Day I

RMT is a very vast subject, with many applications. To try to pretend to give lectures in 2 weeks would be foolish from my part. So, instead I am going to be selfish and present some of the work I have done during the last years that one could entitled very bravely

Statistical Mechanics
of
Random Matrices

If you could go away with a main goal of these lectures will be: the art of ~~stealing~~ using tools of statistical mechanics (of disordered systems) to solve problems of interest in (in this case) Random matrices

Train of thought is the following

problem of interest in random matrices $\xrightarrow{\text{map}}$ $\boxed{\text{problem in statistical mechanics of disordered systems (spin glasses)}}$

solve it by using:

* concepts of stat. mech
  * partition functions
  * free energies
  * ...
* tools of disordered systems
  * cavity method
  * replica method
* some mathematical tools
  * saddle point method
  * multidimensional Gaussian integrals
  * useful representations for Dirac deltas & Heaviside

Content of these lectures

functions

I. Some useful mathematical tools | Day 1

2. tools from disordered systems | Days 2,3

3. problems in RMs mapped into problems of statistical mechanics → Days 4,5

4. Particular problems

days 1,2 | 4a) spectral density of directed and undirected random graphs

days 3,4 | 4b) Large deviation theory for the spectrum of random graphs

1st week | 2nd week

Day 5 → exam!

I Some mathematical tools

The saddle-point method

This method is also referred to it as steepest descent or Laplace method.

Suppose that:

$$I_N = \int_a^b dx\, e^{-N S(x)}$$

S is a real function

study the asymptotic behaviour of I_N for large N. One can show that

$$I_N \asymp e^{-Nf(x_0)}$$

$x_0 : f'(x)|_{x=x_0} = 0$

$f''(x)|_{x=x_0} > 0$

$x_0 \in (a,b)$

and this symbol means:

$$-\lim_{N\to\infty} \frac{1}{N} \ln I_N = f(x_0)$$

<u>Proof:</u> Expand $f(x)$ around x_0

$$f(x) = f(x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \sum_{n\geq 3} \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n$$

this means that

$$I_N = \int_a^b dx \, \exp[-Nf(x)]$$

$$= \int_a^b dx \, \exp\Big[-Nf(x_0) - \frac{N}{2}|f''(x_0)|(x-x_0)^2 - \sum_{n\geq 3} \frac{N}{n!} f^{(n)}(x_0)(x-x_0)^n\Big]$$

$$= e^{-Nf(x_0)} \int_a^b dx \exp\left(-\frac{N}{2}|f''(x_0)|(x-x_0)^2 + R(x)\right)$$

Do the following change of variables

$$y = \sqrt{N}(x-x_0)$$

$$= e^{-Nf(x_0)} \int_{\sqrt{N}(a-x_0)}^{\sqrt{N}(b-x_0)} dy/\sqrt{N} \; e^{-|f''(x_0)|y^2/2 + \tilde{R}(y)}$$

$$= e^{-Nf(x_0)} \int_{\sqrt{N}(a-x_0)}^{\sqrt{N}(b-x_0)} dy/\sqrt{N} \; e^{-|f''(x_0)|y^2/2} \sum_{n=0}^{\infty} \frac{\tilde{R}^n(y)}{n!}$$

since $x_0 \in (a,b) \Rightarrow \int_{\sqrt{N}(a-x_0)}^{\sqrt{N}(b-x_0)} \longrightarrow \int_{-\infty}^{\infty}$

$$I_N \simeq e^{-Nf(x_0)} \underbrace{\int_{-\infty}^{\infty} \frac{dy}{\sqrt{N}} e^{-|f''(x_0)|y^2/2}}\; \underbrace{\sum_{n=0}^{\infty} \frac{\tilde{R}^n(y)}{n!}}$$

Gaussian measure + perturbative terms

Wick's theorem
(Feynman diagrams
etc.)

leading term

$$I_N \simeq e^{-Nf(x_0)}\sqrt{\frac{2\pi}{N|f''(x_0)|}}$$

If f is complex function $f: \mathbb{C} \to \mathbb{C}$

$$I_N = \int_{a,\gamma}^{b} dz\, e^{-Nf(z)}$$

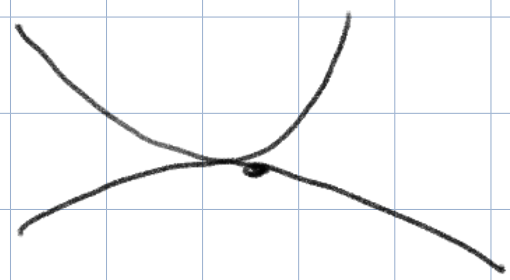
the same idea applies by deforming the path γ so that is goes through a point z_0 such that

$$f'(z_0) = 0$$

and then you further deform the path so that the imaginary part of

$$f''(z_0)(z-z_0)^2$$

is constant for a while so that the Laplace method can be applied to the real part.



the point at which $f'(z_0)=0$ has the form of a saddle (thus the name)

This is also called stationary phase approximation

The equation(s)/conditions for which

$$f'(x_0)=0$$

are called saddle-point equations

Of course this is quickly generalizable to multivariate case (and path integrals)

$$I_N = \int d^N\vec{x}\, e^{-Nf(\vec{x})} \simeq e^{-Nf(\vec{x}_0)}$$

with $\nabla f(\vec{x})\big|_{\vec{x}=\vec{x}_0} = 0$ saddle point equations

and $\underbrace{\nabla_i \nabla_j f(\vec{x})\big|_{\vec{x}=\vec{x}_0}}_{\text{Hessian matrix}} > 0$

Multidimensional Gaussian integrals

Very important in condensed matter (QFT, CFT, etc). In our case we worry about integral expressions of determinants

Suppose that we have a definite positive symmetric matrix A of size $N \times N$

$$A = A^T \qquad T \text{ means to transpose}$$

and $A > 0$ (so eigenvalues are positive)

Then

$$\frac{1}{\sqrt{\det A}} = \int_{\mathbb{R}^N} \frac{d^N \vec{x}}{(2\pi)^{N/2}} \, e^{-\frac{1}{2}\vec{x}^T \cdot A \vec{x}} \qquad d^N\vec{x} = \prod_{i=1}^{N} dx_i$$

or more generally

$$\frac{1}{\sqrt{\det A}}\, e^{\frac{1}{2}\vec{b}^T A^{-1}\vec{b}} = \int_{\mathbb{R}^N} \frac{d^N\vec{x}}{(2\pi)^{N/2}}\, e^{-\frac{1}{2}\vec{x}^T A\vec{x}+\vec{b}\cdot\vec{x}}$$

Btw, this is also called Hubbard-Stratonovich transformation

In physics these are usually called generating or external fields

Proof: Start with $\vec{b}=\vec{0}$. Since $A=A^T$ then

$$\Delta = O^T A O \qquad O \equiv \text{orthogonal transformation}$$

and $\Delta = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$

since $A>0 \Rightarrow \lambda_i > 0 \;\forall i=1,\ldots,N$

Define a transformation $\vec{x}\to\vec{x}\,'$

$$\vec{x} = O\vec{x}\,'$$

this implies that

$$\vec{x}^T A \vec{x} = \vec{x}'^T O^T A O \vec{x}' =$$

$$= \vec{x}'^T \Delta \vec{x}' = \sum_{i=1}^{N} d_i x_i'^2$$

also $$d^N \vec{x} = \left| \frac{\partial \vec{x}}{\partial \vec{x}'} \right| d^N \vec{x}'$$

↳ Jacobian of transformation

But $\frac{\partial \vec{x}}{\partial \vec{x}'} = O$ and $\det O = \pm 1$

Therefore

$$\int_{\mathbb{R}^N} d^N \vec{x}\, e^{-\frac{1}{2}\vec{x}^T A \vec{x}}$$

$$= \int_{\mathbb{R}^N} d^N \vec{x}'\, e^{-\frac{1}{2}\sum_{i=1}^{N} d_i x_i'^2}$$

$$= \prod^{N} \int dx\, e^{-d_i x^2/2} = \prod^{N} \sqrt{\frac{2\pi}{d_i}}$$

$i=1 \quad K \qquad\qquad i=1$

$$= \frac{(2\pi)^{N/2}}{\sqrt{\prod_{i=1} \lambda_i}} = \frac{(2\pi)^{N/2}}{\det \Lambda}$$

$$= \frac{(2\pi)^{N/2}}{\det A}, \quad \text{since } \det \Lambda = \det A$$

<u>Exercise</u>. Do the case for $\vec{b} \neq \vec{0}$

Suppose now that A is an $N \times N$ complex matrix with nonvanishing determinant. Then

$$\frac{1}{\det A} = \int \prod_{i=1}^{N} \frac{dz_i \, d\bar{z}_i}{2\pi i} \exp\left[-\sum_{i,j=1}^{N} \bar{z}_i A_{ij} z_j\right]$$

with $z_i, \bar{z}_i \in \mathbb{C}$

or more generally

$$\frac{\exp\left(\sum_{i,j} \bar{b}_i A^{-1}_{ij} b_j\right)}{\det A} =$$

$$= \int \left[\prod_{i=1}^{N} \frac{d\bar{z}_i \, dz_i}{2\pi i}\right] \exp\left[-\sum_{i,j=1}^{N} \bar{z}_i A_{ij} z_j + \sum_i \left(b_i \bar{z}_i + \bar{b}_i z_i\right)\right]$$

these are generating fields

<u>Exercise: prove this</u> (Book of Zinn-Justin)

<u>Other useful results for our mappings</u>

Sokhotski-Plemelj identity

$$\lim_{\eta \to 0^+} \frac{1}{x - i\eta} = \mathcal{P}\frac{1}{x} + i\pi\,\delta(x)$$

and a prescription for the Heaviside step function

$$\Theta(-x) = \lim_{\eta \to 0^+} \frac{1}{2\pi i}\left[\log(x + i\eta) - \log(x - i\eta)\right]$$