stuckrad-Vogel)

The following are equivalent:
(i) $(R, \mathfrak{m})$ is Buchsbaum.
(ii) For every system of parameters $x_1, \ldots, x_d$ we have

$$(x_1, \ldots, x_{i-1}) : x_i = (x_1, \ldots, x_{i-1}) : \mathfrak{m}$$

for all $i \leq d$.

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for all $i \leq d$.

Theorem (Stuckrad-Vogel)

Let $(R, \mathfrak{m})$ be a Buchsbaum local ring of dimension d. Then for all parameter ideal Q we have

$$\ell(R/Q) - e(Q) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H^i_{\mathfrak{m}}(R)).$$

Buchsbaum rings

Definition (Schenzel)

Let $(R, \mathfrak{m})$ be a local ring of dimension d. We define

$$\mathfrak{b}(R) = \bigcap_{x_1,\ldots,x_d} \mathrm{Ann}\Big(\frac{(x_1,\ldots,x_{i-1}) : x_i}{(x_1,\ldots,x_{i-1})}\Big)$$

where $x_1,\ldots,x_d$ runs over all systems of parameters of R.

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where $x_1,\dots,x_d$ runs over all systems of parameters of R.

Theorem

$(R, \mathfrak{m})$ is a Buchsbaum local ring iff $\mathfrak{b}(R) \supseteq \mathfrak{m}$.

F-rational rings

Definition (Fedder-Watanabe)

An equidimensional local ring $(R, \mathfrak{m})$ is called *F*-**rational** if every parameter ideal is tight closed, i.e. $Q^* = Q$ for all Q

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Theorem

1. (Hochster-Huneke) A local ring $(R, \mathfrak{m})$ is F-rational. Then it is normal and Cohen-Macaulay.
2. (Smith) A local ring $(R, \mathfrak{m})$ is F-rational if and only if it is Cohen-Macaulay and $0^*_{H^d_{\mathfrak{m}}(R)} = 0$.

F-rational rings

Conjecture (Watanabe-Yoshida, 2000)

Let $(R, \mathfrak{m})$ be an unmixed local ring of dimension d and of characteristic $p > 0$. Then
(1) For every parameter ideal Q we have $e(Q) \geq \ell(R/Q^*)$.
(2) If we have $e(Q) = \ell(R/Q^*)$ for some parameter ideal Q, then R is F-rational.

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Theorem (Goto-Nakamura, 2001)

The conjecture of Watanabe and Yoshida is true.

F-rational rings

$$\begin{array}{ccc} F\text{-rational} & \Longrightarrow & ??? \\ \Downarrow & & \Downarrow \\ \text{Cohen-Macaulay} & \Longrightarrow & \text{Buchsbaum.} \end{array}$$

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There are partial answers for ??? of Bhatt, Ma and Schewde (2018), and Q (2018).

Tight Buchsbaum rings

Definition

Let $(R, \mathfrak{m})$ be an equidimension local ring of characteristic $p > 0$ and dimension d. The *parameter test ideal* is defined

$$\tau_{par}(R) = \bigcap_{Q} (Q : Q^*),$$

where Q runs over all parameter ideals.

Tight Buchsbaum rings

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Let $(R, \mathfrak{m})$ be an equidimension local ring of characteristic $p > 0$ and dimension d. The *parameter test ideal* is defined

$$\tau_{par}(R) = \bigcap_{Q} (Q : Q^*),$$

where Q runs over all parameter ideals.

Remark

(1)R is F-rational iff $\tau_{par}(R) = R$.
(2) R is F-rational on puncture spectrum iff $\tau_{par}(R)$ is an $\mathfrak{m}$-primary ideal.

Question

Question (Watanabe)

Suppose $\mathfrak{m} \subseteq \tau_{par}(R)$. Is it true that $e(Q) - \ell(R/Q^*)$ does not depend of the choice of Q?

Tight Buchsbaum rings

Theorem (Ma-Q, 2022)

Let $(R, \mathfrak{m})$ be an unmixed local ring of characteristic $p > 0$ and dimension d. Let following are equivalent:

1. R is a **tight Buchsbaum ring** i.e., the difference $e(Q) - \ell(R/Q^*)$ is independent of Q.
2. $\mathfrak{m} \subseteq \tau_{par}(R)$, i.e., $\mathfrak{m}Q^* \subseteq Q$ for every Q.

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2. $\mathfrak{m} \subseteq \tau_{par}(R)$, i.e., $\mathfrak{m}Q^* \subseteq Q$ for every Q.

Moreover, if R is tight Buchsbaum then for every Q we have

$$e(Q) - \ell(R/Q^*) = \sum_{i=0}^{d-1} \binom{d}{i} \ell(H^i_{\mathfrak{m}}(R)) + \ell(0^*_{H^d_{\mathfrak{m}}(R)}).$$

Tight Buchsbaum rings

$$\begin{array}{ccc} F\text{-rational} & \Longrightarrow & \text{tight Buchsbaum} \\ \Downarrow & & \Downarrow \\ \text{Cohen-Macaulay} & \Longrightarrow & \text{Buchsbaum.} \end{array}$$

Example of Tight Buchsbaum rings

Example

A Buchsbaum ring with $0^*_{H^d_{\mathfrak{m}}(R)} = 0$ is a tight Buchsbaum ring. For example, $k[[a,b,c,d]]/(a,b) \cap (c,d)$, and $k[[x^4, x^3y, xy^3, y^4]]$, $\mathrm{char}\, p > 0$, are tight Buchsbaum.

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Example

Let $R = k[[X,Y,Z]]/(X^3 + Y^3 - Z^3)$, $\mathrm{char}\, p > 3$. We have that R is Cohen-Macaulay and $0^*_{H^2_{\mathfrak{m}}(R)} \cong k$. Then R is tight Buchsbaum.

Hilbert polynomials for Buchsbaum rings

Theorem (Schenzel)

Let $(R, \mathfrak{m})$ be a Buchsbaum local ring of dimension d. Then for any parameter ideal Q we have:

$$\ell(R/Q^{n+1}) = \sum_{i=0}^{d} (-1)^i e_i(Q) \binom{n+d-i}{d-i},$$

where

$$e_i(Q) = (-1)^i \sum_{j=0}^{d-i} \binom{d-i-1}{j-2} \ell(H^j_{\mathfrak{m}}(R))$$

for $i = 1, 2, \cdots, d$.

Tight Hilbert polynomials for tight Buchsbaum rings

Theorem (Dubey-Q-Verma, 2023)

Let $(R, \mathfrak{m})$ be a tight Buchsbaum local ring of dimension d. Then for any parameter ideal Q we have:

$$\ell(R/(Q^{n+1})^*) = \sum_{i=0}^{d} (-1)^i e_i^*(Q) \binom{n+d-i}{d-i},$$

where

$$e_1^*(Q) = \sum_{j=0}^{d-1} \binom{d-2}{j-2} \ell(H_{\mathfrak{m}}^j(R)) + \ell(0^*_{H_{\mathfrak{m}}^d(R)})$$

and

$$e_i^*(Q) = (-1)^{i-1} \Big[\sum_{j=0}^{d-i} \binom{d-i-1}{j-2} \ell(H_{\mathfrak{m}}^j(R)) + \ell(H_{\mathfrak{m}}^{d-i+1}(R)) \Big]$$

for $i = 2, \cdots, d$.

Corollary

Let $(R, \mathfrak{m})$ be a tight Buchsbaum local ring with $\mathrm{depth} R \geq 2$. Suppose $e_1^*(Q) = 0$ for some parameter ideal Q. Then R is F-rational.

Tight Hilbert coefficients

Similar to Hayasaka-Hyry's theorem we can prove the following.

Theorem (Ma-Q)

Let $(R, \mathfrak{m})$ be a reduce equidimensional local ring of positive characteristic. Let Q be a parameter ideal. Then for all n, we have

$$\ell(R/(Q^{n+1})^*) \leq e_0(R)\binom{n+d}{d}.$$

Therefore $e_1^*(Q) \geq 0$.

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Therefore $e_1^*(Q) \geq 0$.

If I is a $\mathfrak{m}$-primary ideal and Q is minimal reduction of I, then $e_1^*(I) \geq e_1^*(Q) \geq 0$.

Tight Hilbert coefficients

Similar to Morales-Trung-Villamayor's result, we have

Corollary (Ma-Q)

Let $(R, \mathfrak{m})$ be a reduce equidimensional local ring of positive characteristic. Let Q be a parameter ideal. If $e_1^*(Q) = e_1(Q)$, then R is F-rational.

BCM Hilbert coefficients

In fact, for any balance big Cohen-Macaulay local ring B of R we can define $I^B = IB \cap R$, and hence $e_i^B(Q)$. We have

$$\ell(R/(Q^{n+1})^B) \leq e_0(R)\binom{n+d}{d}.$$

Therefore $e_1^B(Q) \geq 0$.

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$$\ell(R/(Q^{n+1})^B) \leq e_0(R)\binom{n+d}{d}.$$

Therefore $e_1^B(Q) \geq 0$.
Note that $I^B \subseteq \overline{I}$, hence $\overline{e}_1(Q) \geq e_1^B(Q) \geq 0$.

Normal Hilbert coefficients

We can prove the following result.

Theorem (Ma-Q)

Let $(R, \mathfrak{m})$ be a reduce equidimensional local ring satisfying the S_2 condition. If $\overline{e}_1(Q) = 0$ for some Q, then R is regular and $\mu(\mathfrak{m}/Q) \leq 1$.

Conjecture (Huneke, Ma-Q)

Let $(R, \mathfrak{m})$ be a reduced and S_2 local ring of characteristic p. Suppose $e_1^*(Q) = 0$ for some parameter ideal Q. Then R is F-rational.

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Let $(R, \mathfrak{m})$ be a reduced and S_2 local ring of characteristic p. Suppose $e_1^*(Q) = 0$ for some parameter ideal Q. Then R is F-rational.
In general, let B is a balance big Cohen-Macaulay algebra of R. Suppose $e_1^B(Q) = 0$ for some parameter ideal Q. Then R is BCM_B-rational.

THANK YOU VERY MUCH FOR YOUR ATTENTION