Workshop on Commutative Algebra and Algebraic Geometry in Prime Characteristics

11 May 2023, ICTP Trieste, Italy

Tight closure

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A message from Mel Hochster for the participants:

‘I would like to thank the organizers, and the participants for listening. I count myself lucky to have had so many wonderful collaborators, colleagues, students and mathematical friends. I wish everyone much joy in their pursuit of mathematics!’

Throughout:

p is a prime integer, q is a power of p

R is a ring, k is a field, most of the time of characteristic p

R° is the set of all elements of R not in any minimal prime ideal

I is an ideal

$I^{[q]} = (x^q : x \in I)$ is a **Frobenius power** of I (eth power if $q = p^e$)

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Proof: (Kunz) Frobenius is faithfully flat on regular rings.

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Rings in which all ideals are tightly closed are called **F-regular**.

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Hochster's question: Can you find a proof of the following fact without using tight closure?

Let $R = k[x, y]$ be the polynomial ring in variables x, y over a field k. Then for any $f, g, h \in R$,

$$f^2g^2h^2 \in (f^3, g^3, h^3).$$

(Here, $I = (f^3, g^3, h^3)$, $l = 2$, $n = 0$; fgh is integral over I.)

Colon capturing: Let R be module-finite over a regular domain A and let $x_1, \ldots, x_n$ be elements of A that are parameters in R. Then

$$(x_1, \ldots, x_{n-1})R :_R x_n \subseteq ((x_1, \ldots, x_{n-1})R)^*.$$

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Colon capturing II: Let R be module-finite over and torsion-free over a regular domain A. Let I, J be ideals of A. Then

$$IR :_R JR \subseteq ((I :_A J)R)^*, \qquad IR \cap JR \subseteq ((I \cap J)R)^*.$$

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Monomial conjecture: If $x_1, \ldots, x_d$ is a system of parameters of R, then for all non-negative integers t,

$$x_1^t \cdots x_d^t \notin (x_1^{t+1}, \ldots, x_d^{t+1}).$$

Direct summand conjecture: A regular local ring is a direct summand of every module-finite extension.

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Vanishing of Tor: Let $A \subseteq R \subseteq S$ be excellent equicharacteristic rings such that A and S are regular domains and R is module-finite over A. Then for any finitely generated A-module M and any integer $i \geq 1$, the map $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, S)$ is zero.

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Phantom intersection theorem

Improvement of the syzygy theorem of Evans and Griffith: Let R be Noetherian local of characteristic p, let M be a finitely generated kth syzygy theorem of finite projective dimension, and let x be a minimal generator of M. Then the depth of the order ideal $\{f(x) : f \in \mathrm{Hom}_R(M, R)\}$ is at least k.

Different notions of tight closure:

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- (integral closure, regular closure, plus closure)

Test elements

An element $c \in R^\circ$ is a **test element** if for every ideal I in R, every $x \in I^*$ and every power q of p, $cx^q \in I^{[q]}$.

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Consequence: persistence of tight closure under ring homomorphisms.

Phantom homology, phantom acyclicity criterion: Suppose that R is a homomorphic image of a Cohen-Macaulay ring and is locally equidimensional. Let $\mathbb{G}$ be a finite free complex that satisfies the standard rank and height conditions after tensoring with $R/\sqrt{0}$. Then for all e, all (higher) cycles of $F^e\mathbb{G}$ are in the tight closure of the boundaries (i.e., the homologies of $F^e\mathbb{G}$ are phantom).

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Also:

Phantom acyclicity with denominators, what elements annihilate the homologies.

Under (mild) conditions on the ring, there exist test elements that annihilate all higher Koszul homology on parameters.

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- Hilbert-Kunz function, Hilbert-Kunz multiplicity

- F-signature

- F-threshold