Castelnuovo-Mumford regularity of powers of an ideal

Le Tuan Hoa

Institute of Mathematics Hanoi (IMH)
Vietnam Academy of Science and Technology

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Content

I. Motivation

R: standard graded algebra over a field k, $\mathfrak{m} := R_+$,
M: finitely generated graded R-module,
If E is an Artinian graded R-module, we set

$$a(E) := \sup\{t |\ E_t \neq 0\}.$$

The Castelnuovo-Mumford regularity:

$$\operatorname{reg} M := \max\{a(H^i_{R_+}(M)) + i |\ i \geq 0\}.$$

reg M controls the complexity of the graded structure of M[1]

[1] D. Bayer and D. Mumford, *What can be computed in algebraic geometry?*, Computational algebraic geometry and commutative algebra (Cortona, 1991), 1-48, Cambridge Univ. Press, 1993.

I. Motivation

$0 \neq I \subset R$: graded ideal *that is not nilpotent*. Then:
- $\mathrm{reg}(I^n) = reg(R/I^n) + 1$ for all $n \gg 0$;
- If R is a polynomial ring, then the above equality holds for all $n \geq 1$.
- In general, $\mathrm{reg}(I)$ is very big compared to the maximal generating degree $d_{max}(I)$ of I. However,

Theorem A [a b c]. *There are integers $d > 0$ and $e \geq 0$ such that*

$$\mathrm{reg}(I^n M) = dn + e \ \forall n \gg 0.$$

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[a] D. Cutkosky, J. Herzog and N.V. Trung, *Asymptotic behavior of the Castelnuovo-Mumford regularity*, Compositio Math. **118** (1999), 243 - 261.

[b] V. Kodiyalam, *Asymptotic behaviour of Castelnuovo-Mumford regularity*, Proc. Amer. Math. Soc. **128** (2000), 407 - 411.

[c] N.V. Trung and H-J. Wang, *On the asymptotic linearity of Castelnuovo-Mumford regularity*, J. Pure Appl. Alg. **201** (2005), 42 - 48.

I. Motivation

Remarks:
- The slope d is called the *asymptotic degree* of I w.r.t. M. It is the smallest number d such that $I^nM = I_{\leq d}I^{n-1}M$ for large n, where $I_{\leq d}$ denotes the ideal generated by the elements of I having degree at most d.
- d is one of the generating degrees of I. In particular, $d \leq d_{max}(I)$. If I is equigenerated, i.e. generated in degree δ, then $d = \delta$.
- The intercept e remains mysterious.

Problem: *When does* $\operatorname{reg} I^nM$ *become a linear function, or equivalently, give an upper bound on*

$$\text{reg-stab}(I;M) = \min\{n_0 |\ \operatorname{reg}(I^nM) = dn + e \ \ \forall n \geq n_0\}.$$

I. Motivation

Hard problem. Few results: even in the case $\ell(R/I) < \infty$

* D. Berlekamp, *Regularity defect stabilization of powers of an ideal*, Math. Res. Lett. **19** (2012), 109 - 119.
* D. Eisenbud and B. Ulrich, *Notes on regularity stabilization*, Proc. Amer. Math. Soc. **140** (2012), 1221 - 1232.
* M. Chardin, *Regularity stabilization for the powers of graded M-primary ideals*, Proc. Amer. Math. Soc. **143** (2015), 3343 - 3349.

No explicit bound for reg-stab(I), except:

Theorem 3.1 in Berlekamp: *Let I be an $\mathfrak{m}$-primary monomial ideal of $S = KX_1, ..., X_r]$, with asymptotic degree d, and the number of generators of type X_i^d is equal to s. Then*

$$\text{reg-stab}(I) \leq \max\{r,\ (r-1)[s(d-1)-1]+1\}.$$

I. Motivation

Higher dimensional case

Similarly,

$$\mathrm{reg}(\overline{I^n}M) = dn + \bar{e} \ \forall n \gg 0.$$

Let

$$\overline{\text{reg-stab}}(I, M) = \min\{n_0 | \ \mathrm{reg}(\overline{I^n}M) = dn + \bar{e} \ \ \forall n \geq n_0\}.$$

Theorem 3.13 in [a] *For any monomial ideal in $S = k[X_1, ..., X_r]$,*

$$\overline{\text{reg-stab}}(I) \leq (r+1)(r+2)r^r d_{\max}(I)^{2r^2}.$$

[a] H, *Asymptotic behavior of Integer Programming and the stability of the Castelnuovo-Mumford regularity*, Math. Programming; **193**(2022), 157 - 194.

I. Motivation

- It is unclear if the above bound is close to be optimal.

- However, it is no known bound for reg-stab(I) even if I is a monomial ideal.

- In the worst case, even for monomial ideals, e as well as reg-stab(I) should be at least $O(d_{max}(I)^{r-2})$(Theorem 2.7 in[2]).

[2] H., *Maximal generating degrees of powers of homogeneous ideals*, Acta Math. Vietnam. **47**(2022), 19-37

I. Motivation

Question: Why is it so difficult to study/bound e and reg-stab$(I; M)$?

A way to answer this question is to consider:

Problem 1: *Study the behavior of the whole function* reg I^nM*!*

Equivalently,

Problem 1': *Study the behavior of the function*

$$e_n := e_n(I, M) := \text{reg}\, I^nM - dn, \ \ n \geq 1,$$

which is called defect sequence *of the function* reg I^nM [a].

[a] D. Berlekamp, Math. Res. Lett. **19** (2012), 109 - 119.

I. Motivation

$+$ D. Eisenbud and J. Harris[3]: Assume M is generated in degree 0, $\dim M > 0$, $\dim M/IM = 0$, and I is *equigenerated*.
Then, $\{e_n\}$ is a weakly decreasing sequence of non-negative integers.

-D. Eisenbud and B. Ulrich[4]. Under the same assumption and $H^0_{R_+}(M) = 0$, then $e_n - e_{n-1} \leq d$.

[3] *Powers of ideals and fibers of morphisms*, Math. Res. Lett. **17** (2010), 267 - 273.

[4] Proc. Amer. Math. Soc. **140** (2012), 1221–232.

I. Motivation

$+$ If $\dim R/I > 0$ and I is equigenerated, the sequence $\{e_n\}$ needs not be weakly decreasing.

- Even if $M = R$ is a polynomial ring, B. Sturmfels[5] found examples with $e_1 = 0 < e_2$.

- A. Conca[6] gave examples with $e_1 = \cdots = e_n = 0 < e_{n+1}$ for an arbitrary n.

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[5] *Four counterexamples in combinatorial algebraic geometry*, J. Algebra **230** (2000), 282–294.

[6] *Regularity jumps for powers of ideals* in Commutative Algebra with a focus on Geometric and Homological Aspects. Lecture Notes in Pure Applied Mathematics, **244**, 21–32. Chapman & Hall 2006.

I. Motivation

+ If I is not equigenerated (and $M = R$ is a polynomial ring), D. Berlekamp[7] showed that the sequence $\{e_n\}$ can be initially increasing then later decreasing.

The above partial results suggest that the the sequence $\{e_n\}$ could be arbitrary!

Main results of this talk confirm this guess!

For simplicity: $M = R$; $0 \neq I \subset R$: graded ideal *that is not nilpotent*. We study 3 functions: $\operatorname{reg} I^{n-1}/I^n$, $\operatorname{reg} R/I^n$ and $\operatorname{reg} I^n$. (in both papers [8] and [9], the defect sequence of the function $\operatorname{reg} I^n M$ was studied via the function $\operatorname{reg} M/I^n M$.)

[7] Math. Res. Lett. **19** (2012), 109 - 119.
[8] D. Eisenbud and J. Harris, Math. Res. Lett. **17** (2010), 267 - 273.
[9] D. Eisenbud and B. Ulrich, Proc. Amer. Math. Soc. **140** (2012), 1221-232.

II. Equigenerated ideals in a graded ring

Setting:
R any standard graded ring.
$I \subset R$ graded ideal that is not nilpotent.
By Theorem A, $\mathrm{reg}(I^n) = dn + e_n$ with $e_n = e$ for all $n \gg 0$.
Using short exact sequences

$$0 \to I^n \to I^{n-1} \to I^{n-1}/I^n \to 0,$$
$$0 \to I^n \to R \to R/I^n \to 0,$$

one can show

Proposition 2.1. *Let I be an arbitrary graded ideal. Then $\mathrm{reg}\, I^{n-1}/I^n = \mathrm{reg}\, R/I^n = dn + e - 1$ for $n \gg 1$, where d and e are the slope and intercept of the function $\mathrm{reg}\, I^n$ for $n \gg 1$.*

II. Equigenerated ideals in a graded ring

From now on, in this Part II, we assume in addition that I is *generated by forms of degree d*

Definition 2.2. 1) Set $c_n = \operatorname{reg} I^{n-1}/I^n - dn + 1$ for all $n \geq 1$. We call $\{c_n\}$ the *defect sequence of the function* $\operatorname{reg} I^{n-1}/I^n$.
2) Set $a_n = \operatorname{reg} R/I^n - dn + 1$ for all $n \geq 1$. We call $\{a_n\}$ the defect sequence of the function $\operatorname{reg} R/I^n$.

Remarks. i) $e_n \geq 0$ for all n.
Under the assumption that I is generated by forms of degree d, one can prove:
ii) $a_n \geq 0$, and
iii) $c_n \geq 0$ if $\operatorname{ht} I > 0$.
iv) Although $e_n = a_n = c_n = e$ for all $n \gg 0$, they are different for small n.

II. 1. The function $\operatorname{reg} I^{n-1}/I^n$

The case $\dim R/I = 0$

Proposition 2.1.1. *Let I be an equigenerated ideal with $\dim R/I = 0$. Then the defect sequence of the function $\operatorname{reg} I^{n-1}/I^n$ is weakly decreasing.*

It turns out that this additional constraint is exactly the condition for a convergent sequence of non-negative integers to be the defect sequence of the function $\operatorname{reg} I^{n-1}/I^n$ in the case $\dim R/I = 0$.

Theorem 2.1.2. *A sequence of non-negative integers is the defect sequence of the function $\operatorname{reg} I^{n-1}/I^n$ for an equigenerated ideal I in a standard graded algebra R with $\dim R/I = 0$ if and only it is a weakly decreasing sequence.*

II. 1. The function reg I^{n-1}/I^n

For the proof we give an explicit construction

Proposition 2.1.3. *Let* $\{c_n\}_{n\ge 1}$ *be any weakly decreasing sequence of positive integers and* $d \ge 1$. *Let* m *be the minimum integer such that* $c_n = c_m$ *for* $n > m+1$. *Let* $S = k[x,y]$ *and*

$$Q = (x^{c_1}, x^{c_2}y^d, \dots, x^{c_{m+1}}y^{dm}).$$

Let $R = S/Q$ *and* $I = (y^d, Q)/Q$. *Then for all* $n \ge 1$,

$$\operatorname{reg} I^{n-1}/I^n = dn + c_n - 2.$$

II. 1. The function reg I^{n-1}/I^n

The case $\dim R/I > 0$

No constraint other than the convergence on the defect sequence of the function $\operatorname{reg} I^{n-1}/I^n$.

Theorem 2.1.4. *A sequence of non-negative integers is the defect sequence of the function* $\operatorname{reg} I^{n-1}/I^n$ *of an equigenerated graded ideal* I *in a standard graded algebra* R *with* $\dim R/I \geq 1$ *if and only it is a convergent sequence.*

Theorem 2.1.4'. *A numerical function* $f(n)$ *is the function* $\operatorname{reg} I^{n-1}/I^n$ *of an equigenerated ideal* I *of positive height in a standard graded algebra* R *with* $\dim R/I \geq 1$ *if and only if* $f(n)$ *is asymptotically linear with slope* d *and* $f(n) \geq dn - 1$ *for all* $n \geq 1$.

II. 1. The function reg I^{n-1}/I^n

For the proof we give an explicit construction

Proposition 2.1.5. *Let* $\{c_n\}_{n\geq 1}$ *be any convergent sequence of positive integers and* $d \geq 1$. *Let* m *be the minimum integer such that* $c_n = c_m$ *for all* $n > m+1$. $S = k[x_1, x_2, y_1, ..., y_m]$, $P = (y_1, ..., y_m)$ *and*

$$Q = \left(x_1^{c_1}, x_1 P^d, \sum_{i=1}^{m-1} (x_2^{c_{i+1}}, P^d) y_i^{di}, x_2^{c_{m+1}} y_m^{dm}\right).$$

Let $R = S/Q$ *and* $I = (P^d + Q)/Q$. *Then for all* $n \geq 1$,

$$\operatorname{reg} I^{n-1}/I^n = dn + c_n - 2.$$

II. 2. The function reg R/I^n

The case $\dim R/I = 0$

By D. Eisenbud and J. Harris (Proposition 1.1 in[10]): this defect sequence is weakly decreasing.
A further constraint:

Proposition 2.2.1. *Let* $\{a_n\}$ *be the defect sequence of the function* $\operatorname{reg} R/I^n$ *of an ideal* I *generated by forms of degree* d *with* $\dim R/I = 0$. *Then* $a_n - a_{n+1} \leq d$ *for all* $n \geq 1$.

A complete characterization, which follows from Theorem 2.1.2.

Theorem 2.2.2 *A sequence of non-negative integers* $\{a_n\}$ *is the defect sequence of the function* $\operatorname{reg} R/I^n$ *of an ideal* I *generated by forms of degree* d *in a standard graded algebra* R *with* $\dim R/I = 0$ *if and only if it is weakly decreasing and* $a_n - a_{n+1} \leq d$ *for all* $n \geq 1$.

[10] Math. Res. Lett. **17** (2010), 267 - 273.

II. 2. The function reg R/I^n

The case $\dim R/I > 0$

Theorem 2.2.3. *The defect sequence of the function* $\operatorname{reg} R/I^n$ *of an ideal* I *generated by forms of degree* d *with* $\dim R/I \geq 1$ *can be any convergent sequence of non-negative integers* $\{a_n\}$ *with the property* $a_n - a_{n+1} \leq d$ *for all* $n \geq 1$.

Equivalently:

Theorem 2.2.3'. *The function* $\operatorname{reg} R/I^n$ *of an ideal* I *generated by forms of degree* d *with* $\dim R/I \geq 1$ *can be any numerical asymptotically linear function of slope* d *and* $f(n) \geq dn - 1$ *that is weakly increasing.*

Remark. The above condition is not necessary. Nguyen Dang Hop and Vu Quang Thanh (Remark 5.9 in [11]) have constructed an equigenerated ideal I in a polynomial ring R in $m \geq 4$ variables such that $\operatorname{reg} I = m+3$ and $\operatorname{reg} I^n = 6n$ for $n \geq 2$. Therefore, if $m+3 > 6n$,

$$\operatorname{reg} R/I = \operatorname{reg} I - 1 > \operatorname{reg} I^n - 1 = \operatorname{reg} R/I^n.$$

Proof of Theorem 2.2.3: Explicit construction, which is similar to Proposition 2.1.5.

[11] *Homological Invariants of Powers of Fiber Products*, Acta Mathematica Vietnamica **44** (2019), 617 - 638.

II. 2. The function reg R/I^n

Proposition 2.2.4. *Let* $\{c_n\}_{n\ge 0}$ *be any convergent sequence of positive integers and* $d \ge 1$. *Let* m *be the minimum integer such that* $c_n = c_m$ *for all* $n > m$. *Let* $S = k[x_1, x_2, y_1, ..., y_m]$, $P = (y_1, ..., y_m)$ *and*

$$Q = \left(x_1^{c_0}, x_1x_2, x_1P^d, \sum_{i=1}^{m-1}(x_2^{c_i}, P^d)y_i^{di}, x_2^{c_m}y_m^{dm}\right).$$

Let $R = S/Q$ *and* $I = (P^d + Q)/Q$. *Then for all* $n \ge 1$,

$$\operatorname{reg} R/I^n = \begin{cases} \max\{d(i+1) + c_i - 2 \mid i = 0, ..., n-1\} & \text{if } n \le m, \\ \max\{dn + c_m - 2, d(i+1) + c_i - 2 \mid i = 0, ..., m-1\} & \\ \qquad\qquad \text{if } n > m. \end{cases}$$

II. 3. The function reg I^n

The case $\dim R/I = 0$

By Eisenbud and Harris (Proposition 1.1 in[12]): the defect sequence $\{e_n\}$ is weakly decreasing.
Using construction in Proposition 2.3, we can compute reg I^n.

Proposition 2.3.1. *Let $\{c_n\}_{n\geq 0}$ be any weakly decreasing sequence of positive integers and $d \geq 1$. Let m be the minimum integer such that $c_n = c_m$ for all $n > m$. Let $S = k[x, y]$ and*

$$Q = (x^{c_0}, x^{c_1}y^d, ..., x^{c_m}y^{dm}).$$

Let $R = S/Q$ and $I = (y^d, Q)/Q$. Then for all $n \geq 0$,

$$\operatorname{reg} I^n = \begin{cases} \max\left\{d(i+1) + c_i - 2 \mid i = n, ..., m-1\right\} & \text{if } n < m, \\ dn + c_n - 1 & \text{if } n \geq m. \end{cases}$$

II. 3. The function $\operatorname{reg} I^n$

Theorem 2.3.2. *The defect sequence of the function* $\operatorname{reg} I^n$ *of an ideal* I *generated by forms of degree* d *with* $\dim R/I = 0$ *can be any weakly decreasing sequence* $\{e_n\}$ *of non-negative integers with the property* $e_n - e_{n+1} \geq d$ *for* $n < m$*, where* m *is the least integer such that* $e_n = e_m$ *for all* $n > m$*.*

Remarks. 1) The above condition is not necessary.

2) The condition $e_n - e_{n+1} \geq d$ in Theorem 2.3.2 is opposite to the property $a_n - a_{n+1} \leq d$ in Proposition 2.2.1. If $H^0_{R_+}(R) = 0$, $\{e_n\}$ also has the property $e_n - e_{n+1} \leq d$ for all $n \geq 1$ (Proposition 1.4(1) in [13]. We have $H^0_{R_+}(R) \neq 0$ in the proof of Theorem 2.3.2.

[13] D. Eisenbud and B. Ulrich, Proc. Amer. Math. Soc. **140** (2012), 1221 - 232.

II. 3. The function reg I^n

Question (D. Eisenbud and B. Ulrich): Is the sequence $\{e_n - e_{n+1}\}$ always weakly decreasing?

Example. Let $e_n = e_m + d(m-n) + (m-n)(n+m-1)/2$ for $n < m$ in Theorem 2.3.2. Then $e_n - e_{n+1} = d + n$ for $n < m$. Hence $\{e_n - e_{n+1}\}$ is an increasing sequence for $n < m$. This gives a large class of counter-examples to the above question of D. Eisenbud and B. Ulrich.

II. 4. The function sdeg I^n

Definition 2.4.1. Let $\widetilde{I} = \bigcup_{t \geq 0} I : R_+{}^t$ be the saturation of I. *The saturation degree* sdeg I *of* I is defined by

$$\operatorname{sdeg} I := a(\widetilde{I}/I) + 1 = a(H^0_{R_+}(R/I)) + 1.$$

L. Ein, H. T. Hà and R. Lazarsfeld (see Theorem A in [14]) proved that if $R = \mathbb{C}[x_0, ..., x_r]$ is a polynomial ring over the complex numbers and $I = (f_0, ..., f_p)$ an ideal generated by forms of degree $d_0 \geq \cdots \geq d_p$ such that the projective scheme cut out by the $f_0, ..., f_p$ is nonsingular, then sdeg $I^n \leq d_0 n + d_1 + \cdots + d_r - r$ for all $n \geq 1$.

[14] *Saturation bounds for smooth varieties*, Algebra Number Theory **16** (2022), 1531-1546.

II. 4. The function sdeg I^n

Extending the method in [15] we can show

Theorem 2.4.2. *Let I be a graded ideal and d its asymptotic degree.*
(1) *If $H^1_{R_+}(I^n) = 0$ for $n \gg 1$, then $\operatorname{sdeg} I^n = a(H^0_{R_+}(R)) + 1$ for $n \gg 1$.*
(2) *If $H^1_{R_+}(I^n) \neq 0$ for $n \gg 1$, then $\operatorname{sdeg} I^n$ is asymptotically a linear function with a positive slope $\delta \leq d$. Moreover, $\delta = d$ if $I_{\leq d}$ is generated by forms of degree d.*

[15] N.V. Trung and H-J. Wang, *On the asymptotic linearity of Castelnuovo-Mumford regularity*, J. Pure Appl. Alg. **201** (2005), 42 - 48.

II. 4. The function sdeg I^n

Example. Let $R = k[x_0, ..., x_r]$ and $I = lQ$, where l is a linear form and $Q = (x_0^2, ..., x_r^2, x_0 \cdots x_r)$. Then the projective scheme cut out by the generators of I is nonsingular. By the above result of L. Ein, H. T. Hà and R. Lazarsfeld: sdeg $I^n \le (r+2)n + 2r$ for all $n \ge 1$. On the other hand, the asymptotic degree of I is 3. This follows from the fact that $I^2 = I_{\le 3} I$ and $I_{\le 3} = l(x_0^2, x_1^2, ..., x_r^2)$.
One can show sdeg $I^n = 3n + r - 1$ for all $n \ge 1$.

Proposition 2.4.3 *Let I be an ideal generated by forms of degree d.*

(i) *Assume that $H^1_{R_+}(I^n) \ne 0$ for $n \gg 1$. Then* sdeg $I^n = dn + b$ *for $n \gg 1$ for some $b \ge 0$.*

(ii) *We set $b_n :=$ sdeg $I^n - dn$ if $\widetilde{I^n} \ne I^n$ for all $n \ge 1$ and call $\{b_n\}$ the defect sequence of the function* sdeg I^n.
Assume that $H^0_{R_+}(R) = 0$. Then $b_n \ge 0$ if $\widetilde{I^n} \ne I^n$.

II. 4. The function sdeg I^n

Remarks. - If $H^0_{R_+}(R) \neq 0$, b_n may be a negative number.
- If $\dim R/I = 0$, $\operatorname{sdeg} I^n = \operatorname{reg} R/I^n + 1$ for all $n \geq 1$. Hence, $\{b_n\}$ is the defect sequence of the function $\operatorname{reg} R/I^n$. By Remark after Definition 2.2, $b_n \geq 0$ for all $n \geq 1$.
By Theorem 2.2.2, a sequence of non-negative integers $\{b_n\}$ is the defect sequence of the function $\operatorname{sdeg} R/I^n$ of an ideal I generated by forms of degree d in a standard graded algebra R with $\dim R/I = 0$ if and only if it is weakly decreasing and $b_n - b_{n+1} \leq d$ for all $n \geq 1$.
- It remains to consider the case $\dim R/I \geq 1$.

Theorem 2.4.4. *The defect sequence of the function* $\operatorname{sdeg} I^n$ *of an ideal* I *generated by forms of degree* d *with* $\dim R/I \geq 1$ *can be any convergent sequence of non-negative integers* $\{b_n\}$ *with the property* $b_n - b_{n+1} \leq d$ *for all* $n \geq 1$.

Proof. Use the construction in Proposition 2.2.4.

III. Polynomial ideals

Setting
$S = k[x_1, \dots, x_r]$; $0 \neq I \subset S$: homogeneous ideal, which can be generated in different degrees. $\mathfrak{m} = (x_1, \dots, x_s)$.
In this case $\operatorname{reg} I^n = \operatorname{reg} S/I^n + 1$, so we only study $\operatorname{reg} I^n$.

Small dimension: some restrictions

Proposition 3.1.
(i) If $\dim(S/I) \leq 1$, *then for all* $n, m \geq 1$ *we have* $e_{n+m} \leq e_n + e_m$.
In particular, if $e_{n_0} = 0$ *for some* $n_0 \geq 1$, *then* $e_n = 0$ *for all* $n \gg 0$.

(ii) Assume that $\dim(S/I) = 0$. *For all* $m > n \geq 2$, *we have* $e_m/(m-1) \leq e_n/(n-1)$.
In particular, if $e_{n_0} = 0$ *for some* $n_0 \geq 1$, *then* $e_n = 0$ *for all* $n \geq n_0$.

III. Polynomial ideals

From Proposition 3.1(i), it is clear that not any bounded non-decreasing function can be realized as a defect sequence of the Castelnuovo-Mumford regularity function of an ideal of dimension at most one.

However, we can prove

Theorem 3.2. *Let $f: \mathbb{N} \to \mathbb{N}$ be any non-increasing function. Then there is an $\mathfrak{m}$-primary monomial ideal I such that $e_n = f(n)$ for all $n \geq 1$.*

The construction is quite complicate.

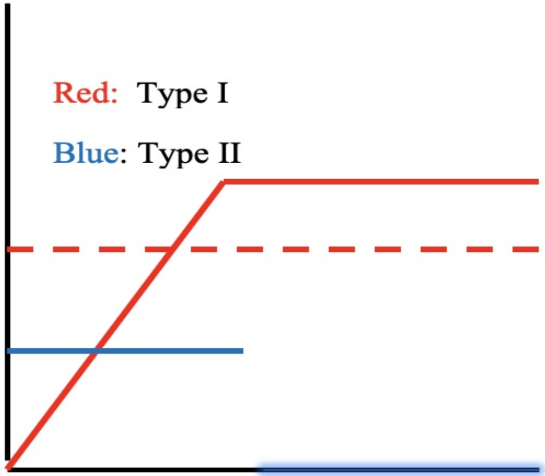
Step 1: Construct a monomial ideal I such that $\operatorname{reg} I^n = dn$ (that is $e_n = 0$) for all $n > n_0$; and $\operatorname{reg}(I^n) \geq dn + \omega$ (that is $e_n > \omega$) for all $1 \leq n \leq n_0$, where d and ω satisfy certain relations $\omega \ll d$.

III. Polynomial ideals

Step 2: Truncate this ideal by a power of $\mathfrak{m}$ and apply Theorem 2.3 in[16]. Then, we get an ideal J whose defect sequence has an elementary type (called type II): $e_n(J) = \textit{constant}$ for $n \leq n_0$ and $e_n(J) = 0$ for $n > n_0$.

[16] D. Eisenbud and B. Ulrich, Proc. Amer. Math. Soc. **140** (2012), 1221 - 232.

III. Polynomial ideals



III. Polynomial ideals

Step 3: Use the so-called fiber product:

Definition 3.3. Assume that x and y are two disjoint sets of variables. Let $I \subset k[\mathsf{x}]$ and $J \subset k[\mathsf{y}]$ be ideals. We set $\mathfrak{m} := (\mathsf{x})$, $\mathfrak{n} := (\mathsf{y})$. The *fiber product*

$$I \times_k J := (I, J, \mathfrak{m}\mathfrak{n}) \subset K[\mathsf{x}, \mathsf{y}].$$

III. Polynomial ideals

Lemma 3.4. *Assume that* $\dim K[\mathrm{x}]/I = \dim[\mathrm{y}]/J = 0$ *and the two functions* $\operatorname{reg} I^n$ *and* $\operatorname{reg} J^n$ *have the same slope* d*. Assume further, that* $e_n(I) \leq d-2$ *and* $e_n(J) - 2$ *for all* $n \geq 1$*. Then for all* $n \geq 1$*, we have*

$$\operatorname{reg}((I \times_k J)^n) = \max\{\operatorname{reg}(I^n),\ \operatorname{reg}(J^n)\}.$$

This implies $\operatorname{reg}(I \times_k J)^n$ *is an asymptotic linear function of slope* d *and*

$$e_n(I \times_k J) = \max\{e_n(I),\ e_n(J)\}.$$

III. Polynomial ideals

Then we can use induction and put together one ideal with non-increasing function defect sequence with another ideal with function defect sequence of type II.

Higher dimension

In this case, we can show

Theorem 3.5. *Given any sequence of positive numbers* $2 \leq n_1 < n_2 < \cdots < n_k$ $(k \geq 1)$ *such that* $n_{i+1} - n_i \geq 2$. *Then there is a monomial ideal such that its defect sequence of the Castelnuovo-Mumford regularity - considered as a numerical function - gets local maxima exactly at points of the set* $\{n_1, ..., n_k\}$.

For the proof, we need a monomial ideal whose defect sequence is of type I.

III. Polynomial ideals

Lemma 3.6 (Subsection 4.2 in[a]). *Let* $d \geq 2$ *and* $b \leq s(d-1) - d$. *Let*

$$J := (X_1^d, \ldots, X_s^d) + (X_1, \ldots, X_s)^{d+b}.$$

Let

$$\begin{aligned} t_0 &= \lfloor \tfrac{s(d-1)+1}{d+b} \rfloor, \\ \delta &= \max\{0,\ s(d-1) + 1 - t_0(d+b) - p\} < b. \end{aligned}$$

Then the slope of $\operatorname{reg} J^n$ *is* d *and*

$$e_n(J) = \begin{cases} bn & \text{if } n \leq t_0, \\ t_0 b + \delta & \text{if } n > t_0. \end{cases}$$

[a] D. Berlekamp, Math. Res. Lett. **19** (2012), 109 - 119.

III. Polynomial ideals

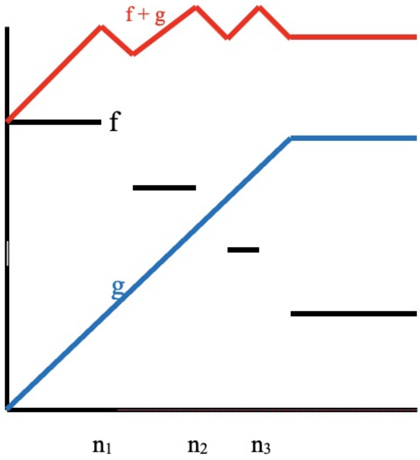
Then, one can put an ideal in Theorem 3.2 and an ideal in Lemma 3.6 together to get a proof of Theorem 3.5, by using the following technique, which is an immediate consequence of Lemma 3.2 in[17]

Lemma 3.7. *Given two non-zero ideals $I \subset K[X]$ and $J \subset K[Y]$, where all variables are different. We consider IJ as an ideal of $K[X, Y]$. Then for all $n \geq 1$, we have*

$$e_n(IJ) = e_n(I) + e_n(J).$$

[17] H. and N. D. Tam, Arch. Math. **94** (2010), 327 - 337.

III. Polynomial ideals



III. Polynomial ideals

Conjecture. *Any convergent sequence of non-negative integers can be realized as a defect sequence of the function Castelnuovo -Mumford regularity.*

THANK YOU FOR YOUR ATTENTION!