Thermodynamics of information

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Expressions are greatly simplified by using the function $h(x) \equiv -x \ln x - (1-x)\ln(1-x)$

Lesson 1. Introduction: Maxwell demons and Szilárd engines.

1.1 Consider a Szilárd engine where the measurement is wrong with probability ϵ. In that case, we cannot perform the whole expansion from $V_0/2$ to V_0 because, if the measurement is wrong, we would compress the particle to zero volume (which takes an infinite amount of work). Then, we have to consider a partial expansion from $V_0/2$ to αV_0 with $1/2 < \alpha < 1$.

a) Calculate the average extracted work as a function of α.

b) Determine the protocol that maximizes the extracted work.

1.2 The work needed to compress reversibly and isothermally an ideal gas from a volume V_{init} to a final volume V_{fin} is $W = kT\ln(V_{\text{init}}/V_{\text{fin}})$. This work is the transfer of energy from the external agent that changes the volume of the gas to the system and is dissipated as heat to the thermal bath, since the internal energy of the gas does not depend on the volume.

Consider now a generic system in contact with a thermal bath at temperature T, which occupies a region in the phase space with volume $\mathcal{V}_{\text{init}}$. In certain isothermal processes, this region is mapped onto a region with volume $\mathcal{V}_{\text{fin}}$. Following Landauer's arguments, show that minimal dissipated heat in this process is $Q_{\text{min}} = kT\ln(\mathcal{V}_{\text{init}}/\mathcal{V}_{\text{fin}})$.

1.3 In this exercise we analyze the physical implementation of a random boolean function: $f(0) = 0$ or 1 with probability $1/2$, and $f(1) = 1$, as shown in the leftmost sketch of Fig. 1.

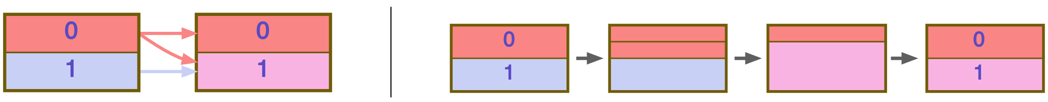

Figure 1: Implementation of a random boolean function.

a) Calculate the change of the Shannon entropy along the process, assuming that the initial information states are equally probable.

b) Consider the implementation described by the sketch in Fig. 1 (*right*). We split state 0 into two regions with the same volume. Then we remove the barrier separating the two lower regions and, finally, we compress the lower region and expand the upper region as shown in the figure. Calculate the dissipated heat if all the manipulations are quasistatic and the two informational states, 0 and 1, are initially occupied with the same probability.

c) Generalize the above procedure to the implementation of an arbitrary random function: $f(i) = j$ with probability $p(j|i)$ and $i, j = 1, 2, \ldots, N$. Show that the minimal dissipated heat to implement the function is

$$Q_{\text{min}} = kT\left[-\ln N - \sum_j p(j)\ln p(j)\right]$$

where $p(j) = \sum_i p(j|i)/N$ is the probability that the output of the function is j if the input is chosen at random among the $i = 1, 2, \ldots, N$ possible states with the same probability $1/N$.

Lesson 2. Information theory.

2.1 Consider the two following distributions, q and p, of a random variable X that takes on two values, 0 and 1:

a) $p(x) = 1/2 \quad$ for $x = 0, 1$;

b) $q(0) = \epsilon \quad$ and $\quad q(1) = 1 - \epsilon$.

Calculate and plot the relative entropies $D(p(x)||q(x))$ and $D(q(x)||p(x))$. Which one is bigger? Could you argue why?

2.2 Calculate the mutual information of the measurement output of a binary variable X. The variable takes on two values, 0 and 1, with probability p and $1 - p$ respectively, and the measurement has a symmetric error ϵ, i.e., the probability of a wrong outcome m is ϵ, independent of the true value X. What is the maximum work that we can extract from a Szilárd engine in a container of volume V if our measurement has an error ϵ? Check that the optimal protocol derived in exercise **1** is able to extract that amount of work.

Lesson 3. Stochastic thermodynamics.

3.1 Consider a particle of mass m in a harmonic potential of frequency ω_0, $V(x) = \frac{m\omega_0^2 x^2}{2}$, in contact with a thermal bath at temperature T. The particle is in equilibrium and the frequency is *suddenly* changed from the initial value ω_0 to a value $\omega_1 > \omega_0$.

a) Calculate the work done to change the frequency.

b) After the change, the particle relaxes to a new equilibrium state dissipating energy to the thermal bath. Calculate this dissipated heat.

c) Compare the work obtained in the first question with the variation of equilibrium free energy.

d) Calculate the entropy production along the whole process.

e) Check the validity of the Crooks theorem and the Jarzynski equality.

Lesson 4. Information and the second law.

4.1 Consider the Szilárd engine with imperfect measurements introduced in exercise 1.1

a) Calculate the mutual information $I(X; M)$ between the system and the measurement outcome, or memory, and the maximum work that can be extracted.

b) Calculate the efficiency $\eta = \frac{W_{\text{extract}}}{TI(X;M)}$ of an information engine involving a protocol that moves the piston from the middle point to a distance α from the wall of the side that is empty according to the measurement.

c) Find the maximum efficiency and the optimal protocol.

4.2 Non-equilibrium free energy can provide a general explanation of the results obtained in exercise 1.3. Consider a symmetric memory with N informational states $j = 1, 2, ..., N$. We implement the following random function: $f(i) = j$ with probability $p(j|i)$. Calculate the minimal work needed to implement the function if the initial probabilistic informational state is the uniform distribution $p_{\text{init}} = 1/N$.

Lesson 5. Fluctuation theorems for feedback processes.

5.1 Write the Horowitz-Vaikuntanathan fluctuation theorem for the optimal Szilárd engine with error that you found in exercises **1** and **2**.

Lesson 6. Optimal Maxwell demons.

6.1 Check that the optimal protocol that you found in exercise **1** is reversible.

Lesson 7. Thermodynamic cost of measurement.

7.1 Let's consider again the Szilárd engine with imperfect measurements introduced in exercise 1.1. We consider that the energy associated with the particule being in the right or left reservoir is the same. Using the results of exercice 4.1 and given that the measurement outcomes are stored in a one bit memory whose states are degenerate in energy and that it initially starts in 0:

a) What is the minimal work needed to perform the measurement?

b) Calculate the minimal work necessary to reset the memory to its initial state.

c) Calculate the minimal work necessary to complete the feedback process.

d) Sum the three former quantities

Lesson 8. Information flows.

8.1 Following [Horowitz, Esposito, PRX **4**, 031015], Sec. IV, analyze the information flow in the optimal Szilárd engine with error that you found in exercises **1** and **2**.

Lesson 9. Creating information.

9.1 In the globally coupled Ising model of N interacting spins and of Hamiltonian

$$H = -\frac{J}{N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} s_i s_j - B\sum_{i=1}^{N} s_i$$

with J the coupling between spins and B the external magnetic field:

a) Give the conditions to witness a spontaneous symmetry breaking.

b) Give the conditions to witness a forced symmetry breaking.

9.2 In the case of a Szilárd engine, can you identify the step corresponding to a spontaneous symmetry breaking ? And a forced symmetry breaking (think about the reverse protocol)?

9.3 Consider a system with Hamiltonian $H(\lambda)$, subjected to an isothermal process at temperature T involving a symmetry breaking, where the control parameter $\lambda(t)$ changes in time with $t \in [t_{ini}, t_{fin}]$. Given that the average work required to complete the process, when the system adopts instance i, is bound by $\langle W\rangle_i^{(SB)} - \Delta F_i \geqslant kT\ln(p_i)$ where $\Delta F_i = F_{fin,i} - F_{init}$ is the change in free energy and p_i is the probability for such instance to be the final configuration:

a) Show that the conformational entropy production is bounded by $\langle S_{\text{prod}}\rangle_i^{(SB)} = \Delta S_i - \frac{\langle Q\rangle_i^{(SB)}}{T} \geqslant k\ln(p_i)$.

b) Given that the symmetry restoration (SR) process is such that its time reversal leads to probabilities $\tilde{p}_i$ that the system adopts instance i, what is the bound on the work necessary to complete this symmetry restoration process ?

c) Summing the weighted quantities computed above, give the expression for the maximal work one can extract from implementing a spontaneous symmetry breaking followed by a forced symmetry breaking.

Lesson 10. Maxwell demons in the phase space.

10.1 In a microcanonical Szilárd engine, by properly choosing the height of the barrier splitting the box in two regions and allowing one of the box wall to move such that the total size of the box increases, one can extract energy from the trapped particle without measurement via a cyclic protocol $\lambda(t)$. In this context, the energy of the particle decreases after each cycle. Using the result of exercise 8.3.c), show that a typical Szilárd engine which does not use information from the measurement cannot extract useful work (keep in mind that the relative entropy is always a positive quantity).