On the resurgence and asymptotic resurgence of homogeneous ideals

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Introduction

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Known in the case of space monomial curves (Grifo)
In general, the question is open.

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Definition: (Bocci-Harbourne) **resurgence** of I

$$\rho(I) := \sup \left\{ \frac{s}{t} \ : \ s, t \in \mathbb{N} \text{ and } I^{(s)} \not\subseteq I^t \right\}.$$

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Corollary: If I is a radical ideal in a regular ring, $\rho(I) \leq \text{bh}(I)$.

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Corollary: If I is a radical ideal in a regular ring, $\rho(I) \leq \mathrm{bh}(I)$.

Note: if $\frac{s}{t} > \rho(I)$, then $I^{(s)} \subseteq I^t$.

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asymptotic resurgence of I

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Theorem: (Guardo-Harbourne-Van Tuyl) If is a homogeneous ideal in a finitely generated graded K-algebra, then

$$1 \leq \frac{\alpha(I)}{\hat{\alpha}(I)} \leq \rho_a(I) \leq \rho(I),$$

$\alpha(I) = \min\{\deg f \ : \ f \in I\}$,

$\hat{\alpha}(I) = \lim_{s\to\infty} \frac{\alpha(I^{(s)})}{s}$, Waldschmidt constant.

Introduction

Theorem: (Grifo, Grifo-Huneke-Mukundan) Stable Harbourne conjecture is true if $\rho(I) < \text{bh}(I)$ or $\rho_a(I) < \text{bh}(I)$.

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Question: Can one compute the resurgence and asymptotic resurgence in finite steps?

(DiPasquale-Drabkin): If $\rho_a(I) < \rho(I)$, then **YES**.

If the symbolic Rees algebra is Noetherian, then $\rho(I)$ is a rational number.

Resurgence of Homogeneous Ideals

Theorem

Suppose S is Noetherian, $(0) \neq I \subsetneq R$ such that $R_s(I) = S[It, I^{(n)}t^n]$ and $\exists P$ such that $PI^{(n)} \subset I^n \& I^{(n)} \subset P^k I^{n-1}$ for some $k \geq 1$. Then $I^{(nkq+nq)} \subset I^{nkq+nq-q}$ for all $q \in \mathbb{N}$ and $\rho(I) \leq \frac{nk+n}{nk+n-1}$.

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Upper bound is tight: $I = (x_1x_2, \ldots, x_{2n-1}x_1) \subset K[x_1, \ldots, x_{2n-1}]$. Then $\rho(I) = \rho_a(I) = \frac{2n}{2n-1}$.

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Theorem

If I, J nonzero proper ideals generated in disjoint set of variables in a polynomial ring, then

1. $\rho(IJ) = \max\{\rho(I), \rho(J)\}$.
2. $\rho_a(IJ) = \max\{\rho_a(I), \rho_a(J)\}$.

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Suppose I, J *nonzero proper homogeneous ideals generated in disjoint set of variables in a polynomial ring. If* $I^{(s)} = I^s$ *for all* $s \geq 1$*, then* $\rho(I + J) = \rho(J)$*.*

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$I_j \subset K[x_{j1}, \ldots x_{jn_j}]$, *nonzero proper homogeneous ideals,* $p_j = \min\{t \ : \ I_j^{(t)} \neq I_i^t\}$, $j = 1, \ldots k$. *If* $\rho(I_j) = 1$ *for all* $j = 1, \ldots, k$, *then* $\rho(I + J) = \rho(J)$.

$$\rho(I_1 + \cdots + I_k) = \max\left\{\frac{p_1 + \cdots + p_r}{p_1 + \cdots + p_r - r + 1} \ : \ 2 \leq r \leq k\right\}$$

Resurgence of Homogeneous Ideals

Theorem:(DiPasquale-Drabkin) If $I \subset K[x_1, \dots, x_\ell]$ is a squarefree monomial ideal of big height h, then $\rho_a(I) \leq h - \frac{1}{\ell}$.

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If $I \subset K[x_1, \dots, x_\ell]$ is a squarefree monomial ideal of big height h, then $I^{(hr-h)} \subseteq I^r$ for all $r \geq \chi(I) = \min\{d \ : \ (x_1 \cdots x_\ell)^{d-1} \in I^d\}$. In particular, $\rho_a(I) \leq h - \frac{1}{\chi(I)}$.

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Observation: I is squarefree monomial $\Rightarrow$ I is cover ideal of a hypergraph $\mathcal{H}$. Then, $\chi(I) := \chi(\mathcal{H})$ is the chromatic number of $\mathcal{H}$.

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Hence $\chi(I) \leq \ell$, but a large bound in general.

For example, if $\mathcal{H} = K_{1,n}$, then $\chi(\mathcal{H}) = 2$.

Resurgence of Cover Ideals

G - a finite simple graph, $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$.

Edge ideal of G, $I(G) = (x_i x_j \; : \; \{x_i, x_j\} \in E(G)) \subset K[x_1, \ldots, x_n]$.

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$w \subset V(G)$ is a Vertex cover of G, if $w \cap e \neq$ for all $e \in E(G)$.

Cover ideal of G, $J(G) := (\prod_{x_j \in w} x_j \ : \ w$ is a vertex cover of $G)$.

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Cover ideal is the Alexander Dual of the edge ideal.

$$J(G) = \bigcap_{\{x_i, x_j\} \in E(G)} (x_i, x_j).$$

Cover ideals are radical, height 2, unmixed ideals.

Resurgence of Cover Ideals

Question:(Grifo) If I is a radical ideal in a regular ring R, then for given $C > 0$, does there exist an $N > 0$ such that $I^{(hr-C)} \subset I^r$ for all $r \geq N$?

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Theorem

Let G be a graph and $c \in \mathbb{N}$. Then

- $J(G)^{(2r-2c)} \subset J(G)^r$ *for every* $r \geq c\chi(G)$,
- $J(G)^{(2r-2c-1)} \subset J(G)^r$ *for every* $r \geq c\chi(G) + 1$.

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Theorem

Let $\omega(G) :=$ *max size of a clique in* G *and* $\alpha(G) :=$ *max size of an independent set in* G*. Then*

$$\max\left\{2 - \frac{2}{\omega(G)}, 2 - \frac{2\alpha(G)}{n}\right\} \leq \rho_a(G) \leq \rho(G) \leq 2 - \frac{2}{\chi(G)}.$$

Resurgence of Cover Ideals

Corollary

If G is a perfect graph $(\chi(G) = \omega(G))$, then
$\rho_a(G) = \rho(G) = 2 - \frac{2}{\chi(G)}$.

The earlier lower bound and the above Corollary was also simultaneously proved by Grisalde, Seceleanu and Villarreal using very different techniques.

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If G is a perfect graph ($\chi(G) = \omega(G)$), then $\rho_a(G) = \rho(G) = 2 - \frac{2}{\chi(G)}$.

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- Herzog-Hibi-Trung: G is bipartite if and only if $J(G)^s = J(G)^{(s)}$ for all $s \geq 1$.

Theorem

$\rho(J(G)) = 1 \iff$ *G is bipartite* $\iff \rho_a(G) = 1$.

Resurgence of Cover Ideals

Theorem

1. $\rho_a(J(C_{2n+1})) = \rho(J(C_{2n+1})) = \dfrac{\alpha(J(G))}{\hat{\alpha}(J(G))} = \dfrac{2n+2}{2n+1}$.
2. $J(C_{2n+1})^{(2nt+2t)} \subseteq J(C_{2n+1})^{2nt+t}$.

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2. $J(C_{2n+1})^{(2nt+2t)} \subseteq J(C_{2n+1})^{2nt+t}$.
3. *If G is non-bipartite cactus graph, then $\rho(J(G)) = \rho_a(J(G)) = \frac{n+1}{n}$, where n is the number of vertices of a smallest induced odd cycle in G.*

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Theorem

Let $G = G_1 \cup G_2$ be a clique-sum of G_1 and G_2. Then :

1. *For any $t \geq 1$, $J(G)^t = J(G_1)^t \cap J(G_2)^t$.*
2. *For any $s \geq 1$, $J(G)^{(s)} = J(G_1)^{(s)} \cap J(G_2)^{(s)}$.*
3. $\rho(J(G)) = \max\{\rho(J(G_1)), \rho(J(G_2))\}$.
4. $\rho_a(J(G)) = \max\{\rho_a(J(G_1)), \rho_a(J(G_2))\}$.

Resurgence of Cover Ideals

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Theorem

1. *If $G = K_m^c * H$, where H is a non-trivial bipartite graph on n vertices, then $\rho(J(G)) = \frac{\alpha(J(G))}{\hat{\alpha}(J(G))}$ if and only if $m = \alpha(J(H)) = \frac{n}{2}$.*
2. *If $G = K_{n_1,\ldots,n_k}$, then $\rho(J(G)) = \frac{\alpha(J(G))}{\hat{\alpha}(J(G))}$ if and only if $n_1 = \cdots = n_k$.*

Resurgence of Edge Ideals

- G - graph on $\{x_1, \ldots, x_n\}$.
- Edge ideal $I(G) := (x_i x_j \ : \ \{x_i, x_j\}$ is an edge of $G)$.
- Simis-Vasconcelos-Villarreal: G is bipartite if and only if $I(G)^{(s)} = I(G)^s$ for all $s \geq 1$.

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Theorem

$\rho(I(G)) = 1 \iff$ *G is bipartite* $\iff \rho_a(I(G)) = 1.$

Resurgence of Edge Ideals

- G - clique-sum of bipartite graphs and cycles of length $2n+1$.
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Theorem

$$\rho(I(G)) = \begin{cases} \dfrac{2n+2}{2n+1} & \text{if } k = 1, \\[2ex] \dfrac{kn+k}{kn+1} & \text{if } k \geq 2. \end{cases}$$

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2. Stable Harbourne conjecture
3. Identify important classes of ideals for which the (Stable) Harbourne conjecture is true
4. Generalize the results to cover ideals of hypergraphs
5. Bounds for resurgence and asymptotic resurgence for edge ideals.

THANK YOU!!!